Invariant foliations near normally hyperbolic equilibria for quasilinear parabolic problems

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Let $X_0$, $X_1$ be Banach spaces such that $X_1 \overset{d}{\to} X_0$. Consider the abstract quasilinear problem

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)), \quad t > 0, \quad u(0) = u_0. \quad (1)$$

For $p \in (1, \infty)$ let $\emptyset \neq V \subset X_\gamma := (X_0, X_1)_{1-1/p, p}$ be open such that

$$(A, F) \in C^1(V; \mathcal{L}(X_1, X_0) \times X_0).$$

$A \in \mathcal{L}(X_1, X_0)$ has maximal regularity of type $L_p$ ($A \in \mathcal{MR}_p(X_0)$) if for each $f \in L_p(\mathbb{R}^+; X_0)$ there is a unique

$$v \in W_p^1(\mathbb{R}^+; X_0) \cap L_p(\mathbb{R}^+; X_1)$$

solving the linear problem

$$\dot{v}(t) + Av(t) = f(t), \quad t > 0, \quad v(0) = 0.$$
If $A(u) \in \mathcal{MR}_p(X_0)$ for each $u \in V$, then (1) generates a local semiflow in $V \subset X_\gamma$.

Set of equilibria:

$$\mathcal{E} := \{u \in V \cap X_1 : A(u)u = F(u)\}.$$

Given $u_\ast \in \mathcal{E}$, we assume that there is an open $U \subset \mathbb{R}^m$, $0 \in U$, $\Psi \in C^1(U; X_1)$ such that

- $\Psi(U) \subset \mathcal{E}$ & $\Psi(0) = u_\ast$,
- $\text{rank } \Psi'(0) = m$,
- $A(\Psi(\zeta))\Psi(\zeta) = F(\Psi(\zeta))$, $\zeta \in U$.

Let $A_0 := A(u_\ast) + [A'(u_\ast) \cdot u_\ast - F'(u_\ast)]$ denote the full linearization of (1) at $u_\ast \in \mathcal{E}$. 

Such an equilibrium $u_*$ is **normally hyperbolic** if

1. the tangent space $T_{u_*} \mathcal{E}$ for $\mathcal{E}$ at $u_*$ is given by $N(A_0)$,
2. $0 \in \sigma(A_0)$ is semi-simple, i.e. $X_0 = N(A_0) \oplus R(A_0)$,
3. $\sigma(A_0) \cap i\mathbb{R} = \{0\}$, $\sigma(-A_0) \cap \mathbb{C}_+ \neq \emptyset$.

If $\sigma(-A_0) \cap \mathbb{C}_+ = \emptyset$, then $u_*$ is **normally stable**.

In particular $\{0\}$ is an isolated point in $\sigma(A_0)$.

Any $w \in \mathcal{E}$ close to a normally hyperbolic (stable) equilibrium in $X_\gamma$ is normally hyperbolic (stable) as well.

**Intuition:** At each $w \in \mathcal{E}$ close to $u_*$ in $X_\gamma$ there exists a stable manifold $\mathcal{M}^s_w$ and an unstable manifold $\mathcal{M}^u_w$ such that

$$\mathcal{M}^s_w \cap \mathcal{M}^u_w \cap B_r(u_*) = \{w\}$$

for some $r > 0$. 
We refer to it as the **stable** resp. **unstable foliation** of (1) near \( u_* \in \mathcal{E} \).

Let \( P^j \) be the projection to the spectral sets \( \sigma_j(A_0) \) and \( X^j_i := P^j X_i \) for \( j \in \{c, s, u\} \) and \( i \in \{\gamma, 0, 1\} \).
Existence of the stable foliation

Theorem (Prüss, Simonett, W. 13)

Let $u_* \in \mathcal{E}$ be normally hyperbolic, $A(u_*) \in \mathcal{MR}_p(X_0)$ and

$$(A, F) \in C^1(V; \mathcal{L}(X_1, X_0) \times X_0).$$

Then $\exists \ r > 0$ and

$$\lambda^s \in C(B_{X^s_\gamma}(0, r) \times B_{X^c}(0, r); X_\gamma), \quad \text{with} \quad \lambda^s(0, 0) = u_*,$$

the stable foliation, such that the solution $u(t)$ of (1) with $u(0) = \lambda^s(y_0, \xi)$ converges to $u_* + \xi + \phi(\xi)$ in $X_\gamma$ exponentially as $t \to \infty$.

