Evolutionary Equations with Frictional Boundary Conditions.

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Elasticity with frictional boundary conditions

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The equations

Consider an elastic body $\Omega \subseteq \mathbb{R}^3$. We denote by $u : \mathbb{R} \to L_2(\Omega)^3$ the displacement field and by $\sigma : \mathbb{R} \to L_{2,sym}(\Omega)^{3 \times 3}$ the stress tensor field. The equations of elasticity read as

$$\partial_0^2 u - \text{Div} \sigma = f,$$
$$\sigma = C \text{Grad} u,$$

(1)
The equations

Consider an elastic body $\Omega \subseteq \mathbb{R}^3$. We denote by $u : \mathbb{R} \to L_2(\Omega)^3$ the displacement field and by $\sigma : \mathbb{R} \to L_{2,\text{sym}}(\Omega)^{3 \times 3}$ the stress tensor field. The equations of elasticity read as

$$\begin{aligned}
\partial_0^2 u - \text{Div}\sigma &= f, \\
\sigma &= C \text{Grad}\,u,
\end{aligned}$$

(1)

where $\text{Div}\sigma = \left(\sum_{j=1}^{3} \partial_j \sigma_{ij}\right)_i$ and $\text{Grad}\,u = \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{i,j}$. The operator $C : L_{2,\text{sym}}(\Omega)^{3 \times 3} \to L_{2,\text{sym}}(\Omega)^{3 \times 3}$ is assumed to be selfadjoint and strictly positive definite.
The equations

Consider an elastic body \( \Omega \subseteq \mathbb{R}^3 \). We denote by \( u : \mathbb{R} \to L_2(\Omega)^3 \) the displacement field and by \( \sigma : \mathbb{R} \to L_{2,\text{sym}}(\Omega)^{3 \times 3} \) the stress tensor field. The equations of elasticity read as

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\partial_0^2 u - \text{Div} \sigma = f, \\
\sigma = C \text{Grad} u,
\]

where \( \text{Div} \sigma = (\sum_{j=1}^3 \partial_j \sigma_{ij})_i \) and \( \text{Grad} u = \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{i,j} \). The operator \( C : L_{2,\text{sym}}(\Omega)^{3 \times 3} \to L_{2,\text{sym}}(\Omega)^{3 \times 3} \) is assumed to be selfadjoint and strictly positive definite. We write this as an first order system. Set \( v := \partial_0 u \). Then (1) can be written as

\[
\begin{pmatrix}
\partial_0 \left( \begin{array}{cc} 1 & 0 \\ 0 & C^{-1} \end{array} \right) + \left( \begin{array}{cc} 0 & -\text{Div} \\ -\text{Grad} & 0 \end{array} \right) \\
\end{pmatrix}
\begin{pmatrix}
v \\
\sigma \end{pmatrix}
= 
\begin{pmatrix}
f \\
0 \end{pmatrix}.
\]
Boundary conditions

We consider the following frictional boundary condition (Coulomb’s law for unilateral contact):

\[ \sigma_n (x) \cdot n = F_n (x) \]
\[ | \sigma_n (x) \cdot t | < g (x) \Rightarrow v_t (x) = 0 \]
\[ | \sigma_n (x) \cdot t | = g (x) \Rightarrow \exists \lambda \geq 0 : v_t (x) = - \lambda \sigma_n (x) \cdot t \]
Boundary conditions

We consider the following frictional boundary condition (Coulomb’s law for unilateral contact):
Denote by $n : \partial \Omega \rightarrow \mathbb{R}^3$ the unit outward normal vector field on $\partial \Omega$. For a vector field $w \in L_2(\partial \Omega)^3$ we define the decomposition

$$w_n := \langle w | n \rangle n \text{ and } w_t = w - w_n.$$
Boundary conditions

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Denote by \( n : \partial \Omega \to \mathbb{R}^3 \) the unit outward normal vector field on \( \partial \Omega \). For a vector field \( w \in L_2(\partial \Omega)^3 \) we define the decomposition

\[
\begin{align*}
    w_n &:= \langle w | n \rangle n \quad \text{and} \quad w_t = w - w_n.
\end{align*}
\]

