Parabolic singularly perturbed problems with exponential layers: robust discretizations using finite elements in space on Shishkin meshes

Lena Kaland and Hans-Görg Roos

October 13, 2008

Abstract

A parabolic initial-boundary value problem with solutions displaying exponential layers is solved using layer-adapted meshes. The paper combines finite elements in space, i.e., a pure Galerkin technique on a Shishkin mesh, with some standard discretizations in time. We prove error estimates as well for the $\theta$-scheme as for discontinuous Galerkin in time.

Key words convection-diffusion, transient, finite element, Shishkin mesh, time discretization

2000 MSC 65N12, 65N30, 65N50

1 Introduction

We consider 1D unsteady convection-diffusion problems of the type

\begin{align}
  u_t + Lu &= f(x,t) \quad \text{in } Q = (0,1) \times (0,T], \quad (1.1a) \\
  u(x,0) &= u_0(x) \quad \text{for } x \in (0,1), \quad (1.1b) \\
  u(0,t) &= u(1,t) = 0 \quad \text{for } t \in (0,T]. \quad (1.1c)
\end{align}

Here the differential operator $L$ is given by,

\[ Lu := -\varepsilon u_{xx} + b(x)u_x + c(x)u, \]  \hspace{1cm} (1.2)

$0 < \varepsilon << 1$ is a small parameter and $b$, $c$ are sufficiently smooth with

\[ b(x) > \beta > 0. \]  \hspace{1cm} (1.3)

By changing the dependent variable we may also assume that

\[ c - \frac{1}{2} b_x \geq c_0 > 0. \]  \hspace{1cm} (1.4)
The exact solution of (1.1) has, in general, an exponential boundary layer at \( x = 1 \). Additionally, a discontinuity in the initial-boundary data at the point \( x = 0, t = 0 \) would lead to an interior layer along the subcharacteristics through that point. We assume sufficient compatibility of the data to exclude the existence of an interior layer, see [9].

In recent years many numerical methods have been developed to solve the corresponding stationary problem on layer-adapted meshes, resulting in error estimates that are uniform with respect to the parameter \( \varepsilon \), see [9]. For unsteady problems, however, the situation is different.

Most existing papers deal with low order finite difference schemes, beginning with [10] and the error estimate

\[
| u(x_i, t_j) - u_{i,j} | \leq C(N^{-1} \ln^2 N + \tau) \tag{1.5}
\]

for backward differencing in time and upwind differencing in space on a Shishkin mesh. This result was extended in [5], [1] and [4]; in the last paper defect correction in both space and time is applied to enhance the accuracy of the computed solution.

Concerning finite elements in space on a Shishkin mesh, we only know the pointwise error estimates of [3] using space-time finite elements that are linear and continuous in space but discontinuous in time, while additionally the streamline diffusion stabilization in space is applied.

It is the aim of this paper to combine systematically finite elements in space (based on a Galerkin technique or stabilization on a Shishkin mesh) with some standard discretizations in time. First we shall study the \( \theta \)-scheme, later discontinuous Galerkin in time. For simplicity, we present the results for problems one-dimensional in space but we apply only techniques which can be used in several dimensions as well.

2 The continuous problem

It is well known that for \( f \in L_2(Q) \) and \( u_0 \in L_2(\Omega) \) problem (1.1) has a unique solution \( u \in L_2(0, T; H^1_0(\Omega)) \) with \( u' \in L_2(0, T; H^{-1}(\Omega)) \) (in our case we have \( \Omega = (0, 1) \)).

If we introduce the \( \varepsilon \)-weighted \( H^1 \)-norm defined by

\[
\| v \|^2_\varepsilon := \varepsilon \| v \|^2_1 + \| v \|^2_0, \tag{2.1}
\]

standard arguments lead us to the stability estimate (see [7], Theorem 11.1.1)
\[ \sup_{t \in (0,T)} \| u(t) \|_0 + \left( \int_0^T \| u(t) \|_\varepsilon^2 \, dt \right)^{1/2} \leq C \left( \| f \|_{0,Q} + \| u_0 \|_0 \right). \]

(2.2)

Therefore it is natural that we shall later prove error estimates in "\(L_\infty(L^2)\)"- and "\(\sqrt{\varepsilon}L^2(H^1)\)"-norms or their discrete analogues.