For fixed $\xi$ the function $\lambda^s_\xi(y_0) := \lambda^s(y_0, \xi)$ defines the fibers $\mathcal{M}^s_\xi := \lambda^s_\xi(B_{X^s_\gamma}(0, r))$ near $u_*$, which are locally invariant.

If $u_*$ is normally stable, then $\lambda^s(B_{X^s_\gamma}(0, r) \times B_{X^c}(0, r))$ forms a neighborhood of $u_*$ in $X_\gamma$. 
For $v := u - u_*$, (1) becomes

$$\dot{v}(t) + A_0 v(t) = G(v(t)), \quad t > 0, \quad v(0) = u_0 - u_*,$$

(2)

where $G \in C^1$ and $G(0) = 0$ as well as $G'(0) = 0$.

Assumptions $\Rightarrow \exists \phi_j \in C^1(B_{X^c}(0, \rho_0), X^j_1), j \in \{s, u\}$ s.t.

$$\phi_j(0) = \phi'_j(0) = 0, \quad \{\xi + \phi(\xi) + u_* : \xi \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap \mathcal{W},$$

for some neighborhood $\mathcal{W}$ of $u_*$ in $X_1$, where $\phi := \phi_s + \phi_u$.

Equations of equilibria:

$$P^c G(\xi + \phi(\xi)) = 0,$$

$$P^j G(\xi + \phi(\xi)) = A_j \phi_j(\xi), \quad \xi \in B_{X^c}(0, \rho_0), \quad j \in \{s, u\},$$

with $A_j$ being the part of $A_0$ in $X^j_0$ and $A_c = 0$. 
Define

\[ x = P^c v - \xi = P^c(u - u_*) - \xi, \]
\[ y = P^s v - \phi_s(\xi) = P^s(u - u_*) - \phi_s(\xi), \]
\[ z = P^u v - \phi_u(\xi) = P^u(u - u_*) - \phi_u(\xi). \]

Then

\[
\begin{align*}
\dot{x} &= R_c(x, y, z, \xi), & x(0) &= x_0 - \xi, \\
\dot{y} + A_s y &= R_s(x, y, z, \xi), & y(0) &= y_0 - \phi_s(\xi), \\
\dot{z} + A_u z &= R_u(x, y, z, \xi), & z(0) &= z_0 - \phi_u(\xi),
\end{align*}
\]

with

\[ R_j(x, y, z, \xi) = P^j[G(x + y + z + \xi + \phi(\xi)) - G(\xi + \phi(\xi))], \quad j \in \{c, s, u\}. \]

(3) is the asymptotic normal form of (1) near \( u_* \).
Construction of the stable foliation III

For

$$(x, y, z) \in e^{-\delta t}[H_p^1(\mathbb{R}_+; X^c \times X_0^s \times X^u) \cap L_p(\mathbb{R}_+; X^c \times X_1^s \times X^u)]$$

and $(y_0, \xi) \in X^s_\gamma \times X^c$ close to 0, define

$$H_s((x, y, z), (y_0, \xi))(t) = \begin{bmatrix} x(t) + \int_t^\infty R_c(x(\tau), y(\tau), z(\tau), \xi) d\tau \\ y(t) - L_s(R_s(x, y, z, \xi), y_0 - \phi_s(\xi)) \\ z(t) + \int_t^\infty e^{-A_u(t-\tau)} R_u(x(\tau), y(\tau), z(\tau), \xi) d\tau \end{bmatrix},$$

where $t > 0$ and $L_s(f, y_0)$ denotes the unique solution of

$$\dot{w}(t) + A_s w(t) = f(t), \quad t > 0, \quad w(0) = y_0.$$

Then $H_s$ is $C^1$ w.r.t $(x, y, z, y_0)$ and continuous in $\xi$ with $H_s(0) = 0$ and $D_{(x,y,z)}H_s(0) = I$.