Let \( g \in L_2(\partial \Omega) \) attaining positive (real) values and \( F_n \in L_2(\partial \Omega)^3 \). Then the boundary condition reads as

\[
\begin{align*}
    (\sigma n)_n(x) &= F_n(x) \\
    |(\sigma n)_t(x)| &< g(x) \Rightarrow v_t(x) = 0 \\
    |(\sigma n)_t(x)| &= g(x) \Rightarrow \exists \lambda \geq 0 : v_t(x) = -\lambda (\sigma n)_t(x).
\end{align*}
\]
The relation $h$

$$\begin{align*}
\sigma_n(x) &= F_n(x) \\
|\sigma_t(x)| &< g(x) \Rightarrow \nu_t(x) = 0 \\
|\sigma_t(x)| &= g(x) \Rightarrow \exists \lambda \geq 0 : \nu_t(x) = -\lambda \sigma_t(x). \quad (2)
\end{align*}$$

Define the following relation on $L_2(\partial \Omega)^3$:

$$h := \{(x, y) \mid y_n = F_n, |y_t| \leq g, \Re \langle x_t | y_t \rangle = g |x_t| \text{ a.e.}\}.$$
The relation $h$

\[(\sigma n)_n(x) = F_n(x)\]
\[|(\sigma n)_t(x)| < g(x) \Rightarrow v_t(x) = 0\]
\[|(\sigma n)_t| = g(x) \Rightarrow \exists \lambda \geq 0 : v_t(x) = -\lambda(\sigma n)_t(x). \quad (2)\]

Define the following relation on $L_2(\partial \Omega)^3$:

\[h := \{(x, y) \mid y_n = F_n, \ |y_t| \leq g, \Re \langle x_t | y_t \rangle = g|x_t| \text{ a.e.}\}.\]

Then the boundary conditions (2) are equivalent to $(v, \sigma n) \in h$. 
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Then the boundary conditions (2) are equivalent to $(v, \sigma n) \in h$.

**Lemma**

*The relation $h$ is monotone, i.e. for every $(u, v), (x, y) \in h$ the estimate*

\[\Re \langle u - x | v - y \rangle \geq 0\]

*holds.*
The relation $h$

$h := \{(x, y) | y_n = F_n, |y_t| \leq g, \Re \langle x_t | y_t \rangle = g|x_t| \text{ a.e.}\}$.

Proof.

For $(u, v), (x, y) \in h$ we estimate

$$\Re \langle u - x | v - y \rangle = \Re \langle u_t - x_t | v_t - y_t \rangle$$
The relation $h$

$$h := \{ (x, y) \mid y_n = F_n, \ |y_t| \leq g, \Re \langle x_t | y_t \rangle = g |x_t| \text{ a.e.} \}.$$ 

Proof.

For $(u, \nu), (x, y) \in h$ we estimate

$$\Re \langle u - x | \nu - y \rangle = \Re \langle u_t - x_t | \nu_t - y_t \rangle$$

$$= g(|u_t| + |x_t|) - (\Re \langle x_t | \nu_t \rangle + \Re \langle u_t | y_t \rangle)$$
The relation $h$

$$h := \{(x, y) \mid y_n = F_n, |y_t| \leq g, \Re \langle x_t | y_t \rangle = g |x_t| \text{ a.e.}\}.$$ 

Proof.

For $(u, v), (x, y) \in h$ we estimate

$$\Re \langle u - x | v - y \rangle = \Re \langle u_t - x_t | v_t - y_t \rangle = g(|u_t| + |x_t|) - (\Re \langle x_t | v_t \rangle + \Re \langle u_t | y_t \rangle) \geq g(|u_t| + |x_t|) - g(|u_t| + |x_t|)$$
The relation $h$

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**Proof.**
For $(u, v), (x, y) \in h$ we estimate

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\Re\langle u - x | v - y \rangle = \Re\langle u_t - x_t | v_t - y_t \rangle \\
= g(|u_t| + |x_t|) - (\Re\langle x_t | v_t \rangle + \Re\langle u_t | y_t \rangle) \\
\geq g(|u_t| + |x_t|) - g(|u_t| + |x_t|) \\
= 0,
\]

by Cauchy-Schwarz.
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$$= g(\|u_t\| + \|x_t\|) - (\Re \langle x_t | v_t \rangle + \Re \langle u_t | y_t \rangle)$$

$$\geq g(\|u_t\| + \|x_t\|) - g(\|u_t\| + \|x_t\|)$$

$$= 0,$$

by Cauchy-Schwarz.

