A new variant of the local projection stabilization for convection–diffusion–reaction equations

Petr Knobloch

MATH-NM-09-2009
July 2009
A NEW VARIANT OF THE LOCAL PROJECTION STABILIZATION FOR CONVECTION–DIFFUSION–REACTION EQUATIONS

PETR KNOBLOCH

Abstract. We introduce a new variant of the local projection stabilization for scalar steady convection–diffusion–reaction equations which allows to use local projection spaces defined on overlapping sets. This enables to define the local projection method without the need of a mesh refinement or an enrichment of the finite element space and increases the robustness of the local projection method with respect to the choice of the stabilization parameter. The stabilization term is slightly modified, which leads to an optimal estimate of the consistency error even if the stabilization parameters scale correctly with respect to convection, diffusion and mesh width. We prove that the bilinear form corresponding to the method satisfies an inf–sup condition with respect to the SUPG norm and establish an optimal error estimate in this norm. Moreover, we show that approximation of exponential boundary layers can be significantly improved by increasing the polynomial degree of the approximation on elements of the triangulation at an outflow boundary. The theoretical considerations are illustrated by numerical results.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain with a polyhedral Lipschitz–continuous boundary $\partial \Omega$ and let us consider the convection–diffusion–reaction equation

\begin{equation}
-\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu = f \quad \text{in } \Omega, \quad u = u_b \quad \text{on } \partial \Omega.
\end{equation}

We assume that $\varepsilon$ is a positive constant and $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $u_b \in H^{1/2}(\partial \Omega)$ are given functions satisfying

$$
\sigma := c - \frac{1}{2} \text{div} \mathbf{b} \geq \sigma_0 > 0,
$$

where $\sigma_0$ is a constant. Then the boundary value problem (1.1) has a unique solution in $H^1(\Omega)$.

It is well known that the Galerkin finite element method is not appropriate for solving the problem (1.1) numerically since the discrete solution is typically globally polluted by spurious oscillations if convection dominates diffusion (i.e., $|\mathbf{b}| \gg \varepsilon$).
To enhance the stability and accuracy of the Galerkin method, various stabilization approaches have been developed, see [19] for an overview. In this paper, we concentrate on stabilization by local projections. This technique was originally proposed for stabilizing discretizations of the Stokes problem in which both the pressure and the velocity components are approximated using the same finite element space [1]. Later, the local projection method was extended to stabilization of convection dominated problems [2] and applied to various types of incompressible flow problems (see the review article [5]) and to convection–diffusion–reaction problems, see [12, 14, 15, 16, 18]. Local projection stabilizations preserve the stability properties of the popular residual–based stabilizations [19] but do not require the computation of second order derivatives and can be easily applied to non–steady problems. Moreover, when applied to systems of partial differential equations, it is possible to avoid undesirable couplings between various components of the solution. A further advantage of these techniques is that they are symmetric. Therefore, if they are applied to optimization problems, the operations ‘discretization’ and ‘optimization’ commute [3, 4].

A drawback of all local projection formulations proposed up to now is that they require (significantly) more degrees of freedom than, e.g., residual–based methods. In this paper we remove this drawback by allowing that the sets on which local projection spaces are defined overlap. Although this is a rather simple idea, the corresponding analysis is by no means a straightforward extension of results published before. In contrast to the traditional error analysis, which is based on the construction of a special interpolation operator, we first show that the bilinear form of the local projection method satisfies an inf–sup condition with respect to a norm containing a streamline derivative term. This improved stability of the local projection method enables to perform the error analysis in a similar way as for residual–based methods. Of course, it is also important on its own since it shows that the local projection method is more stable than its coercivity suggests. Let us mention that a similar stability result was already established in [16] as a consequence of a more general inf–sup condition, however, only under certain restrictions on the convection field \( \mathbf{b} \) or the mesh width \( h \).

Our numerical results show that the overlapping of the sets on which local projection spaces are defined significantly increases the robustness of the local projection method with respect to the choice of the stabilization parameter. Roughly speaking, in non–overlapping variants, spurious oscillations appear for both ‘too small’ and ‘too large’ stabilization parameters whereas, for the overlapping variant, ‘large’ values of the stabilization parameter lead to a smearing of the discrete solution.

Since local projection stabilizations are not consistent, an important step in the error analysis is an estimation of the consistency error. It was demonstrated in [14] that, for stabilizations based on local projections of streamline derivatives, the consistency error generally deteriorates the convergence order if the stabilization parameter scales correctly with respect to \( \mathbf{b} \). As a remedy, we propose to define the local stabilization terms using constant approximations of \( \mathbf{b} \), which makes it possible to prove an optimal error estimate with respect to the norm used in the inf–sup condition. Moreover, in contrast to the analyses published before, it is not necessary to assume a higher (often unrealistic) regularity of \( \mathbf{b} \).

We shall also show that the classical variants of the local projection method based on higher order finite elements have to be expected to provide oscillatory
approximations of exponential boundary layers. To cure this problem, we introduce an enrichment strategy on elements of the triangulation adjacent to the boundary which leads to a considerable improvement of the discrete solutions.

The plan of the paper is as follows. In the next section, we formulate assumptions on approximation and projection spaces and define the local projection discretization investigated in this paper. Section 3 is devoted to the proof of the inf–sup condition and, in Section 4, we derive an optimal error estimate. In Section 5, we present examples of finite element spaces satisfying the assumptions of our theory. In particular, we show that classical finite element spaces can be used, without the need of a mesh refinement or bubble enrichment. Then, in Section 6, we discuss the approximation of exponential boundary layers and, finally, in Section 7, we present our numerical results. Throughout the paper, we use standard notation for Sobolev spaces and corresponding norms, see, e.g., [8]. Given a measurable set $M \subset \mathbb{R}^d$, the inner product in $L^2(M)$ or $L^2(M)^d$ is denoted by $(\cdot, \cdot)_M$ and we use the notation $(\cdot, \cdot)_{\Omega}$ instead of $(\cdot, \cdot)_\Omega$.

2. A local projection discretization

Given $h > 0$, let $W_h \subset H^1(\Omega)$ be a finite–dimensional space approximating the space $H^1(\Omega)$ and set $V_h = W_h \cap H^1_0(\Omega)$. Furthermore, let $M_h$ be a set consisting of a finite number of open subsets $M$ of $\Omega$ such that $\Omega = \bigcup_{M \in M_h} M$. We assume that

\begin{align}
\text{(2.1)} & \quad \text{card}\{M' \in \mathcal{M}_h : M \cap M' \neq \emptyset\} \leq C_M \quad \forall \ M \in \mathcal{M}_h \\
\text{(2.2)} & \quad h_M := \text{diam}(M) \leq C'_M h \quad \forall \ M \in \mathcal{M}_h,
\end{align}

where $C_M \geq 1$ and $C'_M \geq 1$ are constants independent of $h$. Moreover, we assume that, for any $M \in \mathcal{M}_h$, there is a nontrivial space $B_M \subset (W_h|_M) \cap H^1_0(M)$ such that $B_M \subset W_h$ if the functions from $B_M$ are extended by zero outside $\Omega$. For any $M \in \mathcal{M}_h$, we introduce a finite–dimensional space $D_M \subset L^2(M)$ and we assume that there exists a positive constant $\beta_{LP}$ independent of $h$ such that

\begin{align}
\text{(2.3)} & \quad \sup_{v \in B_M} \frac{(v, q)_M}{\|v\|_{0,M}} \geq \beta_{LP} \|q\|_{0,M} \quad \forall \ q \in D_M, \ M \in \mathcal{M}_h.
\end{align}

We shall also need the inverse inequality

\begin{align}
\text{(2.4)} & \quad |v_h|_{1,M} \leq C_{inv} \frac{1}{h_M} \|v_h\|_{0,M} \quad \forall \ v_h \in W_h, \ M \in \mathcal{M}_h,
\end{align}

where $C_{inv}$ is a constant independent of $h$.

We have in mind that $W_h$ is a finite element space (see, e.g., [8]) and that the set $\mathcal{M}_h$ is constructed using the triangulation of $\Omega$ on which the space $W_h$ is defined. Various possibilities how the above assumptions can be satisfied will be presented in Section 5.

For any $M \in \mathcal{M}_h$, we denote by $\pi_M$ a continuous linear projection operator which maps the space $L^2(M)$ onto the space $D_M$. We assume that

\begin{align}
\|\pi_M\|_{L^2(M), L^2(M)} \leq C_\pi \quad \forall \ M \in \mathcal{M}_h,
\end{align}

where $C_\pi$ is a constant independent of $h$. For example, $\pi_M$ can be the orthogonal $L^2$ projection for which $C_\pi = 1$. For any $M \in \mathcal{M}_h$, we introduce the so–called
fluctuation operator \( \kappa_M = id - \pi_M \), where \( id \) is the identity operator on \( L^2(M) \). Then
\[
\|\kappa_M\|_{X(L^2(M),L^2(M))} \leq C_\kappa \quad \forall \ M \in \mathcal{M}_h,
\]
where \( C_\kappa = 1 + C_\pi \). An application of \( \kappa_M \) to a vector valued function means that \( \kappa_M \) is applied componentwise.

For any \( M \in \mathcal{M}_h \), we choose a constant \( b_M \in \mathbb{R}^d \) such that
\[
\|b_M\|_0,\infty,\mathcal{M}_h,\|b - b_M\|_{0,\infty,\mathcal{M}_h} \leq C b_M \quad \forall \ M \in \mathcal{M}_h
\]
with a constant \( C_b \) independent of \( h \). In addition, we introduce a function \( \tilde{u}_{bh} \in W_h \) such that
\[
|b_M| \leq \|b\|_{0,\infty,\mathcal{M}_h}, \quad |b - b_M|_{0,\infty,\mathcal{M}_h} \leq C b_M
\]

The Galerkin solution of (1.1) is a function \( u_h \in W_h \) such that
\[
a^G(u_h,v_h) = (f,v_h) \quad \forall \ v_h \in V_h,
\]
where
\[
a^G(u,v) = \varepsilon (\nabla u, \nabla v) + (b \cdot \nabla u, v) + (cu,v).
\]
If convection dominates diffusion, the Galerkin solution is usually globally polluted by spurious oscillations, cf., e.g., [19]. To stabilize the Galerkin discretization, we change the bilinear form \( a^G \) to
\[
a^{LP}_h(u,v) = a^G(u,v) + s_h(u,v),
\]
where \( s_h(u,v) \) is a local projection stabilization term given by
\[
s_h(u,v) = \sum_{M \in \mathcal{M}_h} \tau_M s_M(u,v),
\]
\( \tau_M \) are nonnegative stabilization parameters and
\[
s_M(u,v) = (\kappa_M(b_M \cdot \nabla u),\kappa_M(b_M \cdot \nabla v))_M.
\]
Thus, the local projection discretization of (1.1) considered in this paper reads:

Find \( u_h \in W_h \) such that \( u_h - \tilde{u}_{bh} \in V_h \) and
\[
a^{LP}_h(u_h,v_h) = (f,v_h) \quad \forall \ v_h \in V_h.
\]
Introducing the norms
\[
|||v|||_G = (\varepsilon |v|^2_1,\Omega + \|\sigma^{1/2} v\|_{0,\Omega}^2)^{1/2}, \quad |||v|||_{LP} = (|||v|||_G^2 + s_h(v,v))^{1/2},
\]
we obtain
\[
a^G(v,v) = |||v|||_G^2, \quad a^{LP}_h(v,v) = |||v|||_{LP}^2 \quad \forall \ v \in H^1_0(\Omega).
\]
This shows that the local projection discretization (2.8) has a unique solution.