**Remark 1** In [7], Proposition 11.1.1., we additionally can find an estimate for \( \max_{t \in (0,T)} \| u \|_1 \). But, in our singularly perturbed case, it seems not possible to follow the proof of Proposition 11.1.1 in such a way that the constants arising are independent of \( \varepsilon \) (if moreover, \( \| u \|_1 \) is replaced by \( \| u \|_{\varepsilon} \)). □

Under certain compatibility conditions [9] there exists a classical solution of problem (1.1). Assuming still more compatibility to avoid interior layers, in [11] there are sufficient conditions for the validity of the estimates

\[ \left| \frac{\partial^{k+m} u(x,t)}{\partial x^k \partial t^m} (x,y) \right| \leq C(1 + \varepsilon^{-k} e^{-\beta(1-x)/\varepsilon}) \]  

(2.3)

for certain values of \( k,m \). The estimate (2.3) implies in the one-dimensional case as well (see [9]) the existence of an S-decomposition of the solution: \( u(x,t) = S(x,t) + V(x,t) \) with

\[ \left| \frac{\partial^{k+m} S(x,t)}{\partial x^k \partial t^m} (x,y) \right| \leq C \text{ and } \left| \frac{\partial^{k+m} V(x,t)}{\partial x^k \partial t^m} (x,y) \right| \leq C \varepsilon^{-k} e^{-\beta(1-x)/\varepsilon}. \]  

(2.4)

For the solution decomposition in case of a higher-dimensional parabolic problem in space see [10].

It is well known [9] that the existence of an S-decomposition in the stationary case allows us in a relatively simple way to estimate both interpolation errors and the error of finite element methods on S-meshes.

### 3 The \( \theta \)-scheme for the discretization in time

For the discretization in space of (1.1) we use linear finite elements on a Shishkin mesh. Denoting the corresponding finite element space by \( V^N \subset H^1_0(\Omega) \), the semidiscrete problem is given by

\[ \left( \frac{du^N}{dt}, v \right) + a(u^N, v) = (f, v) \quad \forall v \in V^N, \quad u^N(0) = u_0^N. \] 

(3.1)
Here, the bilinear form $a(\cdot, \cdot)$ is

$$a(w, v) := \varepsilon(w_x, v_x) + (bw_x + cw, v)$$

with $a(v, v) \geq \omega \|v\|_2^2$. The mesh is piecewise uniform in $[0, 1 - \sigma]$ and $[1 - \sigma, 1]$ with the same number of mesh points in these two subintervals and the definition

$$\sigma = \sigma_0 \varepsilon \ln N. \quad (3.2)$$

For the discretization of the stationary problem related to (1.1) with linear finite elements on our Shishkin mesh the following error estimates are well known ([9]):

If $\sigma_0 \geq 2$, then

$$\|u_S - u^N_S\|_\varepsilon \leq C N^{-1} \ln N, \quad (3.3)$$

If $\sigma_0 \geq \frac{5}{2}$, then

$$\|u^I_S - u^N_S\|_\varepsilon \leq C(N^{-1} \ln N)^2, \quad (3.4)$$

Here $u^I_S$ denotes the linear interpolant of the stationary solution $u_S$.

The estimate (3.4) implies in particular the $L_2$ error estimate

$$\|u_S - u^N_S\|_0 \leq C(N^{-1} \ln N)^2. \quad (3.5)$$

For the analysis of the time discretization it is useful to introduce the Ritz projection $\pi u$ of $u$ defined by $\pi u \in V^N$ and

$$a(\pi u, v) = a(u, v) \quad \forall v \in V^N. \quad (3.6)$$

Then, the error $u - \pi u$ satisfies for every $t$ estimates of the type (3.3)-(3.5).