Lemma

The relation $1 + h$ is onto, i.e. for every $f \in L_2(\partial \Omega)^3$ there exists $(x, y) \in h$ such that $x + y = f$. 

The relation $h$

$$h := \{(x, y) \mid y_n = F_n, \left| y_t \right| \leq g, \Re \langle x_t | y_t \rangle = g |x_t| \text{ a.e.}\}.$$

Proof.

(a) $|f_t| \leq g$:

(b) $|f_t| > g$:
The relation $h$

$$h := \{(x, y) \mid y_n = F_n, \ |y_t| \leq g, \Re \langle x_t | y_t \rangle = g |x_t| \text{ a.e.} \}.$$

**Proof.**

(a) $|f_t| \leq g$: Then we set $x_n = f_n - F_n$, $y_n = F_n$ and $y_t = f_t$, $x_t = 0$.

(b) $|f_t| > g$: 

Corollary: The relation $h$ is maximal monotone.
The relation $h$

$$h := \{(x, y) \mid y_n = F_n, |y_t| \leq g, \Re \langle x_t | y_t \rangle = g|x_t| \text{ a.e.}\}.$$ 

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(b) $|f_t| > g$: Then we set $x_n = f_n - F_n$, $y_n = F_n$ and $y_t = g \frac{f_t}{|f_t|}$, $x_t = f_t - y_t = (1 - g \frac{1}{|f_t|})f_t$. 

Corollary

The relation $h$ is maximal monotone.
The relation $h$

$$h := \{(x, y) | y_n = F_n, |y_t| \leq g, \Re\langle x_t | y_t \rangle = g|x_t| \text{ a.e.}\}.$$ 

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$$\Re\langle x_t | y_t \rangle = \frac{g}{|f_t|} \left(1 - g\frac{1}{|f_t|}\right) |f_t|^2 = g|x_t|. $$
The relation $h$

$h := \{(x, y) \mid y_n = F_n, \, |y_t| \leq g, \, \Re\langle x_t | y_t \rangle = g |x_t| \text{ a.e.}\}.$

Proof.

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$$\Re\langle x_t | y_t \rangle = \frac{g}{|f_t|} \left( 1 - g \frac{1}{|f_t|} \right) |f_t|^2 = g |x_t|.$$

Corollary

The relation $h$ is \textit{maximal monotone}.
Problem setting

Recall the equation of elasticity:

\[
\left( \partial_0 \begin{pmatrix} 1 & 0 \\ 0 & C^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\text{Div} \\ -\text{Grad} & 0 \end{pmatrix} \right) \begin{pmatrix} \nu \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.
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\]

In order to deal with the frictional boundary conditions, we have to consider the (nonlinear) restriction

\[ A \subseteq \begin{pmatrix} 0 & -\text{Div} \\ -\text{Grad} & 0 \end{pmatrix} \]

with domain

\[ \mathcal{D}(A) := \{ (v, \sigma) \mid v \in \mathcal{D}(\text{Grad}), \sigma \in \mathcal{D}(\text{Div}), (v, \sigma n) \in h \}. \]
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Evolutionary Equations with Frictional Boundary Conditions.
Time derivative

Let $H$ be a Hilbert space. For $\rho > 0$ we denote by $H_{\rho,0}(\mathbb{R}; H)$ the $L^2$-space defined as the completion of $C^\infty_c(\mathbb{R}; H)$ with respect to the inner product

$$\langle \phi | \psi \rangle_{H_{\rho,0}(\mathbb{R}; H)} := \int_{\mathbb{R}} \langle \phi(t) | \psi(t) \rangle_H \exp(-2\rho t) \, dt.$$
Time derivative

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$$\langle \phi | \psi \rangle_{H_{\rho,0}(\mathbb{R}; H)} := \int_{\mathbb{R}} \langle \phi(t) | \psi(t) \rangle_H \exp(-2\rho t) \, dt.$$ 

We denote by $\partial_{0,\rho}$ the derivative on $H_{\rho,0}(\mathbb{R}; H)$, defined as the closure of

$$C_c^\infty(\mathbb{R}; H) \subseteq H_{\rho,0}(\mathbb{R}; H) \rightarrow H_{\rho,0}(\mathbb{R}; H) \quad \phi \mapsto \phi'.$$
**Time derivative**

