Stability of Quantum Dynamical Semigroups and Fixed points

Sachi Srivastava

University of Delhi
(Joint work with B.V.R. Bhat, ISI Bangalore)

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By a Quantum dynamical semi-group (QDS), we shall mean a family \((\mathcal{T}_t)_{t \geq 0}\) of linear maps on \(B(H)\), \(H\) a complex, separable Hilbert space, such that

1. \(\mathcal{T}_0(X) = X\), \(\mathcal{T}_s(\mathcal{T}_t(X)) = \mathcal{T}_{t+s}(X)\) for all \(X \in B(H)\).
Quantum Dynamical semigroup

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2. \(\mathcal{T}\) is completely positive for \(t \geq 0\).
3. \(\mathcal{T}_t\) is contractive and normal \(t \geq 0\).
4. The map \(t \rightarrow \mathcal{T}_t(X)\) is continuous wrt the weak topology on \(B(H)\) for each \(X \in B(H)\).

Further, if \(\mathcal{T}_t(I) = I\) for all \(t > 0\), then the QDS \(\mathcal{T}\) is called a quantum Markov semigroup.

If \(\mathcal{T}_t\) is of the form \(\mathcal{T}_t(X) = A_t X A_t^*\) for a strongly continuous semigroup of contractions \(\{A_t\}_{t \geq 0}\) on \(H\), then \(\mathcal{T}\) is called an elementary QDS.
By a Quantum dynamical semi-group (QDS), we shall mean a family \((T_t)_{t \geq 0}\) of linear maps on \(B(H)\), \(H\) a complex, separable Hilbert space, such that

- (i) \(T_0(X) = X, T_s(T_t(X)) = T_{t+s}(X)\) for all \(X \in B(H)\).
- (ii) \(T\) is completely positive for \(t \geq 0\)
- (iii) \(T_t\) is contractive and normal \(t \geq 0\).
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The generator of a QDS

The operator $\mathcal{L}$ with domain $\mathcal{D}(\mathcal{L})$ given by

$$\mathcal{D}(\mathcal{L}) = \left\{ X \in B(\mathcal{H}) : \lim_{t \downarrow 0} \frac{\mathcal{T}_t(X) - X}{t} \text{ exits in the weak topology} \right\}$$

$$\mathcal{L}(X) = \lim_{t \downarrow 0} \frac{\mathcal{T}_t(X) - X}{t}$$
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Assumption 2. : QDS is sub-markovian, that is,

$$T_t(I) \leq I \quad \forall t \geq 0.$$
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- Uniformly stable if \(\lim_{t \to \infty} ||\mathcal{T}_t|| = 0\).
When is a QDS "stable"?

We shall call the QDS \((\mathcal{T}_t)_{t \geq 0}\)

- Uniformly stable if \(\lim_{t \to \infty} \|\mathcal{T}_t\| = 0\).
- stable if \(\lim_{t \to \infty} \mathcal{T}_t(I) = 0\) in the strong operator topology.
The two stability notions do not coincide

- Let $\mathcal{H} = L^2(-1, 0)$ and $K$ be the multiplication operator given by $(Kf)(s) = q(s)f(s)$, where $q(s) = s$, $s \in (-1, 0)$.
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- Let $\mathcal{H} = L^2(-1, 0)$ and $K$ be the multiplication operator given by $(Kf)(s) = q(s)f(s)$, where $q(s) = s$, $s \in (-1, 0)$.
- Let $(P_t)_{t \geq 0}$ be the uniformly continuous semigroup generated by $K$, that is, $(P_t f)(s) = e^{tq(s)}f(s)$, for $f \in \mathcal{H}$, $s \in (-1, 0)$, $t \geq 0$. 
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Let $T_t(X) = P_tXP_t^*$, $X \in B(\mathcal{H})$, $t \geq 0$. Then $T_t$ is a quantum dynamical semigroup which is not uniformly stable.
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Let \( (P_t)_{t \geq 0} \) be the uniformly continuous semigroup generated by \( K \), that is, \((P_t f)(s) = e^{tq(s)}f(s)\), for \( f \in \mathcal{H}, s \in (-1, 0), t \geq 0 \).

Let \( T_t(X) = P_tXP_t^*, X \in B(\mathcal{H}), t \geq 0 \). Then \( T_t \) is a quantum dynamical semigroup which is not uniformly stable.

