Bergman projections of vector-valued functions

Alexandru Aleman
joint with Olivia Constantin
Let:
\( \mathbb{D} \) be the unit disc in the complex plane,
\( A \) be the normalized area measure on \( \mathbb{D} \).

The Bergman projection is the orthogonal projection from \( L^2(\mathbb{D}) \) onto the closed subspace consisting of analytic functions in \( \mathbb{D} \). An elementary calculation shows that this singular operator is explicitly defined for every integrable function on the disc by

\[
P_0 f(z) = \int_{\mathbb{D}} \frac{1}{(1 - \zeta z)^2} f(\zeta) dA(\zeta), \quad z \in \mathbb{D}.
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\]
This is the Bergman-space analogue of the well known Riesz projection on spaces of integrable functions on the unit circle $\mathbb{T}$

$$P_{-1}f(z) = \int_{\mathbb{T}} \frac{1}{1 - \zeta z} f(\zeta) |d\zeta|, \quad z \in \mathbb{D}. \quad (2)$$
Even if they are connected by the chain of standard weighted Bergman projections

\[ P_\alpha f(z) = \int_{\mathbb{D}} \frac{1}{(1 - \bar{\zeta}z)^{\alpha+2}} f(\zeta) dA_\alpha(\zeta), \quad z \in \mathbb{D}, \quad (3) \]

where \( \alpha > -1 \) and \( dA_\alpha(\zeta) = (\alpha + 1)(1 - |\zeta|^2)^\alpha dA(\zeta) \),

\( P_0, P_\alpha \) are essentially different from the Riesz projection \( P_{-1} \).

While most of the properties of \( P_{-1} \) are due to the cancellation in the Szegö kernel, the kernels arising in \( P_\alpha \) induce very little cancellation. For example,

\[ P_\alpha^+ f(z) = \int_{\mathbb{D}} \frac{1}{|1 - \bar{\zeta}z|^{\alpha+2}} f(\zeta) dA_\alpha(\zeta), \quad z \in \mathbb{D} \]

has the same mapping properties as \( P_\alpha, \alpha > -1 \).
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It is a challenge to understand the deformation of Bergman projections into the Riesz projection.

Maybe the family of weights should be refined. There is very recent work in this direction by J. A. Pelaez and J. Rättyä.
The vector-valued case

The differences between these projections become more striking in the vector-valued case, that is if we fix a separable Hilbert space $\mathcal{H}$ and consider $P_\alpha$, $P_{-1}$ acting on integrable $\mathcal{H}$-valued functions $f$.

Let $L_{a}^{2,\alpha}(\mathcal{H})$, $\alpha > -1$, equal the space of analytic $\mathcal{H}$-valued functions $f$ in $\mathbb{D}$ with

$$\|f\| = \left(\int_{\mathbb{D}} \|f(z)\|_{\mathcal{H}}^2 \, dA_\alpha(z)\right)^{\frac{1}{2}} < \infty.$$  

(4)

Similarly, when $\alpha = -1$, the corresponding space of analytic functions is the Hardy space $H_{a}^{2}(\mathcal{H})$ with

$$\|f\| = \lim_{r \to 1} \left(\int_{\mathbb{T}} \|f(rz)\|_{\mathcal{H}}^2 \, |dz|\right)^{\frac{1}{2}} < \infty.$$  

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- Let \( L^2_{a,\alpha}(\mathcal{H}), \ \alpha > -1 \), equal the space of analytic \( \mathcal{H} \)-valued functions \( f \) in \( \mathbb{D} \) with

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Then for $\alpha > -1$

$$P_\alpha(L^2(\mathbb{D}, \mathcal{H})) = L^2,\alpha(\mathcal{H}),$$

and

$$P_{-1}(L^2(\mathbb{T}, \mathcal{H})) = H^2(\mathcal{H}).$$
Hankel operators

We want to use Hankel operators that map the Hardy and Bergman spaces into themselves, therefore we introduce them via the following bilinear forms.

Given analytic functions $T : \mathbb{D} \to \mathcal{B}(\mathcal{H})$, and $x, y : (1 + \varepsilon)\mathbb{D} \to \mathcal{H}$ let

$$\langle \Gamma_T x, y \rangle = \lim_{r \to 1} \int_{\mathbb{T}} \langle T(rz) x(r\bar{z}), y(rz) \rangle |dz|,$$  \hspace{1cm} (6)

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The Hardy space case

By a deep factorization result of Sarason the characterization of bounded Hankel operators amounts to finding an intrinsic characterization of Riesz projections of bounded measurable operator-valued functions on the unit circle.