Introduce a mesh in time that is equidistant for simplicity, with mesh width $\tau$ and $\tau \cdot M = T$. Then the $\theta$-scheme is given by

$$\left( \frac{U^m - U^{m-1}}{\tau}, v \right) + a(U^{m-\theta}, v) = (f^{m-\theta}, v) \quad \forall v \in V^N, \quad U^0 = u^N_0 \quad (3.7)$$

with some $0 \leq \theta \leq 1$ and the abbreviation

$$g^{m-\theta} := \theta g^m + (1 - \theta) g^{m-1}.$$

To analyze the $\theta$-scheme, let us introduce $\psi := U - \pi u$. Then $\psi$ satisfies the error equation

$$\left( \frac{\psi^m - \psi^{m-1}}{\tau}, v \right) + a(\psi^{m-\theta}, v) = (W^m, v) \quad \forall v \in V^N$$

with

$$W^m := \frac{(\pi u)^m - (\pi u)^{m-1}}{\tau} - (u')^{m-\theta}. \quad (3.9)$$
Here and in the following \( u' \) denotes the derivative in time. Next, in (3.8) we set \( v = \psi^{m-\theta} \). Further, we use for \( \theta \geq 1/2 \) the inequality
\[
(\psi^m - \psi^{m-1}, \psi^{m-\theta}) \geq (\| \psi^m \|_0 - \| \psi^{m-1} \|_0) || \psi^{m-\theta} ||_0
\]
and get
\[
\| \psi^m \|_0 + \omega \tau || \psi^{m-\theta} ||_\varepsilon \leq || \psi^{m-1} ||_0 + \tau || W^m ||_0.
\]
Summation leads to
\[
\| \psi^M \|_0 + \omega \sum_m \tau || \psi^{m-\theta} ||_\varepsilon \leq || \psi^0 ||_0 + \tau \sum_m || W^m ||_0.
\]
Here \( \psi^0 = u_0^N - \pi u_0 \) is zero if we choose \( u_0^N = \pi u_0 \). We can write \( W^m \) in the form
\[
W^m = \frac{(\pi u - u)^m - (\pi u - u)^{m-1}}{\tau} + \left( \frac{u^m - u^{m-1}}{\tau} - (u')^m - \Theta \right)
\]
or
\[
W^m = \frac{1}{\tau} \int_{t_{m-1}}^{t_m} ((\pi - I)u(s))' ds + \frac{1}{\tau} \int_{t_{m-1}}^{t_m} \left[ u'(s) - (u')^m - \Theta \right] ds.
\]
This gives us
\[
\| W^m \|_0 \leq \begin{cases} C((N^{-1} \ln N)^2 + \tau) & \text{for } \frac{1}{2} < \theta \leq 1, \\ C((N^{-1} \ln N)^2 + \tau^2) & \text{for } \theta = \frac{1}{2} \end{cases}
\]
and consequently
\[
\begin{align*}
\| \psi^M \|_0 + \omega \sum_{m=1}^M \tau || \psi^{m-\theta} ||_\varepsilon & \leq \begin{cases} C((N^{-1} \ln N)^2 + \tau^2) & \text{for } \frac{1}{2} < \theta \leq 1, \\ C((N^{-1} \ln N)^2 + \tau) & \text{for } \theta = \frac{1}{2} \end{cases} \\
\| (u - \pi u)^M \|_0 + \omega \sum_{m=1}^M \tau || (u - \pi u)^{m-\theta} ||_\varepsilon & \leq CN^{-1} \ln N.
\end{align*}
\]
**Theorem 1** Let us assume that for the Ritz projection of \( u \) estimates of the type (3.3)-(3.5) can be proved based on a solution decomposition. Moreover, set \( u_0^N = \pi u_0 \). Then, the error \( \psi = U - \pi u \) of the \( \theta \)-scheme satisfies
\[
\| \psi^M \|_0 + \omega \sum_{m=1}^M \tau || \psi^{m-\theta} ||_\varepsilon \leq \begin{cases} C((N^{-1} \ln N)^2 + \tau^2) & \text{for } \frac{1}{2} < \theta \leq 1, \\ C((N^{-1} \ln N)^2 + \tau) & \text{for } \theta = \frac{1}{2} \end{cases}
\]
and consequently
\[
\begin{align*}
\| \psi^M \|_0 + \omega \sum_{m=1}^M \tau || \psi^{m-\theta} ||_\varepsilon & \leq \begin{cases} C((N^{-1} \ln N)^2 + \tau^2) & \text{for } \frac{1}{2} < \theta \leq 1, \\ C((N^{-1} \ln N)^2 + \tau) & \text{for } \theta = \frac{1}{2} \end{cases} \\
\| (u - \pi u)^M \|_0 + \omega \sum_{m=1}^M \tau || (u - \pi u)^{m-\theta} ||_\varepsilon & \leq CN^{-1} \ln N.
\end{align*}
\]
**Remark 2** Let us assume that instead of (3.7) we use
\[
\left( \frac{U^m - U^{m-1}}{\tau}, v \right) + a_s(U^m - \Theta, v) = (f^m - \Theta, v),
\]
where we replace the Galerkin scheme by some stabilization. Then a consistent stabilization, i.e., where the exact solution \( u \) satisfies
\[
\left( \frac{du}{dt}, v \right) + a_s(u, v) = (f_s, v), \quad (3.14)
\]
allows the same kind of error estimation if the stabilization term is time-independent. Again we have (3.8) with (3.9), if \( \pi u \) now denotes the Ritz projection with respect to the stabilized bilinear form.