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$$C_c^\infty(\mathbb{R}; H) \subseteq H_{\rho,0}(\mathbb{R}; H) \rightarrow H_{\rho,0}(\mathbb{R}; H) \quad \phi \mapsto \phi'.$$

Then $\partial_{0,\rho}$ is a normal operator with $\Re \partial_{0,\rho} = \rho$. In particular, $\partial_{0,\rho}$ is boundedly invertible.
Maximal monotone relations

Let $A \subseteq H \oplus H$ be a binary relation. Then $A$ is called monotone, if

$$\forall (u, v), (x, y) \in A : \Re \langle u - x | v - y \rangle_H \geq 0.$$
Maximal monotone relations

Let \( A \subseteq H \oplus H \) be a binary relation. Then \( A \) is called \textit{monotone}, if

\[
\forall (u, v), (x, y) \in A : \Re \langle u - x | v - y \rangle_H \geq 0.
\]

\( A \) is called \textit{maximal monotone}, if it is monotone and there exists no proper monotone extension, i.e. for every monotone \( B \subseteq H \oplus H \) with \( A \subseteq B \) it follows that \( A = B \).
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Theorem (Minty, 1962)

Let $A \subseteq H \oplus H$ be monotone. TFAE

(i) $A$ is maximal monotone,
Maximal monotone relations

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$A$ is called maximal monotone, if it is monotone and there exists no proper monotone extension, i.e. for every monotone $B \subseteq H \oplus H$ with $A \subseteq B$ it follows that $A = B$.

**Theorem (Minty, 1962)**

Let $A \subseteq H \oplus H$ be monotone. TFAE

(i) $A$ is maximal monotone,

(ii) there is $\lambda > 0$ such that $1 + \lambda A$ is onto (i.e. $\forall z \in H \exists (x, y) \in A : x + \lambda y = z$).
Maximal monotone relations

Let $A \subseteq H \oplus H$ be a binary relation. Then $A$ is called \textit{monotone}, if

$$\forall (u, v), (x, y) \in A : \mathbb{R} \langle u - x | v - y \rangle_H \geq 0.$$  

$A$ is called \textit{maximal monotone}, if it is monotone and there exists no proper monotone extension, i.e. for every monotone $B \subseteq H \oplus H$ with $A \subseteq B$ it follows that $A = B$.

\textbf{Theorem (Minty, 1962)}

\textit{Let $A \subseteq H \oplus H$ be monotone. TFAE}

\begin{enumerate}
  \item $A$ is maximal monotone,
  \item there is $\lambda > 0$ such that $1 + \lambda A$ is onto (i.e. $\forall z \in H \exists (x, y) \in A : x + \lambda y = z$),
  \item for all $\lambda > 0$ the relation $1 + \lambda A$ is onto.
\end{enumerate}
Evolutionary inclusions

Throughout let $M_0, M_1 \in L(H)$, $M_0$ selfadjoint and strictly positive definite on its range and $\mathcal{R}M_1$ strictly positive definite on the kernel of $M_0$. 

Evolutionary inclusions

Throughout let $M_0, M_1 \in L(H)$, $M_0$ selfadjoint and strictly positive definite on its range and $\Re M_1$ strictly positive definite on the kernel of $M_0$.

Lemma

*There exists* $\rho_0, c > 0$ *such that for every* $\rho \geq \rho_0$ *the operator* $\partial_{\rho} M_0 + M_1 - c$ *is maximal monotone on* $H_{\rho,0}(\mathbb{R}; H)$. 
Evolutionary inclusions

Throughout let $M_0, M_1 \in L(H)$, $M_0$ selfadjoint and strictly positive definite on its range and $\Re M_1$ strictly positive definite on the kernel of $M_0$.

**Lemma**

There exists $\rho_0, c > 0$ such that for every $\rho \geq \rho_0$ the operator $\partial_{0,\rho} M_0 + M_1 - c$ is maximal monotone on $H_{\rho,0}(\mathbb{R}; H)$.