But,

\[
\lim_{t \to \infty} T_t(I)f = \lim_{t \to \infty} \int_{-1}^{0} |e^{2ts}f(s)|^2 ds = 0,
\]

for every \( f \in \mathcal{H} \) making \( T \) stable.
The bounded generator

In the case when $\mathcal{T}$ is uniformly continuous,

- the generator $\mathcal{L}$ is given by

\[
\mathcal{L}(X) = KX + XK^* + \sum_{j=1}^{\infty} L_j^* X L_j, \quad X \in B(\mathcal{H}),
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where $K, L_j \in B(\mathcal{H})$ and the sum on the right hand side above converges in strong operator topology.

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In the case when $\mathcal{T}$ is uniformly continuous, the generator $\mathcal{L}$ is given by

$$\mathcal{L}(X) = KX + XK^* + \sum_{j=1}^{\infty} L_j^* XL_j, \quad X \in B(\mathcal{H}),$$

where $K, L_j \in B(\mathcal{H})$ and the sum on the right hand side above converges in strong operator topology. (Gorini et al, '76, Lindbald '76, Christensen and Evans '79,...) We write $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_0$ where $\mathcal{L}_1$ is the completely positive part of the generator, given by

$$\mathcal{L}_1(X) = \sum_{j} L_j^* XL_j, \quad (1)$$

while $\mathcal{L}_0$ is given by

$$\mathcal{L}_0(X) = KX + XK^*, \quad (2)$$

for all $X \in B(\mathcal{H})$. 

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Stability of Quantum Dynamical Semigroups and Fixed points
Can stability of a uniformly continuous QDS be characterised in terms of the "coefficients" of its generator?
A particular case

**Theorem**
Suppose that $\mathcal{L}$ is a bounded operator on $B(\mathcal{H})$ with $\mathcal{L}(I) \leq 0$ and there exists a $b < 0$ such that $\sum_j L_j^* L_j < -bl < -(K + K^*)$, where $K, L_j \in B(\mathcal{H})$ and $\mathcal{L}(X) = KX + XK^* + \sum_j L_j^* XL_j$, $X \in B(\mathcal{H})$. Then the Q.D.S. $\mathcal{T}$ generated by $\mathcal{L}$ is stable.
Let $\mathcal{H} = \mathbb{C}^2$ and $a \in \mathbb{C}$. Set

$$K = \begin{pmatrix} -1 + i & 1 \\ 0 & -1 + i \end{pmatrix}, \quad L = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$  

Then $K + K^* < -\frac{1}{2} I$. Then choosing $a$ so that $|a|^2 < \frac{1}{2}$ ensures that $L^*L < \frac{1}{2} I < -(K + K^*)$. Thus the hypothesis of the above Theorem is satisfied. Therefore, if $L(X) = KX + XK^* + L^*XL$, $X \in B(\mathcal{H})$, then $L$ must generate a stable Q.D.S. Actual computation shows that

$$L \left( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = \begin{pmatrix} (|a|^2 - 2)x + y + z \\ (|a|^2 - 2)y + w \\ (|a|^2 - 2)z + w \\ (|a|^2 - 2)w \end{pmatrix}.$$  

So $L$ may be represented by the $4 \times 4$ matrix

$$\begin{pmatrix} (|a|^2 - 2) & 1 & 1 & 0 \\ 0 & (|a|^2 - 2) & 0 & 1 \\ 0 & 0 & (|a|^2 - 2) & 1 \\ 0 & 0 & 0 & (|a|^2 - 2) \end{pmatrix}.$$
Some observations:

- The Q.D.S. $\mathcal{T}$ is stable if and only if the family $(\mathcal{T}_t)_{t \geq 0}$ has the operator 0 as its only fixed point.
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- For any $X \in B(\mathcal{H})$, if $s - \lim_{t \to \infty} \mathcal{T}_t(X)$ exists, then it is a fixed point, and every fixed point arises in this way.