That means, for instance, that CIP stabilization can be handled without problems but SDFEM is less easy to deal with. □

**Remark 3** In the case of irregular initial values \( (u_0 \text{ is not very smooth}) \) the fact that Crank-Nicolson is not strongly A-stable leads to non-physical oscillations. A possible alternative is the strategy to apply first two implicit Euler steps with step size \( \tau/2 \). This damped Crank-Nicolson method has better properties then the original scheme [8], but allows error estimates of the same type as the original scheme. □

### 4 Discontinuous Galerkin in time

First we describe the combination of a dG method in time with a Galerkin finite element method in space to discretize the problem
\[
\left( \frac{du}{dt}, v \right) + a(u, v) = (f, v) \quad \forall v \in V = H^1_0(\Omega), \quad u(0) = u_0 \in L^2(\Omega).
\]

In the time interval \((t_{m-1}, t_m)\) we use a finite element space \( V_{h,m} \subset H^1_0(\Omega) \) of linear elements for the discretization in space (thus on every time interval we could use a different mesh). Moreover we define
\[
S^q_{h,\tau} = \{ \varphi \in L^2(Q) : \varphi|_{[t_{m-1}, t_m]} \in P_q \text{ with coefficients from } V_{h,m} \},
\]
where \( P_q \) is the space of polynomials of degree \( q \).

For the discontinuous functions in time we introduce the jumps at \( t_m \) by
\[
[\varphi]_m := \lim_{t \to t_m^+} \varphi(t) - \lim_{t \to t_m^-} \varphi(t) = \varphi^+_m - \varphi^-_m.
\]