**Remark 2.1.** It was shown in [14] that the stabilization parameters \( \tau_M \) should satisfy
\[
\tau_M \sim \min \left\{ \frac{h_M}{\|b\|_{0,\infty,\mathcal{M}_h}}, \frac{h^2_M}{\varepsilon} \right\}.
\]
**Remark 2.2.** A standard choice is to use \( b \) instead of \( b_M \) in (2.7). However, it was demonstrated in [14] that then it is generally not possible to obtain optimal convergence results if (2.10) holds. We shall see in the next sections that the use of \( b_M \) leads to an optimal error estimate.
3. Stability of the Local Projection Discretization

One of the most popular finite element techniques for the numerical solution of the problem (1.1) is the streamline upwind/Petrov–Galerkin (SUPG) method proposed in [7]. An important feature of this method is that it provides stability with respect to the norm

$$
\|v\|_{\text{SUPG}} = \left(\|v\|_0^2 + \|\delta^{1/2} \mathbf{b} \cdot \nabla v\|_{0, \Omega}^2\right)^{1/2},
$$

where $\delta$ is a stabilization parameter satisfying a relation of the type (2.10). In this section, we shall show that the local projection method has similar stability properties. More precisely, we shall prove that it is stable with respect to the norm

$$
\|v\|_{\text{LPSD}} = \left(\|v\|_0^2 + s_h(v, v) + \sum_{M \in \mathcal{M}} \tau_M \|\mathbf{b} \cdot \nabla v\|_{0, M}^2\right)^{1/2}.
$$

The letters SD in the notation $\| \cdot \|_{\text{LPSD}}$ refer to the streamline–derivative term.

**Theorem 3.1.** Let the stabilization parameters $\tau_M$ satisfy

$$
0 \leq \tau_M \leq C \varepsilon h_M^2 \left(\max\{\varepsilon, h_M \|\mathbf{b}\|_{0, \infty, M}\}\right)^{-1} \quad \forall M \in \mathcal{M},
$$

with a constant $C_\varepsilon \geq 1$ independent of $h$ and the data of (1.1). Then the bilinear form $a_h^{LP}$ satisfies

$$
\sup_{v_h \in V_h} \frac{a_h^{LP}(u_h, v_h)}{\|v_h\|_{\text{LPSD}}} \geq \beta \|u_h\|_{\text{LPSD}} \quad \forall u_h \in V_h,
$$

where $\beta$ is a positive constant independent of $h$ and $\varepsilon$.

**Proof.** Given $u_h \in V_h$, we shall construct a function $v_h \in V_h$ such that

$$
a_h^{LP}(u_h, v_h) \geq \|u_h\|_{\text{LPSD}}^2 \quad \text{and} \quad \|u_h\|_{\text{LPSD}} \geq \beta\|v_h\|_{\text{LPSD}}.
$$

The inequalities (3.3) immediately imply the inf–sup condition (3.2).

Consider any $M \in \mathcal{M}$. In view of the inf–sup conditions (2.3), there exists $z_M \in B_M$ such that (cf., e.g., [10])

$$
(z_M, q)_M = \tau_M (\mathbf{b} \cdot \nabla u_h, q)_M \quad \forall q \in D_M,
$$

$$
\|z_M\|_{0, M} \leq \beta_{LP}^{-1} \tau_M \|\mathbf{b} \cdot \nabla u_h\|_{0, M}.
$$

Consequently,

$$
(z_M, \pi_M(\mathbf{b} \cdot \nabla u_h))_M = \tau_M (\mathbf{b} \cdot \nabla u_h, \pi_M(\mathbf{b} \cdot \nabla u_h))_M
$$

$$
= \tau_M \|\mathbf{b} \cdot \nabla u_h\|_{0, M}^2 - \tau_M (\mathbf{b} \cdot \nabla u_h, \kappa_M(\mathbf{b} \cdot \nabla u_h))_M
$$

and hence

$$
(z_M, \mathbf{b} \cdot \nabla u_h)_M = \tau_M \|\mathbf{b} \cdot \nabla u_h\|_{0, M}^2 - \tau_M (\mathbf{b} \cdot \nabla u_h, \kappa_M(\mathbf{b} \cdot \nabla u_h))_M
$$

$$
+ (z_M, \kappa_M(\mathbf{b} \cdot \nabla u_h))_M.
$$

Thus, denoting $z_h = \sum_{M \in \mathcal{M}} z_M$ (with $z_M = 0$ in $\Omega \setminus M$), we get

$$
a_h^{LP}(u_h, z_h) = \sum_{M \in \mathcal{M}} \left(\tau_M \|\mathbf{b} \cdot \nabla u_h\|_{0, M}^2 - \tau_M (\mathbf{b} \cdot \nabla u_h, \kappa_M(\mathbf{b} \cdot \nabla u_h))_M
$$

$$
+ (z_M, \kappa_M(\mathbf{b} \cdot \nabla u_h))_M + \varepsilon (\nabla u_h, \nabla z_M)_M + (c u_h, z_M)_M\right) + s_h(u_h, z_h).
$$
Using (3.5), (2.4) and (3.1), we derive for any $M \in \mathcal{M}$

\begin{align}
(3.7) \quad & \|z_M\|_{0,M} \leq \beta_{LP}^{-1} \tau_M \|b\|_{0,\infty,M} |u_h|_{1,M} \leq C_T \ C_{\text{inv}} \beta_{LP}^{-1} \|u_h\|_{0,M}, \\
(3.8) \quad & |z_M|_{1,M} \leq C_{\text{inv}} h_M^{-1} \|z_M\|_{0,M} \leq C_{\text{inv}} h_M^{-1} \beta_{LP}^{-1} \tau_M \|b\|_{0,\infty,M} |u_h|_{1,M} \\
& \quad \leq C_T \ C_{\text{inv}} \beta_{LP}^{-1} |u_h|_{1,M}, \\
(3.9) \quad & \varepsilon^{1/2} |z_M|_{1,M} \leq \varepsilon^{1/2} C_{\text{inv}} h_M^{-1} \|z_M\|_{0,M} \leq C_T^{1/2} C_{\text{inv}} \beta_{LP}^{-1/2} \tau_M^{1/2} \|b\cdot\nabla u_h\|_{0,M}.
\end{align}

Moreover, it follows from the triangular inequality and (2.5), (2.6), (2.4) and (3.1) that

\begin{align}
(3.10) \quad & \tau_M \left(\|\kappa_M(b \cdot \nabla u_h)\|_{0,M} - \|\kappa_M(b_M \cdot \nabla u_h)\|_{0,M}\right)^2 \\
& \quad \leq \tau_M \left(\|\kappa_M((b - b_M) \cdot \nabla u_h)\|_{0,M}^2 \right) \\
& \quad \leq C_2^2 \tau_M \|b - b_M\|_{0,\infty,M}^2 |u_h|_{1,M}^2 \\
& \quad \leq 2 C_3^2 \ v h M \|b\|_{0,\infty,M} \|b\|_{1,\infty,M} |u_h|_{1,M}^2 \\
& \quad \leq C_1 \|u_h\|_{0,M}^2,
\end{align}

where $C_1 = 2 C_2^2 \ 2 C_T \ C_{\text{inv}}^2 \ |b|_{1,\infty,\Omega}$. Applying the Schwarz inequality to the terms on the right-hand side of (3.6), using (3.5), (3.10), (3.9) and (3.7) and taking into account that $\beta_{LP} \leq 1$, we deduce that

\begin{align}
& a_h^{LP}(u_h, z_h) \geq s_h(u_h, z_h) + \sum_{M \in \mathcal{M}} \tau_M \|b \cdot \nabla u_h\|_{0,M}^2 - C_2 \sum_{M \in \mathcal{M}} \|\sigma^{1/2} u_h\|_{0,M}^2 \\
& \quad - C_3 \sum_{M \in \mathcal{M}} \tau_M^{1/2} \|b \cdot \nabla u_h\|_{0,M} \left(\varepsilon |u_h|_{1,M}^2 + \|\sigma^{1/2} u_h\|_{0,M}^2 + \tau_M s_m(u_h, u_h)\right)^{1/2}
\end{align}

with $C_2 = C_T C_{\text{inv}} \|c\|_{0,\infty,\Omega} \sigma_0^{-1} \beta_{LP}^{-1}$ and $C_3 = (2 + C_T^{1/2} C_{\text{inv}}^2 + 2 C_1^{1/2} \sigma_0^{-1/2}) \beta_{LP}^{-1}$. In view of (2.1), we obtain

\begin{align}
(3.11) \quad & \sum_{M \in \mathcal{M}} \|\sigma^{1/2} u_h\|_{0,M}^2 \leq C_{\mathcal{M}} \|\sigma^{1/2} u_h\|_{0,\Omega}^2, \quad \sum_{M \in \mathcal{M}} |u_h|_{1,M}^2 \leq C_{\mathcal{M}} \|u_h\|_{1,\Omega}^2
\end{align}

and hence, using the inequality $ab \leq \frac{1}{4} a^2 + b^2$ valid for any $a, b \in \mathbb{R}$, we infer that

\begin{align}
(3.12) \quad & a_h^{LP}(u_h, z_h) \geq s_h(u_h, z_h) + \frac{3}{4} \sum_{M \in \mathcal{M}} \tau_M \|b \cdot \nabla u_h\|_{0,M}^2 - C_4 \|u_h\|_{L^2}^2
\end{align}

with $C_4 = (C_2 + C_3^2) \ C_{\mathcal{M}}$. Now let us estimate the term $s_h(u_h, z_h)$. We have

\begin{align}
(3.13) \quad & s_h(u_h, z_h) \leq \sqrt{s_h(u_h, u_h)} \sqrt{s_h(z_h, z_h)} \leq \sqrt{s_h(z_h, z_h)} \|u_h\|_{L^2}.
\end{align}