Further, $C \in B(\mathcal{H})$ is a fixed point of the uniformly continuous QDS $\mathcal{T}$ if and only if $L(C) = 0$. So, the set of fixed points $F(\mathcal{T}) = \{ X \in B(\mathcal{H}) : \mathcal{T}_t(X) = X, t \geq 0 \}$ is an object of interest. (For a single completely positive map, G. Popescu (2003), and for Markovian semigroups, Fagnola and Umanita have looked at the set $F(\mathcal{T})$.)
Fixed points

Some observations:

- The Q.D.S. $T$ is stable if and only if the family $(T_t)_{t \geq 0}$ has the operator 0 as its only fixed point.
- For any $X \in B(H)$, if $\lim_{t \to \infty} T_t(X)$ exists, then it is a fixed point, and every fixed point arises in this way.
- Further, $C \in B(H)$ is a fixed point of the uniformly continuous QDS $T$ if and only if $\mathcal{L}(C) = 0$.
- So, the set of fixed points
  \[ \mathcal{F}(T) = \{ X \in B(H) : T_t(X) = X, t \geq 0 \}, \]
  is an object of interest.
  (For a single completely positive map, G. Popescu (2003), and for Markovian semigroups, Fagnola and Umanita have looked at the set $\mathcal{F}(T)$. )
Suppose that $K, L_i$ are self adjoint operators in $B(\mathcal{H})$ and $\mathcal{L}$ given by $\mathcal{L}(X) = KX + XK^* + \sum_i L_i XL_i^*$, $X \in B(\mathcal{H})$ generates a submarkovian quantum dynamical semigroup $T$. If $\ker K \neq \{0\}$, then for every $x_0 \in \ker K$, $C := |x_0\rangle\langle x_0|$ is a fixed point for $T$. Thus, $T$ cannot be stable.
Theorem (...Bhat, 1996,..)

Every quantum dynamical semigroup on $B(\mathcal{H})$ admits a unique (upto unitary equivalence) minimal dilation consisting of $e_0$ semigroups.
Minimal dilations

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- Suppose $\mathcal{T} = \{T_t : t \geq 0\}$ is a quantum dynamical semigroup on $B(\mathcal{H})$. If $\hat{\mathcal{H}}$ is a Hilbert space containing $H$ as a closed subspace.
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- Let $\theta = \{\theta_t : t \geq 0\}$ is an $e_0$ semigroup on $B(\hat{\mathcal{H}})$,
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- Let $\theta = \{\theta_t : t \geq 0\}$ is an $e_0$ semigroup on $B(\hat{\mathcal{H}})$,
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- Let \( \theta = \{\theta_t : t \geq 0\} \) is an \( e_0 \) semigroup on \( B(\hat{\mathcal{H}}) \),
- that is, a quantum dynamical semigroup, consisting of \( \ast \)-endomorphisms of \( B(\hat{\mathcal{H}}) \),
- satisfying,

\[
\mathcal{T}_t(X) = P\theta_t(X)P, \; t \geq 0, \; X \in B(\mathcal{H}) = PB(\hat{\mathcal{H}})P \subset B(\hat{\mathcal{H}}),
\]

where \( P \) is the orthogonal projection of \( \hat{\mathcal{H}} \) onto \( H \),
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where \( P \) is the orthogonal projection of \( \hat{\mathcal{H}} \) onto \( \mathcal{H} \),
- then \( \theta \) is called a dilation of \( \mathcal{T} \).
The dilation $\theta$ is said to be minimal if the closed linear span of
$$\{\theta_{r_1}(X_1)\ldots\theta_{r_n}(X_n)u : r_i \geq 0, X_i \in B(\mathcal{H}), u \in \mathcal{H}, 1 \leq i \leq n, n \geq 0\}$$
is all of $\hat{\mathcal{H}}$. 
Theorem

If a quantum dynamical semigroup is stable, then its minimal dilation is also stable.
Theorem

Let $\mathcal{T}$ be a uniformly continuous quantum dynamical semigroup on $B(\mathcal{H})$. A positive operator $C \in B(\mathcal{H})$ is a fixed point for $\mathcal{T}$ if and only if there exists a quantum dynamical semigroup $\beta$ on $B(\mathcal{H})$, such that $\beta_t(I) = I$, and

$$\mathcal{T}_t(C^{\frac{1}{2}}XC^{\frac{1}{2}}) = C^{\frac{1}{2}}\beta_t(X)C^{\frac{1}{2}}$$

for all $t \geq 0$ and $X \in B(\mathcal{H})$. 
idea of proof

- Let $C \in B(\mathcal{H})$ satisfy $\mathcal{T}_t(C) = C$ for all $t \geq 0$ and $\theta$ be the minimal $e_0$ dilation of $\mathcal{T}$ acting on $B(\hat{\mathcal{H}})$, where $\hat{\mathcal{H}}$ is a Hilbert space and $\mathcal{H} \subset \hat{\mathcal{H}}$. 