**Theorem (Solution Theory)**

Let $\rho \geq \rho_0$, $A \subseteq H_{\rho,0}(\mathbb{R}; H) \oplus H_{\rho,0}(\mathbb{R}; H)$ be maximal monotone and autonomous (i.e. $(u, v) \in A \Rightarrow \forall h \in \mathbb{R} : (\tau_h u, \tau_h v) \in A$). Then

$\left(\partial_{0,\rho} M_0 + M_1 + A\right)^{-1} : H_{\rho,0}(\mathbb{R}; H) \to H_{\rho,0}(\mathbb{R}; H)$

is a Lipschitz-continuous, causal mapping.
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Boundary data spaces

Let $H_0, H_1$ be Hilbert spaces and $\hat{G} : \mathcal{D}(\hat{G}) \subseteq H_0 \to H_1$ and $\hat{D} : \mathcal{D}(\hat{D}) \subseteq H_1 \to H_0$ be densely defined, closed linear operators with

$$\hat{G} \subseteq - (\hat{D})^* =: G \text{ and } \hat{D} \subseteq - (\hat{G})^* =: D.$$
Boundary data spaces

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\[
\mathcal{G} \subseteq - (\mathcal{D})^* =: G \quad \text{and} \quad \mathcal{D} \subseteq - (\mathcal{G})^* =: D.
\]

Example

Set $H_0 = L_2(\Omega)$ and $H_1 = L_2(\Omega)^n$, $\hat{\mathcal{G}} = \text{grad}$ with domain $\overset{\mathcal{D}\text{grad}}{\overline{C_c^\infty(\Omega)}}$ and $\hat{\mathcal{D}} = \text{div}$ with domain $\overset{\mathcal{D}\text{div}}{\overline{C_c^\infty(\Omega)^n}}$. 
Boundary data spaces

Let $H_0, H_1$ be Hilbert spaces and $\hat{G} : \mathcal{D}(\hat{G}) \subseteq H_0 \rightarrow H_1$ and $\hat{D} : \mathcal{D}(\hat{D}) \subseteq H_1 \rightarrow H_0$ be densely defined, closed linear operators with $\hat{G} \subseteq - (\hat{D})^* =: G$ and $\hat{D} \subseteq - (\hat{G})^* =: D$.

Example

Set $H_0 = L_2(\Omega)$ and $H_1 = L_2(\Omega)^n$, $\hat{G} = \text{grad}$ with domain $\overline{C_c^\infty(\Omega)}^{\mathcal{D}_{\text{grad}}}$ and $\hat{D} = \text{div}$ with domain $\overline{C_c^\infty(\Omega)}^{\mathcal{D}_{\text{div}}}$. Then $G = \text{grad}$ with domain $\{ f \in L_2(\Omega) \mid \text{grad} f \in L_2(\Omega)^n \}$ and $D = \text{div}$ with domain $\{ \Phi \in L_2(\Omega)^n \mid \text{div} \Phi \in L_2(\Omega) \}$.
Boundary data spaces

We decompose $\mathcal{D}_G$ and $\mathcal{D}_D$ in the orthogonal subspaces

\[
\mathcal{D}_G = \mathcal{D}_\mathcal{G} \oplus \mathcal{BD}(G) \\
\mathcal{D}_D = \mathcal{D}_\mathcal{D} \oplus \mathcal{BD}(D).
\]
Boundary data spaces

We decompose $\mathcal{D}_G$ and $\mathcal{D}_D$ in the orthogonal subspaces

$$\mathcal{D}_G = \mathcal{D}_\ddot{G} \oplus \mathcal{B\mathcal{D}}(G)$$
$$\mathcal{D}_D = \mathcal{D}_\ddot{D} \oplus \mathcal{B\mathcal{D}}(D).$$

We denote the restrictions of $G$ and $D$ to $\mathcal{B\mathcal{D}}(G)$ and $\mathcal{B\mathcal{D}}(D)$ by $\dot{G}$ and $\dot{D}$, respectively.
Boundary data spaces

We decompose $\mathcal{D}_G$ and $\mathcal{D}_D$ in the orthogonal subspaces

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We denote the restrictions of $G$ and $D$ to $\mathcal{B}\mathcal{D}(G)$ and $\mathcal{B}\mathcal{D}(D)$ by $\dot{G}$ and $\dot{D}$, respectively.