Using (3.10) with $z_h$ instead of $u_h$ and applying (2.5), we get

\begin{align}
(3.14) \quad & s_h(z_h, z_h) \leq 2 C_3^2 \sum_{M \in \mathcal{M}} \tau_M \|b \cdot \nabla z_h\|_{0,M}^2 + 2 C_1 \sum_{M \in \mathcal{M}} \|z_h\|_{0,M}^2.
\end{align}
Furthermore, using (2.1), (2.4), (3.1) and (3.5), we derive

\[
(3.15) \quad \sum_{M \in M_h} \tau_M \| b \cdot \nabla z_h \|^2_{0,M} \leq C_{\mathcal{M}} \sum_{M \in M_h} \sum_{M' \in M \setminus M' \neq \emptyset} \tau_M \| b \cdot \nabla z_{M'} \|^2_{0,M} \\
\leq C_{\mathcal{M}} \sum_{M, M' \in M_h, M \cap M' \neq \emptyset} \tau_M \| b \|^2_{0,\infty,M \cap M'} \| z_{M'} \|^2_{1,M \cap M'} \\
\leq C_{\mathcal{T}} C_{\mathcal{M}} C_{\beta} \sum_{M, M' \in M_h, M \cap M' \neq \emptyset} \| b \|^2_{0,\infty,M \cap M'} \| z_{M'} \|^2_{0,M} \| z_{M'} \|^2_{1,M \cap M'} \\
\leq C_{\mathcal{T}} C_{\mathcal{M}}^2 C_{\beta} \sum_{M' \in M_h} \| b \|^2_{0,\infty,M} \| h_{M'}^{-1} \| z_{M'} \|^2_{0,M} \\
\leq C_{\mathcal{T}} \sum_{M \in M_h} \tau_M \| b \cdot \nabla u_h \|^2_{0,M},
\]

where \( C_{\mathcal{T}} = C_{\mathcal{T}} C_{\mathcal{M}} C_{\beta} \beta_{L_p}^{-1} \). The assumption (2.1) implies that

\[
\sum_{M \in M_h} \| z_h \|^2_{0,M} \leq C_{\mathcal{M}} \sum_{M, M' \in M_h, M \cap M' \neq \emptyset} \| z_{M'} \|^2_{0,M} \leq C_{\mathcal{M}} \sum_{M \in M_h} \| z_{M'} \|^2_{0,M},
\]

which, in view of (3.7) and (3.11), gives

\[
(3.16) \quad \| z_h \|^2_{0,\Omega} \leq \left( \sum_{M \in M_h} \| z_h \|^2_{0,M} \right)^{1/2} \leq C_{\mathcal{T}} C_{\mathcal{M}} \sigma_0^{-1/2} \| z_h \|^2_{1,\Omega}.
\]

Analogously, using (3.8) instead of (3.7), we derive

\[
(3.17) \quad \| z_h \|^2_{1,\Omega} \leq C_{\mathcal{T}} C_{\mathcal{M}} \| u_h \|^2_{1,\Omega}.
\]

Substituting (3.15) and (3.16) into (3.14), we obtain

\[
(3.18) \quad \dot{s}_h(z_h, z_h) \leq 2 C_{\mathcal{T}} C_{\mathcal{M}}^2 \sum_{M \in M_h} \tau_M \| b \cdot \nabla u_h \|^2_{0,M} + 2 C_{\mathcal{T}} C_{\mathcal{M}} \| u_h \|^2_{0,L_p}.
\]

Combining this inequality with (3.13) and (3.12) and using once again the inequality \( a b \leq \frac{1}{4} a^2 + b^2 \), we arrive at

\[
a_{h}^{L_p}(u_h, z_h, z_h) \geq \frac{1}{2} \sum_{M \in M_h} \tau_M \| b \cdot \nabla u_h \|^2_{0,M} - C_{6} \| u_h \|^2_{0,L_p}
\]

with \( C_6 = (C_2 + C_{\mathcal{T}}^2) C_{\mathcal{M}} + 2 C_{\mathcal{T}}^2 C_{\mathcal{M}} + C_{\mathcal{T}} (2 C_{\mathcal{M}} \sigma_0^{-1})^{1/2} \). Thus, employing (2.9), we see that \( v_h \in V_h \) given by

\[
v_h := 2 z_h + (1 + 2 C_6) u_h
\]

satisfies the first inequality in (3.3). The second inequality in (3.3) is a simple consequence of (3.15)–(3.18). \( \square \)
4. Error analysis

In this section, we shall investigate the error of the solution of the local projection discretization (2.8) with respect to the norm \(\|\cdot\|_{LPSD}\). Our considerations will be based on the following estimate which is similar to Strang’s lemmas (see, e.g., [8]).

**Lemma 4.1.** Let \(u \in H^1(\Omega)\) be the weak solution of (1.1) and let \(u_h\) be the solution of the local projection discretization (2.8). Then, under the assumption of Theorem 3.1, we have

\[
\beta \|u - u_h\|_{LPSD} \leq \inf_{w_h \in W_h^b} \left\{ \beta \|u - w_h\|_{LPSD} + \sup_{v_h \in V_h} \frac{a_h^{LP}(u - w_h, v_h)}{\|v_h\|_{LPSD}} \right\}
\]

\[
+ \sup_{v_h \in V_h} \frac{s_h(u, v_h)}{\|v_h\|_{LPSD}},
\]

where

\[
W_h^b = \{w_h \in W_h; w_h - \tilde{u}_h \in V_h\}.
\]

**Proof.** The weak solution of (1.1) satisfies \(a^G(u, v) = (f, v)\) for any \(v \in H^1_0(\Omega)\). Therefore,

\[
a_h^{LP}(u - u_h, v_h) = s_h(u, v_h) \quad \forall v_h \in V_h
\]

and hence also

\[
a_h^{LP}(w_h - u_h, v_h) = a_h^{LP}(w_h - u, v_h) + s_h(u, v_h) \quad \forall v_h \in V_h, w_h \in W_h^b.
\]

Now (4.1) follows by applying (3.2) and the triangular inequality.

In the following two lemmas, we establish estimates of the terms on the right-hand side of (4.1).

**Lemma 4.2.** Let the stabilization parameters \(\tau_M\) satisfy (3.1) and

\[
h_M \|b\|_{0,\infty,M} \leq C_\tau \max\{\varepsilon, \tau_M \|b\|_{0,\infty,M}^2\} \quad \forall M \in \mathcal{M}_h.
\]

Then there exists a constant \(C\) independent of \(h\) and \(\varepsilon\) such that, for any \(w \in H^1(\Omega)\),

\[
\|w\|_{LPSD} + \sup_{v \in H^1_0(\Omega)} \frac{a_h^{LP}(w, v)}{\|v\|_{LPSD}} \leq C (\varepsilon + h \|b\|_{0,\infty,\Omega} + h^2 \|\sigma\|_{0,\infty,\Omega})^{1/2} \left( \sum_{M \in \mathcal{M}_h} \{|w|^2_{1,M} + h^2 \|w\|^2_{0,M}\} \right)^{1/2}.
\]

**Proof.** Consider any \(w \in H^1(\Omega)\) and \(v \in H^1_0(\Omega)\). Integrating by parts and using the Schwarz inequality, we obtain

\[
a_h^{LP}(w, v) = \varepsilon (\nabla w, \nabla v) - (w, b \cdot \nabla v) + ((c - \text{div} b) w, v) + s_h(w, v)
\]

\[
\leq -(w, b \cdot \nabla v) + (1 + \sigma_0^{-1} \|c - \text{div} b\|_{0,\infty,\Omega}) \|w\|_L^L \|v\|_L^{LP}.
\]

Furthermore, in view of (2.5), (2.6), (3.1) and (2.2), we get

\[
\|w\|^2_{LPSD} \leq \|\sigma\|_{0,\infty,\Omega} \|w\|^2_{0,\infty,\Omega} + 2 C_\tau^2 C_{\mathcal{M}} (\varepsilon + h \|b\|_{0,\infty,\Omega}) \sum_{M \in \mathcal{M}_h} |w|^2_{1,M}.
\]
Let the stabilization parameters \( h_M \) of Lemma 4.3.

\[ \frac{1}{\varepsilon} \left( \sum_{M \in \mathcal{M}_h} h_M^{-2} \| \nabla v \|^2_{0,M} \right) \left( \sum_{M \in \mathcal{M}_h} h_M^{-2} \| \nabla v \|^2_{0,M} \right) = \frac{1}{\varepsilon} \left( \sum_{M \in \mathcal{M}_h} h_M^{-2} \| \nabla v \|^2_{0,M} \right) \leq \frac{1}{\varepsilon} \left( \sum_{M \in \mathcal{M}_h} h_M^{-2} \| \nabla v \|^2_{0,M} \right) \leq C_\varepsilon \varepsilon^2 \| v \|^2_{1,M}. \]

Consider any \( M \in \mathcal{M}_h \). If \( h_M \| b \|_{0,\infty,M} \leq C_\varepsilon \varepsilon \), then
\[ h_M^2 \| b \cdot \nabla v \|^2_{0,M} \leq C_\varepsilon^2 \varepsilon^2 \| v \|^2_{1,M} \]
and hence also
\[ h_M^2 \| b \cdot \nabla v \|^2_{0,M} \leq C_\varepsilon^2 \varepsilon^2 \| v \|^2_{1,M} \]
for any \( M \in \mathcal{M}_h \). If \( C_\varepsilon \varepsilon^2 \leq C_\varepsilon \varepsilon^2 \| b \|_{0,\infty,M} \) due to (4.2) and hence \( h_M^2 \leq C_\varepsilon^2 h_M \| b \|_{0,\infty,M} \tau_M \) (since \( C_\varepsilon \geq 1 \)). Therefore, (4.3) holds also in this case and we deduce using (2.1) and (2.2) that
\[ \sum_{M \in \mathcal{M}_h} h_M^2 \| b \cdot \nabla v \|^2_{0,M} \leq C_\varepsilon^2 C_\varepsilon \| b \|_{0,\infty,M} \| v \|^2_{1,M}. \]
which completes the proof. \( \square \)

**Lemma 4.3.** Let the stabilization parameters \( \tau_M \) satisfy (3.1). Then, for any \( u \in H^1(\Omega) \), we have
\[ \sup_{v \in H^1(\Omega)} \frac{s_h(u, v)}{\| v \|_{LPSD}^2} \leq C h^{1/2} \frac{\| b \|_{0,\infty,\Omega} \inf_{\mathcal{M}_h \in [D_M]^d} \| \nabla u - q \|^2_{\Omega}}{\| b \|_{0,\infty,\Omega}} \frac{1}{\| v \|_{LPSD}^2} \]
where \( C = C_\varepsilon \| C_\varepsilon \|_{0,\infty,\Omega}^{1/2} \).