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Stability of Quantum Dynamical Semigroups and Fixed points
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Let $\mathcal{K} := \text{Range}C^{\frac{1}{2}}$. For $t \geq 0$, define

$$W_t : \mathcal{K} \rightarrow \hat{\mathcal{H}}, \text{ by setting}$$

$$W_t(C^{\frac{1}{2}}h) = \theta_t(C^{\frac{1}{2}})h, \quad h \in \mathcal{H}.$$
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Let $C \in B(\mathcal{H})$ satisfy $\mathcal{T}_t(C) = C$ for all $t \geq 0$ and $\theta$ be the minimal $e_0$ dilation of $\mathcal{T}$ acting on $B(\hat{\mathcal{H}})$, where $\hat{\mathcal{H}}$ is a Hilbert space and $\mathcal{H} \subset \hat{\mathcal{H}}$.

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Define, for $t \geq 0$, $\gamma_t : B(\mathcal{K}) \rightarrow B(\mathcal{K})$ by

$$\gamma_t(X) = PW_t^* \theta_t(X) W_t P,$$

, for all $X \in B(\mathcal{K})$. Here $P := P_\mathcal{K}$ is the orthogonal projection of $\hat{\mathcal{H}}$ onto $\mathcal{K}$. 
Let $C \in B(\mathcal{H})$ satisfy $\mathcal{T}_t(C) = C$ for all $t \geq 0$ and $\theta$ be the minimal $e_0$ dilation of $\mathcal{T}$ acting on $B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\mathcal{H} \subset \mathcal{H}$.

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Extend $(\gamma_t)_{t \geq 0}$ to a quantum dynamical semigroup on $B(\mathcal{H})$. 

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Theorem

Let $(T_t)_{t \geq 0}$ be an elementary quantum dynamical semigroup acting on $B(\mathcal{H})$ and let $(\theta_t)_{t \geq 0}$ be its minimal dilation acting on $B(\hat{\mathcal{H}})$. A positive operator $C \in B(\mathcal{H})$ is a fixed point of $T$ for all $t \geq 0$, if and only if $C = P_\mathcal{H}D|\mathcal{H}$, where $D$ is a positive fixed point of $\theta$. 

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Idea of proof

Use a commutant lifting theorem for contractive strongly continuous semigroups:

**Theorem**

Let \((R_t)_{t \geq 0}\) and \((S_t)_{t \geq 0}\) be two strongly continuous contraction semigroups acting on the Hilbert spaces \(\mathcal{H}\) and \(\mathcal{K}\) respectively. Suppose that \((V_t)_{t \geq 0}\) and \((W_t)_{t \geq 0}\) are the respective minimal isometric dilations acting on the Hilbert spaces \(\hat{\mathcal{H}}\) and \(\hat{\mathcal{K}}\). If a bounded operator \(C : \mathcal{H} \to \mathcal{K}\) satisfies \(CR_t = S_t C, \ t \geq 0\) then there exists an operator \(\hat{C} : \hat{\mathcal{H}} \to \hat{\mathcal{K}}\) such that \(\hat{C} V_t = W_t \hat{C}, \ t \geq 0\) and \(\|\hat{C}\| = \|C\|, P_\mathcal{H} \hat{C} |_\mathcal{H} = C\).
Idea of proof

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- and the ”intertwining theorem”
Theorem

Let \((T_t)_{t \geq 0}\) be a quantum dynamical semigroup acting on \(B(\mathcal{H})\) and let \((\theta_t)_{t \geq 0}\) be its minimal dilation acting on \(B(\hat{\mathcal{H}})\). A positive, invertible operator \(C \in B(\mathcal{H})\) is a fixed point of \(T_t\) for all \(t \geq 0\), if and only if \(C = P_\mathcal{H}D|\mathcal{H}\), where \(D\) is a positive, invertible fixed point of \(\theta_t\), \(t \geq 0\), such that \(\|C\| = \|D\|\).
References