**Lemma**

The operators $\dot{G}: \mathcal{B}\mathcal{D}(G) \to \mathcal{B}\mathcal{D}(D)$ and $\dot{D}: \mathcal{B}\mathcal{D}(D) \to \mathcal{B}\mathcal{D}(G)$ are unitary with $(\dot{G})^* = \dot{D}$ and $(\dot{D})^* = \dot{G}$. 
The operator $A$

Let $h \subseteq H_{\rho,0}(\mathbb{R}; BD(G)) \oplus H_{\rho,0}(\mathbb{R}; BD(G))$ be an autonomous, maximal monotone relation.
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$$A \subseteq \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix}$$
The operator $A$

Let $h \subseteq H_{\rho,0}(\mathbb{R}; \mathcal{BD}(G)) \oplus H_{\rho,0}(\mathbb{R}; \mathcal{BD}(G))$ be an autonomous, maximal monotone relation. We consider the operator

$$A \subseteq \begin{pmatrix} 0 & D \\ G & 0 \end{pmatrix}$$

with domain

$$\{(u, v) \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_G) \oplus H_{\rho,0}(\mathbb{R}; \mathcal{D}_D) \mid (\pi_{\mathcal{BD}(G)}u, \dot{D} \pi_{\mathcal{BD}(D)}v) \in h \}.$$
The operator $A$

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$$\{(u, v) \in H_{\rho,0}(\mathbb{R}; D_G) \oplus H_{\rho,0}(\mathbb{R}; D_D) | (\pi_{BD(G)}u, \dot{D} \pi_{BD(D)}v) \in h}\}.$$

**Theorem**

*The (nonlinear) operator $A$ is autonomous and maximal monotone.*
Sketch of Proof

A straight forward computation yields that $A$ is monotone and autonomous.
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A straightforward computation yields that $A$ is monotone and autonomous. Moreover, $A$ is closed.

In order to show the maximal monotonicity, it suffices to prove that $1 + A$ has a dense range. For doing so, let $f \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_G)$ and $g \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_D)$ and define

$$\tilde{u} := (1 - D\hat{G})^{-1}f - D(1 - \hat{G}D)^{-1}g \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_{D\hat{G}})$$

$$\tilde{v} := (1 - \hat{G}D)^{-1}g - \hat{G}(1 - D\hat{G})^{-1}f \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_D).$$
Sketch of Proof

A straightforward computation yields that $A$ is monotone and autonomous. Moreover, $A$ is closed. In order to show the maximal monotonicity, it suffices to prove that $1 + A$ has a dense range. For doing so, let $f \in H_{\rho,0}(\mathbb{R}; D_{\hat{G}})$ and $g \in H_{\rho,0}(\mathbb{R}; D_{\hat{D}})$ and define

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Then

$$\tilde{u} + D\tilde{v} = f$$

$$\tilde{v} + \hat{G}\tilde{u} = g.$$
Moreover, we set

\[ u := \nu_{BD(G)} (1 + h)^{-1} \left( - \dot{D} \pi_{BD(D)} \dot{G} \tilde{u} \right) + \tilde{u} \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_G) \]

\[ \nu := -\nu_{BD(D)} \dot{G} (1 + h)^{-1} \left( - \dot{D} \pi_{BD(D)} \dot{G} \tilde{u} \right) + \tilde{\nu} \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_D) \]
Moreover, we set

\[ u := \nu_{BD}(G)(1 + h)^{-1} \left(- \dot{\nu} \pi_{BD}(D) \tilde{G} \tilde{u}\right) + \tilde{u} \in H_{\rho,0}(\mathbb{R}; D_G) \]

\[ v := -\nu_{BD}(D) \dot{G} (1 + h)^{-1} \left(- \dot{\nu} \pi_{BD}(D) \tilde{G} \tilde{u}\right) + \tilde{v} \in H_{\rho,0}(\mathbb{R}; D_D) \]

Then

\[ u + Dv = \tilde{u} + D\tilde{v} = f \]

\[ u + Gv = \tilde{v} + G\tilde{u} = g. \]
Moreover, we set

\[ u := \iota_{\mathcal{BD}(D)}(1 + h)^{-1} \left(- \dot{\mathcal{D}} \pi_{\mathcal{BD}(D)} \hat{G} \tilde{u}\right) + \tilde{u} \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_\mathcal{G}) \]

\[ v := -\iota_{\mathcal{BD}(D)} \hat{G} (1 + h)^{-1} \left(- \dot{\mathcal{D}} \pi_{\mathcal{BD}(D)} \hat{G} \tilde{u}\right) + \tilde{v} \in H_{\rho,0}(\mathbb{R}; \mathcal{D}_\mathcal{D}) \]