**Proof.** For any \( u, v \in H^1(\Omega) \), we have
\[ s_h(u, v) \leq \sup_{v \in H^1(\Omega)} \frac{\| v \|_{LPSD}^2}{\| v \|_{LPSD}^2} \]
and hence it suffices to estimate \( \tau_M s_M(u, v) \) with an arbitrary \( M \in \mathcal{M}_h \). Consider any \( q_M \in [D_M]^d \). Since \( \kappa_M q_M = 0 \), we obtain using (2.5) and (2.6)
\[ s_M(u, v) \leq \frac{\| b \|_{0,\infty,\Omega} \| \nabla u - q \|^2_{0,M}}{\| b \|_{0,\infty,\Omega} \| \nabla u - q \|^2_{0,M}} \]

Therefore, applying (3.1) and (2.2), we obtain
\[ \tau_M s_M(u, v) \leq C_\varepsilon^2 C_\varepsilon \| b \|_{0,\infty,\Omega} \| \nabla u - q \|^2_{0,M} \]
which proves the lemma. \( \square \)

To prove convergence results for the solution of the local projection discretization (2.8), we have to introduce some approximation properties of the spaces \( W_h \) and \( D_M \). We shall assume that there exist interpolation operators \( i_h \in L(\mathcal{H}(\Omega), W_h) \cap \mathcal{L}(\mathcal{H}(\Omega), W_h) \) and \( j_M \in L(\mathcal{H}(M), D_M) \), \( M \in \mathcal{M}_h \), such that, for some constants \( l \in \mathbb{N} \) and \( C > 0 \), we have
\[ \left( \sum_{M \in \mathcal{M}_h} \frac{\| v - i_h v \|^2_{1,M} + h_M^2 \| v - i_h v \|^2_{0,M}}{\| v - i_h v \|^2_{0,M}} \right)^{1/2} \leq C h^k \| v \|_{k+1,\Omega} \]

\[ \forall v \in H^{k+1}(\Omega), k = 1, \ldots, l \]
and
\[ \| q - j_M q \|_{0,M} \leq C h_M^k \| q \|_{k,M} \quad \forall q \in H^k(M), M \in \mathcal{M}_h, k = 1, \ldots, l. \]
Now we are in a position to prove an a priori error estimate for the local projection discretization (2.8).

**Theorem 4.4.** Let the stabilization parameters $\tau_M$ satisfy (3.1) and (4.2). Let the spaces $W_h$ and $D_M$ possess the approximation properties (4.4) and (4.5). Let the weak solution of (1.1) satisfy $u \in H^{k+1}(\Omega)$ for some $k \in \{1, \ldots, l\}$. Finally, let $\tilde{u}_b \in H^2(\Omega)$ be an extension of $u_b$ and let $\tilde{u}_{bh} = i_h \tilde{u}_b$. Then the solution $u_h$ of the local projection discretization (2.8) satisfies the error estimate

$$|||u - u_h|||_{L^p S D} \leq C (\varepsilon + h \|b\|_{0, \infty, \Omega} + h^2 \|\sigma\|_{0, \infty, \Omega})^{1/2} h^k |u|_{k+1, \Omega},$$

where the constant $C$ is independent of $h$ and $\varepsilon$.

**Proof.** Combining Lemmas 4.1–4.3 and setting $w_h = i_h u$ and $q_M = j_M(\nabla u)$, the theorem follows by applying (4.4), (4.5), (2.1) and (2.2). \hfill \Box

**Remark 4.5.** Estimates of the type (4.6) can be proved for various stabilized finite element methods applied to the problem (1.1) (e.g., the SUPG method) and are known to be optimal, see, e.g., [19]. If we define the stabilization term $s_h(u, v)$ using $b$ instead of $b_M$, then Theorem 3.1 and Lemmas 4.1 and 4.2 still hold but the consistency error cannot be estimated as in Lemma 4.3. Assuming (3.1) and $b \cdot \nabla u \in H^k(\Omega)$ with $k \in \{1, \ldots, l\}$, we obtain

$$\sup_{v_h \in V_h} s_h(u, v_h) \|v_h\|_{L^p S D} \leq C h^k \left( \sum_{M \in \mathcal{M}_h} \min \left\{ \frac{\|b \cdot \nabla u\|^2_{k, M}}{\sigma_0}, \frac{h_M \|b \cdot \nabla u\|^2_{h, M}}{\|b\|_{0, \infty, M}} \right\} \right)^{1/2},$$

see [14, 15]. Thus, if $b \neq 0$ in $\Omega$, the optimal convergence order can be still proved but generally we only have the suboptimal convergence order $k$. Moreover, for small $\sigma_0$, the accuracy of the discrete solution may be significantly worse than for $s_h$ defined using $b_M$, see also Example 7.1 in Section 7.

**Remark 4.6.** The assumptions (3.1) and (4.2) are fulfilled for $\tau_M$ satisfying (2.10). Another possibility is to use

$$\tau_M \sim \frac{h_M}{\|b\|_{0, \infty, M}} \quad \text{if} \quad \frac{h_M \|b\|_{0, \infty, M}}{\varepsilon} \geq 1,$$

$$\tau_M = 0 \quad \text{otherwise}.$$

5. **Examples of spaces $W_h$ and $D_M$**

In this section, we present several examples of spaces $W_h$ and $D_M$ satisfying the assumptions made in Sections 2 and 4. For the ease of exposition, we confine ourselves to the two–dimensional case. In one or three dimensions, the spaces can be constructed analogously.

First let us recall some basic notions (cf. [6, 8, 10]) which will be used in the following. Given $h > 0$, a set $\mathcal{T}_h$ will be called a triangulation of $\Omega$ if it consists of a finite number of mutually disjoint open subsets $T$ of $\Omega$ such that $\Omega = \bigcup\{T \in \mathcal{T}_h\}$ and $h_T := \text{diam}(T) \leq h$ for any $T \in \mathcal{T}_h$. We shall assume that either all elements of $\mathcal{T}_h$ are triangles or all elements of $\mathcal{T}_h$ are convex quadrilaterals and that the intersection of the closures of any two different elements of $\mathcal{T}_h$ is either empty or consists of
A NEW VARIANT OF LPS FOR CONVECTION–DIFFUSION–REACTION EQUATIONS

A triangulation \( \mathcal{T}_h \) consisting of triangles is shape–regular if

\[
\frac{h_T}{\rho_T} \leq C_{\mathcal{T}} \quad \forall \ T \in \mathcal{T}_h,
\]

where \( \rho_T \) is the diameter of the circle inscribed in \( T \) and \( C_{\mathcal{T}} \) is a constant independent of \( h \) which is common to the considered family of triangulations. If \( \mathcal{T}_h \) consists of quadrilaterals, then it is shape–regular if (5.1) holds for any triangle whose vertices coincide with three vertices of some element of \( \mathcal{T}_h \). We denote by \( \hat{T} \) a reference element, which is either a triangle or a square, depending on the type of elements in \( \mathcal{T}_h \). If \( \hat{T} \) is a triangle, we set

\[
i_{\mathcal{T}_h,l}(\hat{T}) := P_l(\hat{T})
\]

where \( P_l(\hat{T}) \) is the space of polynomials on \( \hat{T} \) of degree \( \leq l \). If \( \hat{T} \) is a square, then we set

\[
q_{\mathcal{T}_h,l}(\hat{T}) := Q_l(\hat{T})
\]

where \( Q_l(\hat{T}) \) is the space of polynomials on \( \hat{T} \) of degree \( \leq l \) in each variable. For any \( T \in \mathcal{T}_h \), there exists a one–to–one mapping \( F_T \in \mathbb{R}^2_{\mathcal{R}_1(\hat{T})} \) such that

\[
F_T(\hat{T}) = T.
\]

Given \( l \in \mathbb{N} \), we define the space

\[
X_{\mathcal{T}_h,l} = \{ v \in C(\Omega) : v \circ F_T \in R_l(\hat{T}) \ \forall \ T \in \mathcal{T}_h \}.
\]

Then \( X_{\mathcal{T}_h,l} \) does not depend on the choice of the mappings \( F_T \), we have \( X_{\mathcal{T}_h,l} \subset H^1(\Omega) \) and \( X_{\mathcal{T}_h,l} \) satisfies the inverse inequality

\[
|v_h|_{1,T} \leq C h_T^{-1} \|v_h\|_{0,T} \quad \forall \ v_h \in X_{\mathcal{T}_h,l}, \ T \in \mathcal{T}_h.
\]

Moreover, the Lagrange interpolation operator \( i_{\mathcal{T}_h,l} \in \mathcal{L}(C(\Omega), X_{\mathcal{T}_h,l}) \) satisfies

\[
i_{\mathcal{T}_h,l} \in \mathcal{L}(C(\Omega) \cap H^1_0(\Omega), X_{\mathcal{T}_h,l} \cap H^1_0(\Omega))
\]

and

\[
|v - i_{\mathcal{T}_h,l} v|_{1,T} + h_T^{-1} \|v - i_{\mathcal{T}_h,l} v\|_{0,T} \leq C h_T^{k+1} \|v\|_{k+1,T} \quad \forall \ v \in H^{k+1}(T), \ T \in \mathcal{T}_h, \ k = 1, \ldots, l.
\]

Finally, denoting by \( j_{\mathcal{T},l} \) the orthogonal projection of \( L^2(T) \) onto \( P_l(T) \), we also have

\[
\|q - j_{\mathcal{T},l} q\|_{0,T} \leq C h_T^k \|q\|_{k,T} \quad \forall \ q \in H^k(T), \ T \in \mathcal{T}_h, \ k = 0, \ldots, l + 1.
\]

In all the inequalities (5.3)–(5.5), the constant \( C \) depends only on \( l \) and \( C_{\mathcal{T}} \) from (5.1).

Now let us discuss the construction of the spaces \( W_h \) and \( D_M \). The original local projection stabilization \([1, 2]\) was designed as a two–level method. Given a shape–regular triangulation of \( \Omega \), the elements of this triangulation are considered as the set \( \mathcal{M}_h \) introduced in Section 2. Then this triangulation is refined as depicted in Fig. 1, i.e., each triangle is divided in three triangles by connecting its vertices with

\[\text{Figure 1. Relation between the meshes } \mathcal{M}_h \text{ (bold lines) and } \mathcal{T}_h \text{ (bold and fine lines) in the two–level method.}\]
the barycenter and each quadrilateral is divided in four quadrilaterals by connecting midpoints of opposite edges. Let us denote the resulting triangulation by $T_h$. Given $l \in \mathbb{N}$, we set

$$W_h = X_{T_h, l}, \quad D_M = P_{l-1}(M) \quad \forall \ M \in \mathcal{M}_h.$$  

Then in view of Lemma 5.1 below and according to what was said above, all the assumptions of Section 2 as well as (4.4) and (4.5) are satisfied (note that $X_{M, l} \subset X_{T_h, l}$ and that triangulations $T_h$ assigned to a shape–regular family $\{\mathcal{M}_h\}$ also form a shape–regular family).

Another choice of the spaces $W_h$ and $D_M$ (a one–level method) was proposed in [17]. Here, given a shape–regular triangulation of $\Omega$, the elements of this triangulation are again considered as the set $M_h$ but the space $W_h$ is constructed on this triangulation $M_h$ as well. However, the spaces $W_h$ and $D_M$ defined by (5.6) with $T_h = M_h$ do not satisfy the inf–sup conditions (2.3) in general. Indeed, the validity of the inf–sup conditions (2.3) would imply that $\dim B_M \geq \dim D_M$ but this cannot be satisfied if $M$ is a triangle or $l < 5$. Therefore, the space $X_{M, l}$ is enriched elementwise by bubble functions. More precisely, introducing a polynomial bubble function $\tilde{\varphi} \in H^1_0(\hat{T}) \setminus \{0\}$ (cubic if $\hat{T}$ is a triangle and biquadratic if $\hat{T}$ is a square), we set

$$W_h = \{v \in C(\overline{\Omega}) \ ; \ v \circ F_M \in R_l(\hat{T}) + \tilde{\varphi} \cdot R_{l-1} (\hat{T}) \ \forall \ M \in \mathcal{M}_h\},$$

$$D_M = P_{l-1}(M) \quad \forall \ M \in \mathcal{M}_h.$$  

Then the inf–sup conditions (2.3) hold with $B_M = (W_h|_M) \cap H^1_0(M)$, see [17] or the proof of Lemma 5.1 below. The remaining assumptions of Section 2 as well as (4.4) and (4.5) are clearly satisfied as well.