The scalar case is resolved by the celebrated Fefferman duality theorem and the result says that this class coincides with the set of functions $f \in H^2$ whose boundary values have bounded mean oscillation:

$$
\sup_I \frac{1}{|I|} \int_I |f(z) - f_I||dz| < \infty,
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where $f_I = \frac{1}{|I|} \int_I f(z)|dz|$, (8)

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- When trying to extend this to the operator-valued setting the difficulty is self-evident; one has find the right quantity to replace the modulus in the formula defining bounded mean oscillation.

- If we choose the operator norm instead, the resulting condition is too strong, it will not be satisfied by all Riesz projections of bounded operator-valued functions.

- The second natural choice is to consider the so-called ”so” condition

\[ \|f\|_{BMO_{so}} = \sup_{\|x\|_{\mathcal{H}} = 1} \sup_{\|l\|} \int_{I} \|f(z)x - f_{I}x\|_{\mathcal{H}}|dz| < \infty. \] (9)
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Nazarov, Pisier, Treil, Volberg (2002) proved that there are \( d \times d \) matrix-symbols \( T \) with \( \| T \|_{BMO_{so}} = \| T^* \|_{BMO_{so}} = 1 \) for which
\[
\| \Gamma_T \| \gtrsim \log d.
\]

This difficult open problem has a solution in an important special case, the Hardy space on the polydisc \( \mathbb{D}^n \). Ferguson and Lacey \((n = 2, 2004)\) and Lacey and Terwilleger \((\text{general } n, 2009)\) proved the weak factorization of \( H^1(\mathbb{D}^n) \) as the projective tensor product space \( H^2(\mathbb{D}^n) \otimes H^2(\mathbb{D}^n) \).

Equivalently, it holds that a Hankel operator is bounded on \( H^2(\mathbb{D}^n) \) if and only its holomorphic symbol belongs to the dual of \( H^1(\mathbb{D}^n) \), which is the product \( BMO \) space, as identified by Chang and Fefferman.
Nazarov, Pisier, Treil, Volberg (2002) proved that there are $d \times d$ matrix-symbols $T$ with $\| T \|_{BMO_0} = \| T^* \|_{BMO_0} = 1$ for which

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Infinitely many variables?

A variant of the above result for the corresponding spaces on the infinite polydisc $\mathbb{D}^\infty$ would be of great importance for the study of Dirichlet series

$$\sum_n a_n n^{-s}.$$ 

However

the constants in the work of Lacey-Terwilleger:

BMO-norm/Hankel norm go to infinity with $n$.

Very recently, J. Ortega and K. Seip proved that these constants grow exponentially in $n$. 
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The characterization of bounded Hankel operators on Bergman spaces is the same as the well-known scalar result.

Theorem

\[ \Gamma_T \text{ extends to a bounded linear operator on } L^{2, \alpha}_a(\mathcal{H}) \text{ if and only if } \]
\[ \sup_{z \in \mathbb{D}} (1 - |z|^2) \| T'(z) \| < \infty. \]

The proof can be obtained by imitating the scalar methods. Alternatively, one can use the duality theory of Bergman spaces as developed by Arregui and Blasco.
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The proof can be obtained by imitating the scalar methods. Alternatively, one can use the duality theory of Bergman spaces as developed by Arregui and Blasco.
An operator $T$ on a Hilbert space $\mathcal{H}$ is called similar to a contraction if there exist a contraction $S$ ($\|S\| \leq 1$) and an invertible operator $V$ on $H$ such that $T = V^{-1}SV$.

By von Neumann’s inequality such an operator will satisfy for any analytic polynomial $p$

$$\|p(T)\| \leq \|V\|\|V^{-1}\| \sup_{z \in \mathbb{D}} |p(z)|,$$

i.e. $T$ is polynomially bounded.

The similarity problem, i.e. whether a polynomially bounded operator is similar to a contraction, was a long-standing open problem in operator theory posed by Halmos in his "Ten problems in Hilbert space" in 1970.
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Attempting to find a counterexample, Peller considered the following type of operators (sometimes called Foias-Williams/Peller type operators or Foguel-Hankel operators)

\[
R_f = \begin{pmatrix} S^* & \Gamma_f \\ 0 & S \end{pmatrix}
\]  

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acting on the direct sum $H^2 \oplus H^2$, where $S$ is the shift (multiplication by the independent variable) operator on $H^2$, and $\Gamma_f$ is the Hankel operator with symbol $f$.

- $R_f$ is polynomially bounded if $f' \in BMO$ (Peller)
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- If $R_f$ is polynomially bounded then $f' \in BMO$ (Aleksandrov and Peller)
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In 1996 Pisier solved the similarity problem in the negative. His example is an operator of the form

$$R_T = \begin{pmatrix} S^* & \Gamma_T \\ 0 & S \end{pmatrix}$$

with $T$ analytic and operator valued, acting on the direct sum of Hardy spaces with values in an infinite dimensional Hilbert space.