Then our discretization is given by: Find \( U \in S^q_{h,\tau} \) with
\[
\sum_m \int_{t_{m-1}}^{t_m} (U', \varphi) + a(U, \varphi) dt + \sum_{m=2}^M [(U|_{t_{m-1}}, \varphi^+_m) + (U^+_0, \varphi^+_0)]
= \int_0^{t_M} (f, \varphi) dt + (u_0, \varphi^+_0) \quad (4.1)
\]
for all \( \varphi \in S^q_{h,\tau} \).
If one introduces
\[ B(u, v) := \sum_m \int_{t_{m-1}}^{t_m} (u', v) + a(u, v)dt + \sum_2^M ([u]_{m-1}, v'_{m-1}) + (u_0, v_0), \] (4.2)
then integration by parts results in
\[ B(u, v) := \sum_m \int_{t_{m-1}}^{t_m} (-u, v') + a(u, v)\}dt - \sum_1^{M-1} (u_m, [v]_m) + (u_M, v_M). \] (4.3)
The combination of (4.2) and (4.3) allows the estimate
\[ B(v, v) \geq \|v\|^2_{\text{dG}} \quad \text{with} \] (4.4)
\[ \|v\|^2_{\text{dG}} := \omega \sum_m \int_{t_{m-1}}^{t_m} \|v\|^2dt + \frac{1}{2} \|v_0\|^2 + \frac{1}{2} \sum_1^{M-1} \|v_m\|^2 + \frac{1}{2} \|v_M\|^2. \]
Let us next denote by \( \pi u \in S_{Q, \tau}^q \) some interpolant of \( u \) in space and time. We are interested in estimating \( U - \pi u \) because then to bound the error itself we have only to estimate additionally the interpolation error. The exact solution \( u \) satisfies
\[ \sum_m \int_{t_{m-1}}^{t_m} \{(u', \varphi) + a(u, \varphi)\}dt = \int_{0}^{t_M} (f, \varphi)dt, \]
or (again integration by parts)
\[ \sum_m \int_{t_{m-1}}^{t_m} \{-u, \varphi'\} + a(u, \varphi)\}dt + (u_M, \varphi_M) - \sum_m (u_m, [\varphi]_m) \]
\[ = \int_{0}^{t_M} (f, \varphi)dt. \]
It follows that we have the error equation
\[ B(U - \pi u, \xi) = \sum_m \int_{t_{m-1}}^{t_m} \{-u - \pi u, \xi\} + a(u - \pi u, \xi)\}dt \]
\[ + ((u - \pi u)_{M-1}, \xi_M) - \sum_2^M ((u - \pi u)_{m-1}, [\xi]_m) \]
\[ + (u_0 - \pi u, \xi_0^+). \] (4.5)
Based on (4.4) the choice \( \xi := U - \pi u \) allows us to estimate \( \|U - \pi u\|_{\text{dG}} \) if one is able to bound the right-hand-side of (4.5) by some suitable quantity multiplied by \( \|\xi\|_{\text{dG}} \).
Remark 4 (A duality trick)
A second possibility for error estimation is the following (see [12]): We introduce the auxiliary function \( Z \in S_{q,h,\tau} \) defined by
\[
B(\varphi, Z) = (\varphi_M, \psi) \tag{4.6}
\]
with some given \( \psi \in L_2(\Omega) \). Setting \( \psi = (U - \pi u)_M, \varphi := U - \pi u \) we get for the \( L_2 \) error at the time \( t = t_M = T \)
\[
\| (U - \pi u)_M \|_0^2 = B(U - \pi u, Z); \tag{4.7}
\]
again we can reformulate \( B(U - \pi u, Z) \) as in (4.5). The final error estimate is the result of two steps:
- estimate \( B(U - \pi u, Z) \) by terms containing the interpolation error multiplied by some norm \( ||| \cdot ||| \) of \( Z \)
- prove the a priori estimate \( ||| Z ||| \leq C \| \psi \|_0 \) for the solution of (4.8) with respect to the norm \( ||| \cdot ||| \).
We remark that \( Z \) solves a backward homogeneous problem in time. □

If now \( u_{0,h} \) is the \( L_2 \) projection of \( u_0 \), one term of (4.7) vanishes. Next we have to answer the crucial question: How to choose the interpolant of \( u \) in space and time?

Let us first study the case \( q = 0 \), i.e., piecewise constant approximation in time. If we now denote by \( R_u \) the Ritz projection with respect to \( a(\cdot, \cdot) \), then the choice
\[
\pi u \big|_{(t_{m-1},t_m)} = \frac{1}{\tau} \int_{t_{m-1}}^{t_m} (Ru)(t) dt \tag{4.8}
\]
leads to the simplified error equation
\[
B(U - \pi u, \xi) = ((u - \pi u)_M, \xi_M) - \sum_{m=1}^{M-1} ((u - \pi u)_m, [\xi]_m). \tag{4.9}
\]
The definition (4.6) of the \( dG \) norm and Cauchy-Schwarz result in
\[
\| U - \pi u \|_{dG} \leq C \left\{ \| (u - \pi u)_M \|_0^2 + \sum_{m=1}^{M-1} \| (u - \pi u)_m \|_0^2 \right\}^{1/2}. \tag{4.10}
\]