There are also other possibilities to define the spaces $W_h$ and $D_M$ in both the one–level and the two–level framework, see [17] for details. A common feature of all these constructions is that they lead to a (significant) increase of the number of degrees of freedom in comparison with applying, e.g., a residual–based stabilization [19] for which we could simply use a finite element space consisting of piecewise polynomials of degree $l$ (in the sense of (5.2)) on a given triangulation. This increase of the number of degrees of freedom, either due to a refinement of the given triangulation or due to an enrichment by bubble functions, is a consequence of the fact that the sets in $\mathcal{M}_h$ are assumed to be non–overlapping. We shall demonstrate in the following that our theory, which enables to use sets $\mathcal{M}_h$ consisting of overlapping subsets of $\Omega$, makes it possible to satisfy the assumptions on the spaces $W_h$ and $D_M$ without introducing additional degrees of freedom.

Let $\mathcal{T}_h$ be a shape–regular triangulation of $\Omega$. We shall assume that any element of $\mathcal{T}_h$ has at least one vertex in $\Omega$. Let $x_1, \ldots, x_{N_h}$ be the vertices of $\mathcal{T}_h$ lying in $\Omega$ and let us denote

$$M_i = \text{int} \bigcup_{T \in \mathcal{T}_h, x_i \in T} T, \quad i = 1, \ldots, N_h,$$

where ‘int’ denotes the interior of the respective polygon. We set

$$\mathcal{M}_h = \{M_i\}_{i=1}^{N_h}.$$  

Then we can define the spaces $W_h$ and $D_M$ like in the two–level method by (5.6). Let us emphasize once more that we use only the triangulation $\mathcal{T}_h$ we were given at the beginning.
Let us discuss the validity of the assumptions on $\mathcal{M}$, $W_h$ and $D_M$ made in this paper. Since the number of elements of $\mathcal{T}_h$ sharing a common vertex is bounded by a constant depending only on $C_{\mathcal{T}}$ from (5.1), the assumption (2.1) is satisfied. Moreover, the shape-regularity of $\mathcal{T}_h$ implies that

$$C_{\mathcal{M}, \mathcal{T}} h_M \leq h_T \leq h_M \quad \forall T \in \mathcal{T}_h, M \in \mathcal{M}, T \cap M \neq \emptyset,$$

where $C_{\mathcal{M}, \mathcal{T}}$ is a positive constant again depending only on $C_{\mathcal{T}}$ from (5.1). The assumption (2.2) obviously holds with $C'_{\mathcal{M}} = 2$. The validity of (2.4) and (4.4) is a direct consequence of (5.3), (5.4) and (5.11). In view of (5.1) and (5.11), any set $M_i$ is star-shaped with respect to the ball with the center $x_i$ and diameter $h_M, C_{\mathcal{M}, \mathcal{T}} / C_{\mathcal{T}}$ and hence (4.5) holds as well, see, e.g., [6]. Finally, the inf-sup conditions (2.3) hold according to the following lemma and hence all the assumptions on $\mathcal{M}_h, W_h$ and $D_M$ are satisfied.

**Lemma 5.1.** Let $\mathcal{T}_h$ be a triangulation of $\Omega$ and let $\mathcal{M}_h$ be given by (5.9) and (5.10). Consider any $l \in \mathbb{N}$. Then the spaces $W_h$ and $D_M$ defined by (5.6) satisfy the inf-sup conditions (2.3) with $B_M = (W_h|_M) \cap H^1_0(M)$ and a constant $\beta_{LP}$ depending only on $l$.

**Proof.** Consider any $M \in \mathcal{M}_h$ and let $i \in \{1, \ldots, N_h\}$ be such that $M = M_i$. Let $\varphi \in X_{\mathcal{T}_h, 1} \cap H^1_0(\Omega)$ satisfy $\varphi(x_i) = 1$ and $\varphi(x_j) = 0$ for any $j \neq i, j = 1, \ldots, N_h$. For any $T \in \mathcal{T}_h$ such that $T \subset M$, we set $\hat{\varphi}_T := \varphi \circ F_T$ and $\hat{J}_T = \text{det} DF_T$ where $DF_T$ is the Jacobi matrix of $F_T$. Note that $\hat{\varphi}_T \in \mathcal{R}_l(\hat{\mathcal{T}})$ equals 1 at one vertex of $\hat{T}$ and vanishes at the remaining vertices of $\hat{T}$ so that $\hat{\varphi}_T$ is one of three (resp. four) fixed functions on $\hat{T}$ in the triangular (resp. quadrilateral) case. Setting $\|\hat{u}\|_* = \|\hat{\varphi}_T \hat{u}\|_{0,1,\hat{T}}$, the functional $\| \cdot \|_*$ is a norm on $L^1(\hat{T})$ and we have

$$\|\hat{u}\|_* \geq C_1 \|\hat{u}\|_{0,1,\hat{T}} \quad \forall \hat{u} \in \mathcal{R}_l(\hat{T})$$

with $C_1 > 0$ due to the equivalence of norms on finite-dimensional spaces. Now consider any $q \in D_M$. Since $\varphi|_M \in H^1_0(M)$ and $\hat{q}_T := q \circ F_T \in \mathcal{R}_{l-1}(\hat{T})$ for any $T \in \mathcal{T}_h$ such that $T \subset M$, the function $v := \varphi q$ is an element of $B_M$. Using (5.12), we derive for any $T \subset M$

$$(v, q)_T = \|\varphi q^2\|_{0,1,T} = \|\hat{q}_T \hat{J}_T\|_* \geq C_{2l-1} \|\hat{q}_T \hat{J}_T\|_{0,1,\hat{T}} = C_{2l-1} \|q\|_{0,T}^2.$$ 

Moreover, $\|\varphi\|_{0,\infty,M} = 1$ and hence $\|v\|_{0,M} \leq \|q\|_{0,M}$. Thus, (2.3) holds with $\beta_{LP} = C_{2l-1}$. \hfill $\square$

6. Approximation of exponential boundary layers

If convection strongly dominates diffusion, the boundary condition in (1.1) prescribed at outflow and characteristic boundaries (where $b$ points outward from $\Omega$ or is parallel to $\partial \Omega$, respectively) does not influence the solution of (1.1) significantly in most of $\Omega$. Consequently, in the vicinity of outflow and characteristic boundaries, the solution is forced to change its character abruptly and boundary layers develop. At outflow boundaries, these layers are usually called exponential boundary layers, see [19] for more details.

If the solution of (1.1) is approximated by a numerical method based on a mesh which does not resolve the boundary layers, then it is clear that the boundary conditions at outflow and characteristic boundaries should not influence the approximate solution at interior nodes of the mesh (or the influence should be very small). For
exponential boundary layers, which will be considered in the following, this can be achieved by some kind of upwinding, see again [19].

The two–level approach of the local projection method described in the preceding section is not appropriate for approximating exponential boundary layers since the boundary condition is coupled with the values of the discrete solution at many interior nodes and some of these couplings can be suppressed only by decreasing the stabilization parameter \( \tau \), i.e., by suppressing the stabilization, see also [20]. The same properties have to be expected from the local projection method with overlapping sets from Section 2 be such that any

\[ \Omega \]

or in \( \Omega \)

right triangles forming a strip \( G_h \) of constant width, see Fig. 2. Let the set \( \mathcal{M}_h \) from Section 2 be such that any \( M \in \mathcal{M}_h \) is contained either in \( G_h \) or in \( \Omega \setminus G_h \).

We choose a Cartesian coordinate system whose x axis is perpendicular to \( \partial \Omega \) and directed outward from \( \Omega \). The other axis is denoted by \( y \). The coordinates of the vertical and horizontal lines in Fig. 2 are denoted by \( x_{N-1}, x_N \) and \( y_{i-1}, y_i, y_{i+1} \), respectively. We denote by \( h_1 \) and \( h_2 \) the mesh widths in the horizontal and vertical directions, respectively, so that \( h_1 = x_N - x_{N-1} \) and \( h_2 = y_i - y_{i-1} = y_{i+1} - y_i \).

We introduce functions \( \xi_{N-1}, \xi_N \in P_1(x_{N-1}, x_N) \) satisfying

\[
\xi_{N-1}(x_{N-1}) = 1, \quad \xi_{N-1}(x_N) = 0, \quad \xi_N(x_{N-1}) = 0, \quad \xi_N(x_N) = 1
\]

and functions \( \zeta_i \in C(\mathbb{R}), i \in \mathbb{Z} \), piecewise linear with respect to the intervals \( (y_{j-1}, y_j), j \in \mathbb{Z} \), such that \( \zeta_i(y_j) = \delta_{ij} \forall i, j \in \mathbb{Z} \) where \( \delta_{ij} \) is the Kronecker symbol.
A NEW VARIANT OF LPS FOR CONVECTION–DIFFUSION–REACTION EQUATIONS

Like in [21], we consider the problem (1.1) with constant $b = (b_1, b_2)$ and $c = 0$. We assume that $b_1 > 0$ so that $\partial \Omega$ is an outflow boundary. Furthermore, we assume that the problem (1.1) has a one-dimensional character, i.e., the solution $u$ is independent of $y$ and hence $u_b$ is constant along $\partial \Omega$. Equivalently, we can require that the discrete problem (2.8) with $f = 0$ and $\tilde{u}_b h = 1$ has a solution, say $z_h$,vanishing outside $G_h$. Thus, neglecting the diffusion term, we arrive at the problem:

Find $\tau_M \geq 0$ such that there exists a function $z_h$ satisfying

$$
(6.1) \quad z_h \in W_h, \quad z_h = 1 \quad \text{on} \quad \partial \Omega, \quad z_h = 0 \quad \text{in} \quad \Omega \setminus G_h,
$$

$$
(6.2) \quad (b \cdot \nabla z_h, v_h)_{G_h} + \sum_{M \in \mathcal{M}_h, M \subset G_h} \tau_M (\kappa_M (b \cdot \nabla z_h), b \cdot \nabla v_h)_M = 0 \quad \forall \quad v_h \in V_h.
$$

We define the operators $\pi_M$ as orthogonal $L^2$ projections of $L^2(M)$ onto $D_M$, which enables us to use $b \cdot \nabla v_h$ instead of $\kappa_M (b \cdot \nabla v_h)$ in (6.2). The condition (6.2) is understood in the sense of the corresponding infinite algebraic problem obtained using some common basis functions of the finite element space $V_h$. Therefore, the functions $v_h$ in (6.2) are viewed as functions with compact supports so that the left-hand side of (6.2) is well defined and finite.