Implicit function thm \Rightarrow \exists \Lambda^s : (y_0, \xi) \mapsto (x, y, z)$ defined near 0 s.t.

$H_s(\Lambda^s(y_0, \xi), (y_0, \xi)) = 0$. $\Lambda^s$ is $C^1$ w.r.t. $y_0$ and continuous in $\xi$. 

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Construction of the stable foliation IV

\((x(t), y(t), z(t))\) solves (3) with \(x_0 := x(0) + \xi\) and \(z_0 := z(0) + \phi_u(\xi)\), hence

\[ u(t) := u_* + x(t) + y(t) + z(t) + \xi + \phi(\xi) \]

solves (1). Since \((x(t), y(t), z(t)) \to (0, 0, 0)\) as \(t \to \infty\), then \(u(t) \to u_* + \xi + \phi(\xi) \in \mathcal{E}\) as \(t \to \infty\).

Therefore the map

\[ \lambda^s : (y_0, \xi) \mapsto u(0) = u_* + x(0) + y(0) + z(0) + \xi + \phi(\xi) \]

yields a foliation of (1) near \(u_*\) and

\[ \mathcal{M}_\xi^s := \{ \lambda^s(y_0, \xi) : y_0 \in B_{x^s}(0, r) \} \]

are the fibers over \(B_{x^c}(0, r)\) for some small \(r > 0\).
Existence of the unstable foliation

Theorem (Prüss, Simonett, W. 13)

Let \( u_\ast \in \mathcal{E} \) be normally hyperbolic, \( A(u_\ast) \in \mathcal{MR}_p(X_0) \) and

\[
(A, F) \in C^1(V; \mathcal{L}(X_1, X_0) \times X_0).
\]

Then \( \exists \ r > 0 \) and

\[
\lambda^u \in C(B_{X_\gamma^u}(0, r) \times B_{X_c}(0, r); X_\gamma), \quad \text{with} \quad \lambda^u(0, 0) = u_\ast,
\]

the unstable foliation, such that the solution \( u(t) \) of (1) with \( u(0) = \lambda^u(z_0, \xi) \) converges to \( u_\ast + \xi + \phi(\xi) \) in \( X_\gamma \) exponentially as \( t \to -\infty \).

For fixed \( \xi \) the function \( \lambda^u_\xi(z_0) := \lambda^u(z_0, \xi) \) defines the fibers \( M^u_\xi := \lambda^u_\xi(B_{X_\gamma^u}(0, r)) \) near \( u_\ast \), which are locally invariant.
Copy the technique from the stable foliation: Replace $H_s$ by

$$H_u((x, y, z), (z_0, \xi))(t) = \begin{bmatrix}
    x(t) - \int_{-\infty}^{t} R_c(x(\tau), y(\tau), z(\tau), \xi) d\tau \\
y(t) - \int_{-\infty}^{t} e^{-A_s(t-\tau)} R_s(x(\tau), y(\tau), z(\tau), \xi) d\tau \\
z(t) - L_u(R_u(x, y, z, \xi), z_0 - \phi_u(\xi))
\end{bmatrix},$$

where $t < 0$ and $L_u(f, z_0)$ denotes the unique solution of

$$\dot{w}(t) + A_u w(t) = f(t), \quad t < 0, \quad w(0) = z_0.$$

Then $H_u$ is $C^1$ w.r.t $(x, y, z, z_0)$ and continuous in $\xi$ with $H_u(0) = 0$ and $D_{(x,y,z)}H_u(0) = I$.