Remark 5 The duality trick also leads to (4.12) (but we get an error estimate only for \( \| (U - \pi u)_M \|_0 \) because \( Z \) satisfies
\[
\| Z_M \|_0^2 + \sum_{m=1}^{M-1} \| [Z]_m \|_0^2 \leq \| \psi \|_0^2. \tag{4.11}
\]
For sharpened estimates for \( Z \) see the next remark. □
To estimate the right-hand side of (4.12) we observe that
\[ u(t_m) - (\pi u)(t_m) = u(t_m) - u(\tilde{t}_m) + u(\tilde{t}_m) - Ru(\tilde{t}_m) \]
because (4.10) implies \( \pi u|_{(t_m-1, t_m)} = Ru(\tilde{t}_m) \) for \( \tilde{t}_m \in (t_{m-1}, t_m) \).

On a Shishkin mesh with linear elements we obtain consequently

**Theorem 2** Let us assume that for the Ritz projection of \( u \) estimates of the type (3.3)-(3.5) can be proved based on a solution decomposition. Moreover, set \( u_{0,h} \) to be the \( L^2 \) projection of \( u_0 \). Then, the error \( U - \pi u \) of our discretization method can be estimated by

\[ \|U - \pi u\|_{dG} \leq C \left\{ \tau^2 + \frac{(N^{-1} \ln N)^4}{\frac{1}{4} + \tau^2} \right\}^{1/2}. \]

(4.12)

Here we used

\[ \sum_m \int_\Omega (u(t_m) - u(\tilde{t}_m))^2 = \sum_m \int_\Omega (\int_{\tilde{t}_m}^{t_m} u_t)^2 \leq \tau \int_\Omega \int_0^T u_t^2. \]

**Remark 6** : Using the symmetry of the underlying bilinear form, in [12] we can find a stability estimate that sharpens (4.13), namely

\[ \sum_{m=1}^{M-1} \|[Z]_m\|_0 \leq C L \|\psi\|_0 \]

(4.13)

(here \( L \) depends logarithmically on the mesh in time). Inequality (4.13) allows us to estimate the \( L^2 \) error of \( (U - \pi u)_M \) by

\[ \sup_m \|(U - \pi u)_m\|_0, \]

consequently for a symmetric problem on a standard mesh the resulting \( L^2 \) error is proportional to \( \tau + h^2 \) instead of \( \{\tau^2 + h^4\}/\tau \) which we got before. \( \square \)

To estimate the error \( U - u \) we use the estimate (4.12) and have, additionally, to estimate \( u - \pi u \). We estimated already the second part of the norm (4.4), we have still to bound

\[ \sum_m \int_{t_{m-1}}^{t_m} \|u - \pi u\|_t^2 \leq 2 \sum_m \int_{t_{m-1}}^{t_m} (\|u(t) - u(\tilde{t}_m)\|_t^2 + \|u(\tilde{t}_m) - Ru(\tilde{t}_m)\|_t^2) dt. \]

This gives an error contribution of the order

\[ O(\tau^{1/2} + N^{-1} \ln N + \frac{1}{\tau^{1/2}} (N^{-1} \ln N)^2). \]

(4.14)

Now we start to consider dG methods with \( q > 0 \). We define our interpolant \( \pi u \) now in two steps (see [12, 2]):
• $\tilde{u}$ is the piecewise polynomial in $t$ of degree $q$ with
  \[ \tilde{u}(t^*_m) = u(t_m), \quad \int_{t_{m-1}}^{t_m} (\tilde{u}(t) - u(t))t^l dt = 0 \quad \text{for} \ l \leq q - 1. \]

• $\pi u$ is the $L_2$–projection of $\tilde{u}$ onto our finite element space.