Let us consider spaces $W_h$ and $D_M$ defined by (5.7) and (5.8) with $M_h = T_h$. If $l = 1$, then the problem (6.1), (6.2) is solvable with $\tau_M$ equal to the same value $\tau$ for any $M \subset G_h$. In the quadrilateral case, we obtain

$$
(6.3) \quad \tau = \frac{5}{36} \frac{b_1}{h_1^3} + \frac{b_2}{h_2^3}
$$

and $z_h|_M = \xi_N - 18 \varphi_M$, where $\varphi_M \in Q_2(M) \cap H^1(M)$ is equal to $1/16$ at the barycenter of $M$. In the triangular case, the conditions (6.1) and (6.2) are satisfied.
for

\[ \tau = \frac{1}{20} \frac{b_1}{h_1} + \frac{b_2}{h_2} \]

and \( z_h|_M = \xi_N - 30 \varphi_M \), where \( \varphi_M \in P_3(M) \cap H^1_M \) is equal to 1/27 at the barycenter of \( M \). However, for \( l = 2 \), the problem (6.1), (6.2) is not solvable for any \( \tau_M \geq 0 \). Let us demonstrate it for the quadrilateral case with \( h_1 = h_2 = 1 \) and \( b = (1, 0) \). For \( M = (x_{N-1}, x_N) \times (y_{i-1}, y_i) \) with any \( i \in \mathbb{Z} \), the function \( z_h \) has the general form

\[
z_h|_M = \xi_N + \alpha \xi_N - 1 \xi_N + \beta_i \varphi_i + \gamma_i \varphi_i (\xi_N - \xi_N - 1) + \delta_i \varphi_i (\zeta_i - \zeta_i - 1) + \varepsilon_i \varphi_i (\xi_N - \xi_N - 1) (\zeta_i - \zeta_i - 1),
\]

where \( \xi_j = \xi_j(x) \), \( \zeta_j = \zeta_j(y) \), \( \varphi_i = \xi_N - 1 \xi_N - 1 \zeta_i \) and \( \alpha, \beta_i, \gamma_i, \delta_i \) and \( \varepsilon_i \) are arbitrary real numbers. In view of the one-dimensional character of the approximated problem, we assume that \( \alpha \) (and hence \( z_h(x, y_i) \)) does not depend on \( i \). A simple computation gives

\[
\kappa_M (b \cdot \nabla z_h) = \frac{1}{6} \beta_i \left( \xi_N - \xi_N - 1 \right) + \frac{\partial}{\partial x} \{ \beta_i \varphi_i + \gamma_i \varphi_i (\xi_N - \xi_N - 1) \} + \frac{\partial}{\partial y} (\delta_i \varphi_i (\zeta_i - \zeta_i - 1) + \varepsilon_i \varphi_i (\xi_N - \xi_N - 1) (\zeta_i - \zeta_i - 1)).
\]

Using \( v_h = \varphi_i (\zeta_i - \zeta_i - 1) \) and \( v_h = \varphi_i (\xi_N - \xi_N - 1) (\zeta_i - \zeta_i - 1) \) in (6.2), we obtain two equations for \( \delta_i \) and \( \varepsilon_i \) implying that \( \delta_i = \varepsilon_i = 0 \). Then, using \( v_h = \xi_N - 1 \zeta_i - \zeta_i \) and \( v_h = \xi_N - 1 \zeta_i \), we deduce that \( \alpha = -3 \) and \( \beta_i = 0 \). Using \( v_h = \varphi_i \), we get \( \gamma_i = -25 \) and the function \( z_h \) is completely determined. However, this function does not satisfy (6.2) for \( v_h = \xi_N - 1 \xi_N \zeta_i \).

The above discussion shows that the enrichment used in (5.7) with \( l = 2 \) is not appropriate for solving problems with exponential boundary layers. Therefore, we shall propose another construction of the space \( W_h \). In view of the above-mentioned results of [21] obtained in the one-dimensional case, we intend to satisfy (6.1), (6.2) for a function \( z_h \) such that

\[
z_h|_{G_h} = \xi_N + \alpha \xi_N - 1 \xi_N + \beta \xi_N - 1 \xi_N (\xi_N - \xi_N - 1)
\]

with some \( \alpha, \beta \in \mathbb{R} \), i.e., a function which is independent of \( y \) and polynomial of degree 3 in \( x \) on the interval \( (x_{N-1}, x_N) \). Thus, the space \( X_{\mathcal{A}_h} \) has to be enriched in such a way that it contains functions of this type. In addition, the enrichment has to enable us to proof the inf-sup conditions (2.3). First, let us consider the quadrilateral case. For any \( i \in \mathbb{Z} \), we set \( S_i = (x_{N-1}, x_N) \times (y_{i-1}, y_{i+1}) \) and \( \psi_i = \xi_{N-1} \xi_N \zeta_i \). Then \( \psi_i \in X_{\mathcal{A}_h} \), supp \( \psi_i = \mathcal{N} \), and the space

\[
W_h = X_{\mathcal{A}_h} + \text{span} \bigcup_{i \in \mathbb{Z}} \{ \psi_i \cdot P_l(\Omega) \}
\]

contains functions of the type (6.5). In the strip \( G_h \), let the set \( \mathcal{M}_h \) consist of some of the sets \( S_i \) (such that the union of their closures is \( G_h \)). We define the spaces \( D_M \) by (5.8) with \( l = 2 \), which assures the validity of the inf-sup conditions (2.3). It is easy to show that, for any \( M \in \mathcal{M}_h \) with \( M \subset G_h \) (i.e., \( M \) equal to some of the sets \( S_i \)), we have

\[
\kappa_M (b \cdot \nabla z_h) = \beta b_1 \frac{d}{dx} [\xi_{N-1} \xi_N (\xi_N - \xi_N - 1)] = \beta b_1 \frac{1}{h_1} (6 \xi_{N-1} \xi_N - 1).
\]
For simplicity, we shall now assume that the elements $M$ of $\mathcal{M}_h$ contained in $G_h$ are non-overlapping and that the stabilization parameters $\tau_M$ have the same value $\tau$ for any $M \subset G_h$. Then the equation (6.2) can be written in the form

$$(6.6) \quad \frac{\partial z_h}{\partial x} \cdot v_h + \frac{\tau \beta}{h_1} (6 \xi_{N-1} \xi_N - 1, b \cdot \nabla v_h)_{G_h} = 0 \quad \forall \ v_h \in V_h.$$ 

Since the functions $v_h$ are assumed to have compact supports, it follows from integrating by parts that $b \cdot \nabla v_h$ can be replaced by $b_1 \partial v_h / \partial x$. A basis of $V_h$, restricted to $G_h$ consists of functions of the form $v(x) w(y)$ where $v \in P_0(x_{N-1}, x_N)$, $v(x_N) = 0$ and $w$ is a continuous piecewise polynomial function with a compact support. Using such functions as the functions $v_h$ in (6.6), the equation (6.6) reduces to

$$\int_{x_{N-1}}^{x_N} \frac{\partial z_h}{\partial x} v + \frac{\tau \beta b_1}{h_1} (6 \xi_{N-1} \xi_N - 1) \frac{\partial v}{\partial x} \, dx = 0 \quad \forall \ v \in P_3(x_{N-1}, x_N), v(x_N) = 0.$$ 

It is easy to verify that this condition is satisfied for $\alpha = -3$, $\beta = -5$ and

$$(6.7) \quad \tau = \frac{h_1}{10 b_1}.$$ 

This value of $\tau$ coincides with the optimal parameter of [21] in the limit case $\varepsilon \to 0$.

In the triangular case, let us first denote by $\mathcal{E}_{h,G_h}$ the set of all edges of $\mathcal{E}_h$ contained in $G_h$ (not on $\partial G_h$). For any edge $E \in \mathcal{E}_{h,G_h}$, we denote by $S_E$ the interior of the union of the closures of the two triangles adjacent to $E$ and by $\psi_E \in X_{\mathcal{T}_{h,2}}$ a function with supp $\psi_E = S_E$. Then the space

$$W_h = X_{\mathcal{T}_{h,2}} + \text{span} \left\{ \psi_E \cdot P_1(\Omega) \right\}$$

again contains functions of the type (6.5). The sets $M \in \mathcal{M}_h$ contained in $G_h$ will be those sets $S_E$ which correspond to `diagonal' edges in Fig. 2, i.e., the rectangles $(x_{N-1}, x_N) \times (y_{l-1}, y_l)$. The spaces $D_M$ are again defined by (5.8) with $l = 2$. We shall show that $\tau$ and $z_h$ obtained in the quadrilateral case satisfy (6.2) which is again equivalent to (6.6) with $b \cdot \nabla v_h$ replaced by $b_1 \partial v_h / \partial x$. Integrating by parts, we deduce that (6.6) with this $\tau$ and $z_h$ can be written in the form

$$(6.8) \quad \int_{G_h} (1 - 10 \xi_{N-1} \xi_N)(\xi_N - \xi_{N-1}) \frac{\partial v_h}{\partial x} \, dx \, dy = 0 \quad \forall \ v_h \in V_h.$$ 

For any functions $p, q \in C(\mathbb{R})$ and any triangle $T$ from Fig. 2, we have

$$\int_T p(x) q(y) \, dx \, dy = \pm \int_{x_{N-1}}^{x_N} p(x) Q(a x + b) \, dx,$$

where $Q$ is a suitable primitive function to $q$, $y = a x + b$ is the equation of the `diagonal' edge of $T$ and the sign depends on whether $T$ lies below or above this diagonal edge. Since

$$\int_{x_{N-1}}^{x_N} (1 - 10 \xi_{N-1} \xi_N)(\xi_N - \xi_{N-1}) \, q \, dx = 0 \quad \forall \ q \in P_2(x_{N-1}, x_N),$$

we see that (6.8) holds for any $v_h \in X_{\mathcal{T}_{h,2}}$. To prove (6.8) for functions from the spaces $\left\{ \psi_E \cdot P_1(\Omega) \right\}$, $E \in \mathcal{E}_{h,G_h}$, we again integrate by parts and obtain the condition

$$\int_{S_E} (1 - 5 \xi_{N-1} \xi_N) \psi_E \, q \, dx \, dy = 0 \quad \forall \ q \in P_1(S_E), \ E \in \mathcal{E}_{h,G_h}.$$
This can be easily verified by a direct computation after rewriting the integrals on
the two triangles making up $S_E$ in terms of barycentric coordinates.

By analogy with the case $l = 2$, we can enrich the space $X_{\mathcal{S}_h,1}$ by span\{$\psi_i$\}$\subseteq \mathbb{Z}$ or span\{$\psi_E$\}$E \in \mathcal{E}_{h,\alpha_2}$.* The set $\mathcal{M}_h$ is defined in the same way as above and the spaces $D_M$ are given by (5.8) with $l = 1$. Then we deduce that the conditions (6.1) and (6.2) are satisfied for $z_{h|E} = \xi_N - 3\xi_{N-1}\xi_N$ and

$$\tau = \frac{h_1}{6b_1}. \tag{6.9}$$

An advantage of this new enrichment is that the formula for the ‘optimal’ stabilization parameter is simpler than (6.3) and (6.4) so that it can be easier generalized to other types of meshes.