Then

\[ u + Dv = \tilde{u} + D\tilde{v} = f \]

\[ u + Gv = \tilde{v} + G\tilde{u} = g. \]

Moreover, one can show that

\[ \pi_{\mathcal{BD}(G)}u = (1 + h)^{-1} \left( \dot{\mathcal{D}} \pi_{\mathcal{BD}(D)} v + \pi_{\mathcal{BD}(G)}u \right), \]

which yields \((\pi_{\mathcal{BD}(G)}, \dot{\mathcal{D}} \pi_{\mathcal{BD}(D)} v) \in h\).
Elasticity with frictional boundary conditions

Framework

Boundary conditions

Back to elasticity
Recall the example

\[
\begin{pmatrix}
\partial_0 
& \begin{pmatrix}
1 & 0 \\
0 & C^{-1} \\
\end{pmatrix} \\
\end{pmatrix} + A \begin{pmatrix}
\nu \\
\sigma \\
\end{pmatrix} = \begin{pmatrix}
f \\
0 \\
\end{pmatrix},
\]

where

\[
A \subseteq \begin{pmatrix}
0 & -\text{Div} \\
-\text{Grad} & 0 \\
\end{pmatrix}
\]

\[
\mathcal{D}(A) := \{(\nu, \sigma) | \nu \in \mathcal{D}(\text{Grad}), \sigma \in \mathcal{D}(\text{Div}), (\nu, \sigma n) \in h\},
\]

and \( h \) is maximal monotone on \( L^2(\partial \Omega)^3 \).
Recall the example

\[
\left( \partial_0 \begin{pmatrix} 1 & 0 \\ 0 & C^{-1} \end{pmatrix} + A \right) \begin{pmatrix} \nu \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},
\]

where

\[
A \subseteq \begin{pmatrix} 0 & -\text{Div} \\ -\text{Grad} & 0 \end{pmatrix}
\]

\[
\mathcal{D}(A) := \{(\nu, \sigma) \mid \nu \in \mathcal{D}(\text{Grad}), \sigma \in \mathcal{D}(\text{Div}), (\nu, \sigma n) \in h\},
\]

and \(h\) is maximal monotone on \(L_2(\partial \Omega)^3\). Using the terminology above, we set \(G := -\text{Grad}\) and \(D := -\text{Div}\). In case of a smooth boundary, there exists a continuous embedding

\[
i : \mathcal{BD}(\text{Grad}) \to L_2(\partial \Omega)^3.
\]
Then the boundary condition can be reformulated as

\[
\dot{(\pi_{BD} \text{Grad}) \nu, \text{Div} \pi_{BD} \text{Div} \sigma)} \in \iota^* h =: \tilde{h}.
\]
Then the boundary condition can be reformulated as
\[
\left( \pi_{BD}(\text{Grad}) \nu, \text{Div} \pi_{BD}(\text{Div}) \sigma \right) \in \iota^* h \iota =: \tilde{h}.
\]

**Proposition**

*If* $h$ *is maximal monotone and bounded (i.e. the* $h[B]$ *is bounded for bounded* $B$ *), then* $\tilde{h}$ *is maximal monotone.*
Then the boundary condition can be reformulated as

\[
(\pi_{\mathcal{BD}}(\text{Grad}) \nu, \text{Div} \pi_{\mathcal{BD}}(\text{Div}) \sigma) \in \iota^* h \ell =: \tilde{h}.
\]

**Proposition**

*If* \( h \) *is maximal monotone and bounded (i.e. the* \( h[B] \) *is bounded for bounded* \( B \), *then* \( \tilde{h} \) *is maximal monotone.*

**Corollary**

*For sufficiently large* \( \rho > 0 \) *the inverse relation*

\[
\left( \partial_0 \begin{pmatrix} 1 & 0 \\ 0 & C^{-1} \end{pmatrix} + A \right)^{-1}
\]

*is a Lipschitz-continuous, causal mapping on* \( H_{\rho,0}(\mathbb{R}; L_2(\Omega)^3 \oplus L_{2,\text{sym}}(\Omega)^{3 \times 3}) \).*
Thank you for your attention!

Autonomous Evolutionary Inclusions with Applications to Problems with Nonlinear Boundary Conditions.