Then, we get the error equation
  \[ B(\xi, \xi) = \sum_{m} \int_{t_{m-1}}^{t_m} a(u - \pi u, \xi)dt - \sum_{m=1}^{M-1} ((u - \pi u)_m, [\xi]_m). \] (4.15)

We use the splitting
  \[ u - \pi u = u - \tilde{u} + \tilde{u} - \Pi \tilde{u} \quad (\Pi \text{ denotes the } L_2 \text{ projection in space}). \]

It is known that $\tilde{u}$ approximates $u$ with an accuracy of order $O(\tau^{q+1})$, but we have to keep in mind (compare (2.2)) that derivatives of $\tilde{u}$ with respect to $x$ behave like derivatives of $u$.

What about the $L_2$ projection of $\tilde{u}$ on a Shishkin mesh? First, $L_2$ stability shows that
  \[ \|u - \Pi u\|_0 \leq C \|u - u^f\|_0 \leq C(N^{-1} \ln N)^2 \] (4.16)

Moreover, in 1D the $L_2$ projection is $L_\infty - L_\infty$ stable:
  \[ \|u - \Pi u\|_\infty \leq C \|u - u^f\|_\infty \leq C(N^{-1} \ln N)^2. \] (4.17)

(for the two-dimensional case see [6])

\[ \varepsilon^{1/2} |u - \Pi u|_1 \leq C N^{-1} \ln N \] (4.18)

using
  \[ ((u - \Pi u)', (u - \Pi u)') = \int_\Omega (u - \Pi u) u'', \]

the $L_\infty$ estimate and $\|u''\|_1 \leq C \varepsilon^{-1}$. Note that alternatively one can even use the inverse inequality and

\[ |u - \Pi u|_{1, \tilde{\Omega}} \leq C_{meas \tilde{\Omega}}^{1/2} \|u - \Pi u\|_{\infty, \tilde{\Omega}} \]

on both the coarse and the fine parts of the mesh.

Therefore, in the one-dimensional case we have all the ingredients needed to estimate the right-hand side of (4.15). First we get

\[ |\sum_{m=1}^{M-1} ((u - \pi u)_m, [\xi]_m)| \leq C \left( \frac{(N^{-1} \ln N)^2}{\tau^{1/2}} + \tau^{q+1/2} \right) \|\xi\|_{dG}. \] (4.19)
Next we have to estimate

\[ \sum_m \int_{t_{m-1}}^{t_m} a(u - \tilde{u}, \xi) dt \text{ and } \sum_m \int_{t_{m-1}}^{t_m} a(\tilde{u} - \Pi \tilde{u}, \xi) dt. \] (4.20)

In the first term we use the smallness of \( u - \tilde{u} \) but have difficulties with the convective term. Integration by parts yields on \( (t_{m-1}, t_m) \)

\[ |(u - \tilde{u}, \nabla \xi)| \leq C \left( \frac{\tau^{q+1}}{H} + \tau^{q+1} \ln \frac{1}{2} N \right) \|\xi\|_\varepsilon. \] (4.21)

In the estimate of the second term we use the approximation properties of the \( L^2 \) projection and the standard arguments on Shishkin meshes:

\[ |a(\tilde{u} - \Pi \tilde{u}, \xi)| \leq C N^{-1} \ln N \|\xi\|_\varepsilon. \] (4.22)

The final estimate follows from (4.19), (4.21), (4.22).

**Theorem 3** Let us assume that for the \( L^2 \) projection of \( u \) estimates of the type (3.3)-(3.5) can be proved based on a solution decomposition. Moreover, set \( u_{0,h} \) to be the \( L^2 \) projection of \( u_0 \). Then, the error \( U - \pi u \) of our discretization method can be estimated by

\[ \|U - \pi u\|_{dG} \leq C \max \{ N^{-1} \ln N, \frac{(N^{-1} \ln N)^2}{\tau^{1/2}}, \tau^{q+1/2}, N\tau^{q+1} \}. \]

Remark that for a stabilization technique instead of pure Galerkin one can hope to replace the last term \( N\tau^{q+1} \) by the expression \( N^{1/2}\tau^{q+1} \).

**References**