Analogously as above, we can proceed also if $c$ in (1.1) is a positive constant. Denoting $a = c h_1/b_1$, we derive

$$\tau = \frac{h_1}{6b_1} \left[ 1 + \frac{a}{20} \frac{3a + 4}{a + 3} \right] \tag{6.10}$$

instead of (6.9) and

$$\tau = \frac{h_1}{10b_1} \left[ 1 + \frac{a}{21} \frac{2a^2 + 9a + 12}{a^2 + 8a + 20} \right] \tag{6.11}$$

instead of (6.7).

Based on the discussion in this section, we can formulate a new general definition
of the spaces $W_h$ and $D_M$. Given a shape–regular triangulation $\mathcal{T}_h$ of $\Omega$, we first set

$$\mathcal{E}_h = \{ T \in \mathcal{T}_h : T \cap \partial \Omega \neq \emptyset \}, \quad G_h = \operatorname{int} \bigcup_{T \in \mathcal{E}_h} T. \tag{6.12}$$

Here, for simplicity, we consider the strip $G_h$ along the whole boundary of $\Omega$. We denote by $\partial_{G,h}$ the set of the edges of $\mathcal{T}_h$ contained in $G_h$ (but not on $\partial G_h$) and by $S_E$ the interior of the union of the closures of the two elements of $\mathcal{T}_h$ adjacent to $E \in \mathcal{E}_{h,G_h}$. For any $E \in \mathcal{E}_{h,G_h}$, we denote by $\psi_E \in X_{\mathcal{E}_h,2}$ a function with supp$\psi_E = \overline{S_E}$. In the triangular case, we assume that $\psi_E$ transformed on the reference element is linear in one variable (i.e., it is defined like the functions $\psi_i$ above). We denote by $\mathcal{F}_h$ the set of those elements of $\mathcal{F}_h \setminus G_h$ which have all their vertices on $\partial G_h$. For any $T \in \mathcal{F}_h$, we introduce a function $\varphi_T \in X_{\mathcal{F}_h,2}$ (in the quadrilateral case) resp. $\varphi_T \in X_{\mathcal{F}_h,3}$ (in the triangular case) such that supp$\varphi_T = \overline{T}$. The set $\mathcal{M}_h$ consists of (some of) the sets $\{ S_E \}_{E \in \mathcal{E}_{h,G_h}}$, the elements of $\mathcal{F}_h$ and (some of) the sets $M$, defined by (5.9) for $x_i \not\in G_h$ such that $\overline{\Omega} = \bigcup_{M \in \mathcal{M}_h} \overline{M}$. Now, given $l \in \mathbb{N}$, we set

$$W_h = X_{\mathcal{F}_h,l} + \operatorname{span} \left\{ \psi_E \cdot P_{l-1} (\Omega) \right\} + \operatorname{span} \left\{ \varphi_T \cdot P_{l-1} (\Omega) \right\}, \tag{6.13}$$

$$D_M = P_{l-1} (M), \quad \forall M \in \mathcal{M}_h. \tag{6.14}$$

Like in Section 5, we can show that these spaces $W_h$ and $D_M$ as well as the set $\mathcal{M}_h$ satisfy all the assumptions made in Sections 2 and 4.

In the quadrilateral case, the definition of $W_h$ given in (6.11) may be not convenient from the implementational point of view if $l > 1$. One reason is that the functions $\psi_E$ and $\varphi_T$ are typically constructed as images of functions defined on the reference element but they are multiplied by functions defined in $\Omega$. Another reason
is that the dimension of \( W_h \) depends on the shapes of elements of \( \mathcal{G}_h \). Therefore, it may be practical to use the space

\[
W_h = X_{\mathcal{G}_h, t} + X_{\mathcal{G}_h, t+1} \cap H_0^1(G_h) + \bigoplus_{T \in \mathcal{T}_h} \{ \tilde{w} \circ F_T^{-1} ; \tilde{w} \in \tilde{\varphi} \cdot R_{l-1}(\tilde{T}) \},
\]

where \( \tilde{\varphi} \) is the same as in (5.7). Thus, inside the strip \( G_h \), the polynomial degree is simply increased by 1. This increases the dimension of the space \( W_h \) by \( l \, \text{card} \, \mathcal{G}_h \) in general, which is not significant in comparison with \( \dim W_h \). It is easy to see that the space defined by (6.11) is a subspace of the space from (6.13).

In the triangular case, finite element discretizations can be implemented employing barycentric coordinates without using correspondences to reference finite elements and hence the structure of the space \( W_h \) defined by (6.11) can be less adverse than in the quadrilateral case. Nevertheless, using the space from (6.13) instead of its subspace defined by (6.11) can facilitate the implementation. Note that, for \( l = 2 \), both spaces \( W_h \) have the same dimension and hence coincide.

7. Numerical results

In this section, we present numerical results for three test problems illustrating the properties of the methods discussed in the preceding sections. In all computations, the operators \( \pi_M \) are orthogonal \( L^2 \) projections of \( L^2(M) \) onto \( D_M \). The constant approximations \( b_M \) of \( b \) in \( M \) are defined as values of \( b \) at barycentres of \( M \) if \( M \) are triangles or quadrilaterals. For \( M = M_i \) defined by (5.9) we set \( b_{M_i} = b(x_i) \). These choices of \( b_M \) assure the validity of (2.6).

Let us first consider the following example showing that it is really important to use \( b_M \) instead of \( b \) in the local projection stabilization term (2.7).

**Example 7.1 (Problem without layers).** We consider the problem (1.1) with \( \Omega = (0,1)^2 \), \( \varepsilon = 10^{-12} \), \( b(x,y) = (0, x^2) \) and \( c = 10^{-5} \). The functions \( f \) and \( u_b \) are such that the solution of (1.1) is \( u(x,y) = \sin(x+y) \).

We consider triangulations \( T_h \) of the type depicted on the left in Fig. 3 and set \( M_h = T_h \). The spaces \( W_h \) and \( D_M \) are defined by (5.7) and (5.8) with \( l = 2 \). The stabilization parameters \( \tau_M \) are defined simply by the right–hand side of (2.10) and they are also used for computing the SUPG norm. Table 1 shows errors of the

![Figure 3. Triangulations used for computations presented in Section 7.](image-url)
Example 7.2 (Problem with two interior layers). We consider the problem (1.1) with \( \Omega = (0,1)^2 \), \( \varepsilon = 10^{-7} \), \( b(x,y) = (-y,x) \), \( c = 0 \) and \( f = 0 \). We set \( u_0(x,0) = 1 \) for \( x \in (\frac{1}{3}, \frac{2}{3}) \) and \( u_0(x,y) = 0 \) elsewhere. Moreover, we do not use the Dirichlet boundary condition at the outflow boundary \( (0,1) \times \{ 1 \} \), where we prescribe a homogeneous Neumann boundary condition.

The solution of this example exhibits two interior layers starting from the discontinuities of the inflow profile at \( y = 0 \). We shall consider the unstructured triangulation \( T_h \) depicted in the middle of Fig. 3 and the set \( M_h \) given by (5.9) and (5.10). The spaces \( W_h \) and \( D_M \) are now defined by (5.6) with \( l = 2 \). Note that if the sets \( M \in M_h \) were not allowed to overlap, we could not use such a space \( W_h \) but, as explained in Section 5, the classical approaches would be either to define the space \( W_h \) on a refined triangulation or to add additional bubble functions to the space \( W_h \). In both cases, the number of degrees of freedom would significantly increase.
Example 2: LPS solution for $\tau_0 = 0.03$ (left) and the difference to the SUPG solution (right).

Fig. 4 (left) shows the solution of the local projection discretization (2.8) for

$$(7.1) \quad \tau_M = \tau_0 \min \left\{ \frac{h_M}{\|b\|_{\infty, M}}, \frac{h_M^2}{\varepsilon} \right\}, \quad M \in \mathcal{M}_h,$$

with $\tau_0 = 0.03$. It is interesting that, for this value of $\tau_0$, the discrete solution is very similar to the SUPG solution, see Fig. 4 (right) where the difference between the SUPG solution and the LPS solution is shown. In the SUPG method, we used a stabilization parameter $\delta$ given for any $T \in \mathcal{T}_h$ by

$$\delta|_T = \frac{h_{T, b}}{4|b|} \left( \coth \frac{P e_T}{4} - \frac{1}{P e_T} \right) \quad \text{with} \quad P e_T = \frac{|b| h_{T, b}}{4 \varepsilon},$$

where $h_{T, b}$ is the diameter of $T$ in the direction of $b$ (we refer to [9, 11, 13] for details on the definition of $\delta$). Thus, although this example does not satisfy the assumptions of our theory, the local projection method is competitive to the SUPG method.

If the scaling factor $\tau_0$ is increased, then the spurious oscillations visible in Fig. 4 (left) decrease and the smearing of the discrete solution increases. For the one–level and two–level approaches of the local projection method (see Section 5), i.e., for $\mathcal{M}_h$ with non–overlapping sets $M$, it is also possible to find values of $\tau_0$ for which the discrete solutions are similar to the SUPG solution. However, if $\tau_0$ is increased (or decreased), the spurious oscillations in the discrete solutions become larger and spread over the whole computational domain. Consequently, for the approaches with non–overlapping sets $M$, it is very difficult to find a proper value of $\tau_0$ since both under– and overestimation lead to solutions globally polluted by spurious oscillations. This is a further argument for using the variant with overlapping sets $M$.

Example 7.3 (Problem with parabolic and exponential boundary layers). We consider the problem (1.1) with $\Omega = (0, 1)^2$, $\varepsilon = 10^{-8}$, $b = (1, 0)$, $c = 1$, $f = 1$ and $u_0 = 0$.

The solution of Example 7.3 possesses an exponential boundary layer at $x = 1$ and parabolic boundary layers at $y = 0$ and $y = 1$. Outside the layers, the solution
is very close to the function
\[ u_0(x, y) = 1 - e^{-x}. \]
We shall consider a structured triangulation \( \mathcal{T}_h \) of the type depicted on the right in Fig. 3 consisting of 16 \times 16 equal squares. Our aim is to compare the following three choices of the spaces \( W_h \) and \( D_M \):
- **one–level LPS** with \( \mathcal{M}_h = \mathcal{T}_h \) and spaces \( W_h \) and \( D_M \) defined by (5.7) and (5.8) with \( l = 2 \);
- **two–level LPS** where \( \mathcal{M}_h \) is a triangulation of \( \Omega \) of the type depicted on the right in Fig. 3 consisting of 8 \times 8 equal squares and the spaces \( W_h \) and \( D_M \) are defined by (5.6) with \( l = 2 \);
- **overlapping LPS** where \( \mathcal{M}_h \) is given by (5.9) and (5.10) and \( W_h \) and \( D_M \) again by (5.6) with \( l = 2 \).