Implicit function thm $\Rightarrow \exists \Lambda^u : (z_0, \xi) \mapsto (x, y, z)$ defined near 0 s.t. $H_u(\Lambda^s(z_0, \xi), (z_0, \xi)) = 0$. $\Lambda^u$ is $C^1$ w.r.t. $z_0$ and continuous in $\xi$. 
Example: Mullins-Sekerka problem

$\Omega \subset \mathbb{R}^n$ open, bounded, smooth. Find $\{\Gamma(t)\}_{t \geq 0} \subset \Omega$ satisfying

$$V_\Gamma = [\partial_{\nu_\Gamma} u], \quad \text{on } \Gamma(t), \quad \Gamma(0) = \Gamma_0,$$

where $u$ solves

$$
\begin{cases}
\Delta u = 0 & \text{in } \Omega \setminus \Gamma(t), \\
\partial_\nu u = 0 & \text{on } \partial \Omega, \\
u_\Gamma \text{-normal on } \Gamma(t) \text{ w.r.t. } \Omega_1(t), \ H_\Gamma \text{-mean curvature of } \Gamma(t), \ V_\Gamma \text{-normal velocity of } \Gamma(t), \\
\end{cases}
$$

$$\left[\partial_{\nu_\Gamma} u\right] := \partial_{\nu_\Gamma} u_2 - \partial_{\nu_\Gamma} u_1, \ u_j := u|_{\Omega_j},$$

$$\frac{d}{dt} \int_{\Omega_1(t)} dx = \int_{\Gamma(t)} V_\Gamma \ d\Gamma_t = 0.$$

Equilibria:

$$\mathcal{E} = \left\{ \bigcup_{1 \leq k \leq m} S_{R_k}(x_k) \subset \Omega : S_{R_k}(x_k) \cap S_{R_l}(x_l) = \emptyset, \ k \neq l \right\}.$$
Let \( \bar{B}_{R^*}(x_k) \subset \Omega, \ k = 1, \ldots, m, \ m \in \{1, 2\} \) and \( \bar{B}_{R^*}(x_1) \cap \bar{B}_{R^*}(x_2) = \emptyset \).

Use **Hanzawa transform** (direct mapping method) to parameterize \( \Gamma(t) \) over \( \Gamma_* := \bigcup_{k=1}^{m} S_{R^*}(x_k) \subset \Omega \) via height function \( \rho \) to obtain

\[
\dot{\rho} = N(\rho)S(\rho), \quad t > 0, \quad \rho(0) = \rho_0,
\]

where \( N(\rho) \)-transformed jump of Neumann derivative, \( S(\rho) \) solves

\[
\begin{cases}
A(\rho)v = 0 & \text{in } \Omega \setminus \Gamma_*, \\
\partial_\nu v = 0 & \text{on } \partial \Omega, \\
v = K(\rho) & \text{on } \Gamma_*.
\end{cases}
\]

\( A(\rho) \)-transformed Laplacian, \( K(\rho) \)-transformed mean curvature.

The operator \( B(\cdot) := -N(\cdot)S(\cdot) \) exhibits a quasilinear structure i.e.

\( \exists \ A, F \in C^1 \) s.t. \( B(\rho) = A(\rho)\rho - F(\rho) \).
Note: Equilibrium $\Gamma_*$ corresponds to $\rho_* = 0$ in

$$\dot{\rho} + B(\rho) = 0.$$ 

Full linearization of $B(\cdot)$ at $\rho = 0$ is given by $B'(0)\rho = -[S(A_{\Gamma_*}\rho)]$, where $Sg$ solves

$$\begin{cases}
\Delta v = 0 & \text{in } \Omega \setminus \Gamma_*, \\
\partial_{\nu} v = 0 & \text{on } \partial\Omega, \\
v = g & \text{on } \Gamma_*,
\end{cases}$$

and $A_{\Gamma_*} := -\frac{1}{n-1} \left( \frac{n-1}{R_*^2} + \Delta_{\Gamma_*} \right)$.

One can show that (cf. talk of Jan Prüss)

1. $0 \in \sigma(B'(0))$ is semi-simple with multiplicity $m(n + 1)$;
2. If $m = 1$, then $\sigma(-B'(0)) \setminus \{0\} \subset \mathbb{C}_-$;
3. If $m = 2$, then $\sigma(-B'(0)) \cap \mathbb{C}_+ \neq \emptyset$ and $\sigma(-B'(0)) \cap i\mathbb{R} = \{0\}$;
4. $N(B'(0))$ is isomorphic to $T_{\Gamma_*} \mathcal{E}$.

Hence $\Gamma_*$ is normally hyperbolic if $m = 2$ and normally stable if $m = 1$. 