The infinite dimensional version of this operator was crucial for Pisier’s construction as Paulsen and Davidson showed that no such examples are possible in the finite dimensional case.
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We can of course, consider the same type of operator on the weighted Bergman spaces $L^2_{a,\alpha}(\mathcal{H})$. The following result was obtained by us in 2004. Fergusson and Petrovic proved the scalar version 2 years earlier.

**Theorem**

Let $T : \mathbb{D} \to B(\mathcal{H})$ be a holomorphic operator-valued function with $\sup_{z \in \mathbb{D}} (1 - |z|^2) \| T'(z) \| < \infty$. Then the following are equivalent:

(i) $R_T$ is power bounded ($\sup_n \| R_T^n \| < \infty$);
(ii) $R_T$ is polynomially bounded;
(iii) $R_T$ is similar to a contraction;
(iv) $\sup_{z \in \mathbb{D}} (1 - |z|^2) \| T''(z) \| < \infty$. 

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The Hunt-Muckenhoupt-Wheeden theorem characterizes the weights $w$ for which the Riesz projection is bounded on the weighted space $L^2(\mathbb{T}, w)$ in terms of the $A_2$-condition

$$\sup_I \left( \frac{1}{|I|} \int_I w \right) \left( \frac{1}{|I|} \int_I w^{-1} \right) < \infty.$$ 

The boundedness of the Riesz projection translates to the fact that the angle between the "past" and the "future" of a stationary process with spectral measure $W$ is nonzero.

The most obvious impediment is that the proof in the scalar case relies heavily on the use of maximal functions, a tool that is not available for matrix(operator)-valued weights.
The angle between past and future

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Treil and Volberg (’97) overcome this difficulty by means of wavelet techniques and obtain an analogue of this celebrated theorem for matrix-valued weights. The Riesz projection is bounded on the weighted space $L^2(W)$, where $W$ is a $d \times d$ matrix-valued weight, if and only if

$$\sup_I \left\| \left( \frac{1}{|I|} \int_I W \right)^{1/2} \left( \frac{1}{|I|} \int_I W^{-1} \right)^{1/2} \right\| < \infty.$$ 

This result does not generalize to the case $d = \infty$. Again the constants involved grow like $\log d$ (Gillespie, Pott, Treil, Volberg (2004)).
The Bergman projection in weighted spaces

The Bergman space analogue of the scalar Hunt-Muckenhoupt-Wheeden theorem, was obtained by Békollé and Bonami:

**Theorem**

\[ P_\alpha \text{ is bounded on } L^2(\mathbb{D}, wdA_\alpha) \text{ if and only if} \]
\[
\sup_S \frac{1}{A_\alpha^2(S)} \int_S \omega dA_\alpha \int_S \frac{1}{\omega} dA_\alpha < \infty,
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where the supremum is taken over all Carleson squares

\[ S = \{ z = re^{it} : 1 - h < r < 1, |t - \theta| < h \}, \]

with \( h \in (0, 1), \theta \in [0, 2\pi) \).

The class of such weights is denoted by \( B_2(\alpha) \).
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We consider operator-valued weights $W : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ such that $W(z)$ is a nonnegative operator that is invertible a.e. $z \in \mathbb{D}$.

Our only assumptions are:

1. The operator-valued integrals $\int_{\mathbb{D}} W^{\pm 1} dA_\alpha$ exist (i.e. they define bounded linear operators).
2. $\int_{\mathbb{D}} WdA_\alpha$ is invertible.
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The corresponding $L^2$-space on $\mathbb{D}$ is denoted by $L^2(WdA_\alpha)$ and has the norm

$$\|f\|_{2, W, \alpha}^2 = \int_{\mathbb{D}} \langle W(z)f(z), f(z) \rangle dA_\alpha(z).$$

It is easy to see that the subspace of $L^2(WdA_\alpha)$ consisting of $\mathcal{H}$–valued analytic functions in $\mathbb{D}$ is closed. We shall denote this subspace by $L^2_a(WdA_\alpha)$. 
Let $\alpha > -1$ and assume $W$ is as above. Then the Bergman projection $P_\alpha$ is bounded on $L^2(WdA_\alpha)$ if and only if $W$ belongs to the class $B_2(\alpha)$, i.e. if

$$\sup_{S} \left\| \left( \frac{1}{A_\alpha(S)} \int_S WdA_\alpha \right)^{1/2} \left( \frac{1}{A_\alpha(S)} \int_S W^{-1}dA_\alpha \right)^{1/2} \right\| < \infty$$

where the supremum is taken over all Carleson squares $S$ in $\mathbb{D}$.