Thus, in all three cases, the solution of (2.8) is quadratic along the edges of \( \mathcal{T}_h \). In the interiors of the elements of \( \mathcal{T}_h \), the solution is either biquadratic or, for the one–level LPS, it belongs to the space of biquadratic functions enriched by three bubble functions. Using two bubble functions, i.e., \( P_{l-1}(T) \) instead of \( R_{l-1}(T) \) in (5.7), leads to almost identical results as for three bubble functions.

A stabilized method should be able to provide a good approximation of the solution \( u \) away from the boundary layers. Therefore, we shall investigate the quality of the discrete solutions \( u_h \) at the points
\[ (x_i, y_i) = (i/32, 0.5), \quad i = 0, \ldots, 24. \]
These points are all vertices and midpoints of edges of \( \mathcal{T}_h \) lying on the line \( y = 0.5 \) and having their \( x \) coordinate in the interval \([0, 0.75]\). To measure the oscillations and accuracy of \( u_h \) along \([0, 0.75]\) \times \{0.5\}, we define the quantities
\[
RTV(u_h) = \frac{\sum_{i=1}^{24} |u_h(x_i, y_i) - u_h(x_{i-1}, y_{i-1})|}{\max_{i=1, \ldots, 24} |u_h(x_i, y_i)|},
\]
\[
ERR(u_h) = \sqrt{\sum_{i=1}^{24} (u_h(x_i, y_i) - u_0(x_i, y_i))^2}.
\]
The value \( RTV(u_h) \) represents an approximation of the relative total variation of \( u_h \) along \([0, 0.75] \times \{0.5\} \). Since \( u_h(x_0, y_0) = 0 \), we have \( RTV(u_h) \geq 1 \). The sequence \( \{u_h(x_i, y_i)\}_{i=0}^{24} \) is monotone if and only if \( RTV(u_h) = 1 \). Large \( RTV(u_h) \) indicates that the values \( u_h(x_i, y_i) \) oscillate. The value \( ERR(u_h) \) measures the accuracy of \( u_h \) by comparing \( u_h \) with the limit solution \( u_0 \) given in (7.2).

The stabilization parameters \( \tau_M \) are again defined by (7.1) and we shall discuss how the solutions of the local projection discretization (2.8) are influenced by the choice of the scaling factor \( \tau_0 \). Fig. 5 shows the dependence of \( RTV(u_h) \) and \( ERR(u_h) \) on \( \tau_0 \) for the three choices of the spaces \( W_h \) and \( D_M \) described above. First let us consider the one–level and two–level methods. For \( \tau_0 \leq 10^{-3} \), the solutions possess large oscillations along \([0, 1]\) \times \{0.5\} whose width is 1/16. Away from the parabolic layers, the two–level solution is independent of \( y \) whereas the one–level solution is periodic in the \( y \) direction with the period 1/16. Along horizontal lines (i.e., lines with a constant \( y \) coordinate) crossing midpoints of elements, the
A NEW VARIATION OF LPS FOR CONVECTION–DIFFUSION–REACTION EQUATIONS

Figure 5. Example 7.3: dependence of $RTV(u_h)$ (left) and $ERR(u_h)$ (right) on the scaling factor $\tau_0$.

The width of oscillations is 1/8. Thus, for both methods, the values at vertices lying on a horizontal line do not oscillate. For the two-level LPS, also the values at midpoints of edges lying on any horizontal line do not oscillate whereas, for the one-level LPS, this holds only for mesh lines. As we see from Fig. 5, for increasing $\tau_0$, both the oscillations and errors decrease and a minimum is attained for $\tau_0$ near 0.1. The corresponding solutions are shown in Fig. 6 (top). Since the solutions are symmetric with respect to the line $y = 0.5$, we show the solutions only on $[0, 1] \times [0.5, 1]$, which makes oscillations better visible. The lines in the figures connect values at vertices, midpoints of edges and midpoints of elements. The additional bubble functions of the one-level method are not taken into account in this case. We observe that spurious oscillations are localized along the boundary layers but the width of the numerical boundary layers at the outflow boundary is rather large. Far away from the boundary layers, the quality of the discrete solutions is satisfactory.

If $\tau_0$ is increased above 0.1, then Fig. 5 indicates that, for both the one-level and the two-level methods, the oscillations and errors increase, reach a maximum around $\tau_0 = 1$ and decrease again towards a second minimum which is reached for $\tau_0$ between 10 and 100. The oscillations for $\tau_0 \in (0.1, 10)$ have a different character than for $\tau_0 < 10^{-3}$. For both methods, the solutions depend only slightly on $y$ away from the parabolic layers whereas they possess oscillations in the $x$ direction whose width is 1/16 for the one-level LPS and 1/8 for the two-level LPS. Thus, the width of the oscillations corresponds to the size of the sets $M \in \mathcal{M}_h$. To get an impression, how fast the solutions deteriorate if $\tau_0$ is increased, we show the two wildly oscillating solutions in Fig. 6 (middle) obtained for $\tau_0$ slightly larger than the ‘optimal’ values near 0.1. Fig. 6 (bottom) shows that the values of $\tau_0$ corresponding to the second minima in Fig. 5 (between 10 and 100) lead to worse discrete solutions than in case of the first minima. If $\tau_0$ further increases, the oscillations in parabolic layers become larger, in particular for the one-level LPS. For very large values of $\tau_0$, the solutions again wildly oscillate in the $x$ direction. The width of the oscillations
Figure 6. Example 7.3: solutions of the one–level LPS (left) for $\tau_0 = 0.1$, $\tau_0 = 0.5$ and $\tau_0 = 65$ (top to bottom) and of the two–level LPS (right) for $\tau_0 = 0.05$, $\tau_0 = 0.3$ and $\tau_0 = 15$ (top to bottom).

is $1/8$ for the one–level LPS and $1/4$ for the two–level LPS. Inside the sets $M \in \mathcal{M}_h$, no oscillations occur.
For the overlapping LPS and \( \tau_0 \lesssim 10^{-3} \), the discrete solutions are qualitatively similar as for the two–level LPS. However, as we can see from Fig. 5, the dependence of RTV(\( u_h \)) on \( \tau_0 \) is different after the minimal value RTV(\( u_h \)) = 1 has been attained (for \( \tau_0 \sim 0.014 \)). In particular, we observe that, \( u_h \) is monotone along [0, 0.75] × \{0.5\} for a wide range of \( \tau_0 \) and that RTV(\( u_h \)) < 1.3 for any \( \tau_0 > 0.01 \). A more detailed investigation of the solutions reveals that, for \( \tau_0 \gtrsim 0.1 \), no oscillations at all occur along [0, 0.75] × \{0.5\}. More precisely, for \( \tau_0 \in (0.1, 25) \), the solution \( u_h \) has one minimum and two maxima in (0, 1) × \{0.5\}. For \( \tau_0 \sim 0.1 \), the points where the extrema are attained have their \( x \) coordinate larger than 0.75 and hence \( u_h \) is monotone along [0, 0.75] × \{0.5\} (i.e., RTV(\( u_h \)) = 1). When \( \tau_0 \) is increased, these points shift towards \( x = 0 \) and the distances among them become larger so that the solutions appears smoother. When at least one extremum point is in (0, 0.75) × \{0.5\}, the solution is not monotone along [0, 0.75] × \{0.5\} and RTV(\( u_h \)) > 1, see Fig. 5. For \( \tau_0 \sim 25 \), two extrema disappear and \( u_h \) has only one maximum and no minimum in (0, 1) × \{0.5\}. The \( x \) coordinate of the maximum point is larger than 0.75 so that again RTV(\( u_h \)) = 1. If \( \tau_0 \) is further increased, the maximum point moves towards (0.5, 0.5) and the magnitude of \( u_h \) decreases. For \( \tau_0 = 10^5 \), the maximum of \( u_h \) in \( \Omega \) is less than 0.014. The above discussion is illustrated by the solutions for \( \tau_0 = 0.1 \), 1 and 10 in Fig. 7 (top and bottom left). The value \( \tau_0 = 0.1 \) represents a good choice with respect to both oscillations and accuracy, see also the graph of ERR(\( u_h \)) in Fig. 5.

The detailed discussion to this example clearly shows the important difference between the one–level and two–level LPS one the one side and the overlapping LPS on the other side which was already briefly mentioned in the discussion to Example 7.2. For the former two methods, there is only a small interval of values of \( \tau_0 \) which lead to acceptable discrete solutions. Since \( \tau_0 \) both smaller and larger than these ‘optimal’ values leads to spurious oscillations, it is very difficult to find a proper value of \( \tau_0 \) numerically and a small deviation of \( \tau_0 \) from the ‘optimal’ value may deteriorate the solution considerably. On the other hand, for the overlapping LPS, the properties of the discrete solution depend on \( \tau_0 \) in a monotone way: for increasing \( \tau_0 \), oscillations decrease and smearing increases. This is much more convenient from the practical point of view since, in many applications, a moderate smearing is more acceptable than spurious oscillations.

The ‘optimal’ solution in Fig. 7 (top left) for the overlapping LPS is slightly better than the corresponding solutions for the one–level and two–level LPS in Fig. 6 (top). Nevertheless, it is still not satisfactory since the numerical boundary layer at the outflow boundary is rather wide. A significant improvement can be achieved by introducing the enrichment described in Section 6. Thus, let \( W_h \) be defined by (6.13) with \( l = 2 \) (in this case \( \mathcal{T}_h' = \emptyset \)) and let \( \mathcal{M}_h \) consist of \( \{ S_E \} \subseteq \mathcal{E}_h \) and the sets \( M_i \) defined by (5.9) with \( x_i \notin \overline{G}_h \). Again, \( D_M = P_1(M) \) for any \( M \in \mathcal{M}_h \). Since the sets \( S_E \) overlap, we use one half of the formula (6.7) to define \( \tau_M \), i.e., \( \tau_M = h_1/(20b_1) = 1/320 \). For simplicity, this definition is considered for any \( M \in \mathcal{M}_h \). Using the improved formula (6.10) does not lead to any visible change of the solution. The solution is depicted in Fig. 7 (bottom right) and we observe that now the oscillations at the outflow boundary are localized to one row of elements along this boundary and can be easily removed by postprocessing. Away from this row of elements and the parabolic boundary layers, the discrete solution is very accurate. Although the approximation of the parabolic boundary layers improved
Figure 7. Example 7.3: solutions for the overlapping LPS with \( \tau_0 = 0.1 \) (top left), \( \tau_0 = 1 \) (top right), \( \tau_0 = 10 \) (bottom left) and with the enrichment at boundaries of Section 6 (bottom right).

A little bit, a substantial improvement cannot be expected since only the streamline derivative is used to define the stabilization term. Here different techniques have to be applied, see, e.g., [13].

Acknowledgement

The author is gratefully indebted to Professor Hans–Görg Roos for many fruitful discussions which inspired this work.

References

A NEW VARIANT OF LPS FOR CONVECTION–DIFFUSION–REACTION EQUATIONS


Charles University, Faculty of Mathematics and Physics, Department of Numerical Mathematics, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: knobloch@karlin.mff.cuni.cz

Technische Universität Dresden, Fakultät Mathematik und Naturwissenschaften, Institut für Numerische Mathematik, D-01062 Dresden, Germany