Observe that the theorem holds for infinite dimensional Hilbert space $\mathcal{H}$ as well!
About the proof of the sufficiency part

- Assume that \( \alpha = 0 \).
- Assume without loss that \( W \) is almost constant on pseudo-hyperbolic discs (i.e. replace \( W \) by its averages on such discs).
- Prove with "bare hands" that the Békollé-Bonami condition defining \( B_2(0) \) implies that

\[
P^+_\beta W(z) = \int_D \frac{1}{|1 - \zeta z|^{\beta+2}} W(\zeta) dA(\zeta) \leq C(1 - |z|^2)^{-\beta} W(z),
\]

provided that \( \beta \) is sufficiently large (\( \beta > 4 \) will do in this case). This is a Bergman-space analogue of the \( A_\infty \)-condition.

Alexandru Aleman joint with Olivia Constantin

Bergman projections of vector-valued functions
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The last estimate can be used that $P_\beta$ is bounded on $L^2_a(W)$ when $\beta$ is sufficiently large. (larger than before)

But we want $P_0$ to be bounded!

To close the gap we argue by duality. The boundedness of a projection is equivalent to a certain representation of continuous linear functionals on the space in question. This translates to:
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We know that when $\beta$ is sufficiently large, every continuous linear functional $l$ on $L^2_\alpha(W)$ can be written as

$$l(f) = \int_D \langle f(z), g(z) \rangle_{\mathcal{H}} (1 - |z|^2)^\beta dA(z)$$

with $g \in L^2_\alpha((1 - |z|^2)^2W^{-1})$ fixed (and unique).

We want to prove that every continuous linear functional $l$ on $L^2_\alpha(W)$ can be written as

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By Parseval’s formula (or Stokes’s) we can easily see that if $\beta = N$ is an integer then the function $h$ we are looking for is essentially an $N$-th primitive of the original $g$.

But is it in $L^2_a(W^{-1})$?
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About the proof of the sufficiency part

This will follow once we prove that for every \( W \in B_2(0) \) we have

\[
\|f\|_{L^2_a(W)}^2 \preceq \|f(0)\|^2 + \int_\mathbb{D} \langle Wf'(z), f'(z) \rangle (1 - |z|^2)^2 dA(z),
\]

for all \( f \in L^2_a(W) \).

This inequality is true! Reason: For any \( \gamma > 0 \)

\[
f(z) = P_\gamma (1 - |z|^2)f'(z) + \text{ harmless terms}
\]

and we know already that \( P_\gamma \) is bounded on \( L^2_a(W) \) when \( \gamma \) is large.
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This will follow once we prove that for every $W \in B_2(0)$ we have

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for all $f \in L^2_a(W)$.

This inequality is true! Reason: For any $\gamma > 0$

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and we know already that $P_{\gamma}$ is bounded on $L^2_a(W)$ when $\gamma$ is large
A word about the necessity part

This is much easier to see in the scalar case.

For $\lambda \in \mathbb{D}$ and $\alpha > -1$ let

$$k_\lambda(z) = \frac{1}{(1 - \lambda z)^{\alpha+2}}, \quad \phi_\lambda(z) = \frac{\lambda - z}{1 - \lambda z}.$$ 

The key step is the elementary identity:

$$k_z(\lambda) = k_z(\zeta) \frac{k_\zeta(\lambda)}{k_\zeta(\zeta)} + k_z(\lambda) \sum_n c_n \phi_\zeta(\lambda)^n \overline{\phi_\zeta(z)^n},$$

where $\sum_n |c_n| < \infty$. 
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In other words,

$$P_\alpha f(\lambda) = \sum_n c_n M_{\phi_\xi} P_\alpha M_{\phi_\xi}^* f(\lambda) + Pf(\zeta) \frac{k_\xi(\lambda)}{k_\zeta(\zeta)}$$

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In other words,

\[ P_\alpha f(\lambda) = \sum_n c_n M_{\phi \zeta} P_\alpha M_{\phi \zeta}^* f(\lambda) + Pf(\zeta) \frac{k_\zeta(\lambda)}{k_\zeta(\zeta)} \]

i.e. if \( P_\alpha \) is bounded, then the rank-one term on the right is bounded.
A word about the necessity part

Let \( f(z) = k_\zeta(z)W^{-1}(z) \) Then

\[
\|f\|^2 = \int_{\mathbb{D}} |k_\zeta(z)|^2 W^{-1}(z) dA_\alpha(z)
\]

and for this particular \( f \) the boundedness of the rank-one term on the previous slide gives the inequality

\[
\int |k_\zeta(z)|^2 W(z) dA_\alpha(z) \int_{\mathbb{D}} |k_\zeta(z)|^2 W^{-1}(z) dA_\alpha(z) \lesssim k_\zeta(\zeta)^2
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