Two remarks on numerical methods for singularly perturbed problems

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I. Finite elements for a convection-diffusion problem with a weak layer

1 Introduction

Let us consider the convection-diffusion problem

$$-\varepsilon u'' - bu' + cu = f \qquad \text{in} \qquad (0,1) \tag{1.1}$$

subject to the boundary conditions

$$u'(0) = 0, \qquad u(1) = 0 \tag{1.2}$$

with a small parameter $0 < \varepsilon \ll 1$ and smooth b, c, f satisfying

$$b(x) \geq \beta > 0, \quad c + \frac{1}{2}b' \geq \omega > 0 \tag{1.3}$$

Its solution u has a weak exponential boundary layer at $x = 0$ and u can be represented by

$$u = S + E, \tag{1.4}$$

where

$$\mid S^{(k)}(x) \mid \leq C \quad \text{and} \mid E^{(k)}(x) \mid \leq C\,\varepsilon^{-(k-1)} e^{-\beta x}/\varepsilon \quad \text{for} \quad k = 0, 1, ..., q \tag{1.5}$$

up to an arbitrarily prescribed order q. The layer is called weak because $\mid u'(x) \mid \leq C$ (instead $\mid u'(x) \mid \leq C^{-1}\varepsilon$ for a standard layer), (the layer is very weak if $\mid u'(x) \mid \leq C$ and $\mid u''(x) \mid \leq C$). In the scale of Sobolev semi-norms we have

(1.6) $$|E|_k \leq C\varepsilon^{-(k-1)+1/2} \qquad \text{for} \quad k = 0, 1, ..., q.$$

It is well-known [4] that the uniform boundedness of the first order derivative is sufficient to prove the first order uniform convergence of the upwind difference scheme on a standard mesh. Here we want to analyze finite elements and expect difficulties on a standard mesh because $|E|_2$, $|E|_3$, ... are not uniformly bounded with respect to ε.

The question is: do we need a layer-adapted mesh at $x = 0$ or is it sufficient to use a standard mesh?

2 A first error estimate

Let us define $V = \{v \in H^1(0,1) \quad \text{with} \quad v(1) = 0\}$ and the bilinear form

$$a(v, w) := \varepsilon(v', w') + (cv - bv', w).$$

Then, with a finite element space $V_h \subset V$ the discrete approximation $u_h \in V_h$ of solves

(2.1) $$a(u_h, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in V_h.$$

With respect to the $\varepsilon-$weighted H^1-norm $\| \cdot \|_\varepsilon$ the bilinear form $a(\cdot, \cdot)$ is V-elliptic: $a(v, v) \geq \alpha \| v \|_\varepsilon^2$ with $\alpha = \min(1, \omega)$.

Remark that we could even modify the norm by

$$\| v \|_{\varepsilon,*}^2 := \varepsilon \mid v \mid_1^2 + \| v \|_0^2 + b(0)v^2(0).$$

Because the bilinear form is not uniformly bounded, in general the standard error analysis in the framework of the Cea Lemma cannot be used for singularly perturbed problems.

We assume that our mesh with the meshwidth h allows the application of inverse inequalities (the mesh is quasi-equidistant or uniform), moreover we are interested in the case of dominant convection with $\varepsilon \leq ch$.

Let us start from

$$\alpha \| u^I - u_h \|_\varepsilon^2 \leq a(u^I - u_h, u^I - u_h) = a(u^I - u, v_h) = \varepsilon((u^I - u)', v_h') - (b(u^I - u)', v_h) + (c(u^I - u), v_h).$$

Integration by parts of the convection term yields (with $v_h := u^I - u_h$)

$$\alpha \| u^I - u_h \|_\varepsilon^2 \leq \varepsilon((u^I - u)', v_h') + (b(u^I - u, v_h') + ((c + \frac{1}{2}b')(u^I - u), v_h).$$

For the interpolation error we get based on (1.6) if we use finite elements of the polynomial degree $k \geq 1$

$$\varepsilon^{1/2} \mid E - E^I \mid_1 \leq C h^k \varepsilon^{-(k-1)} \tag{2.2}$$

or

$$\| u - u^I \|_0 \leq C h^{k+1} \varepsilon^{-k+\frac{1}{2}}. \tag{2.3}$$

Because we want to use the L_2 error in combination of an inverse inequality, for linear elements and $k = 1$ it seems that (2.3) is not sufficient. But we have as well for $k = 1$ the estimate

$$\| u - u^I \|_0 \leq C\, h\, |u|_1 \leq C\, h^{3/2}.$$

Alternatively, the splitting

$$\| E - E^I \|_0^2 = \int_0^{x_1} (E - E^I)^2 + \int_{x_1}^1 (E - E^I)^2$$

and the smallness of E for $x \geq x_1$ (based on $\varepsilon \leq ch$) result as well in

$$\| E - E^I \|_0 \leq C\, h^{3/2} \qquad \begin{pmatrix} e^{-\beta x_1/\varepsilon} \leq \cdot h^3 & \text{okay if } x_1 \sim h \\ \Rightarrow x_1 \leq \cdot \varepsilon \ln(1/h) & \end{pmatrix}$$

Now standard estimates in combination with an inverse inequality applied to the convective term yield

Theorem 1 On a quasi-equidistant mesh the error of the finite element method for a problem with a weak layer in the case $\varepsilon \leq ch$ satisfies the uniform estimate

$$\| u - u_h \|_\varepsilon \leq C h^{1/2}. \tag{2.4}$$

Remark 1 If the Galerkin method is replaced by streamline diffusion, then we obtain for $u^I - u_h$ the better estimate

$$\| u^I - u_h \|_{SD} \leq Ch \tag{2.5}$$

because now the convective term can be estimate against the SD-norm and we loose only $h^{1/2}$ from the L_2 error. It follows also $\| u - u_h \|_\varepsilon \leq Ch$ for streamline diffusion. (for the SD-norm the term $h^{1/2} \mid u - u^I \mid_1$ is not so nice).

3 An improved estimate for higher order elements and extremely small ε

Because our bilinear form $a(\cdot,\cdot)$ satisfies

$$|a(v,w)| \leq \beta \|v\|_\varepsilon \|w\|_1, \tag{3.1}$$

it follows

$$\| u - u_h \|_\varepsilon \leq \frac{\beta}{\alpha} \inf_{v_h \in V_h} \| u - v_h \|_1 \, .$$

If now U solves the reduced problem, then we get

$$\| u - u_h \|_\varepsilon \leq C(\| u - U \|_1 + \| U - U^I \|_1) \leq C(\varepsilon^{1/2} + h^k) \tag{3.2}$$

(see [4]).

But for stabilization methods it seems not to be appropriate to start from (3.1), therefore we modify the technique above for some stabilization technique, for instance, streamline diffusion.

Let us introduce

$$a_h(v,w) := a(v,w) + \delta\, h \sum_i (-\varepsilon v'' - bv' + cv, bw')_i \tag{3.3}$$

and

$$f_h(w) := (f,w) + \delta h (f, bw'). \tag{3.4}$$

Then, $a_h(\cdot,\cdot)$ is V_h-elliptic with respect to the modified SD-norm

$$\| v \|^2_{mSD} := \varepsilon |v|_1^2 + \|v\|_0^2 + \delta h \sum_i (bv', bv')_i + b(0) v^2(0). \tag{3.5}$$

Next we introduce the splitting

$$u - u_h = u - U + U - U^I + U^I - u_h. \tag{3.6}$$

We have

$$\|u-U\|_{mSD}\leq C\varepsilon^{1/2}(\varepsilon^{1/2}+h^{1/2}) \tag{3.7}$$

and

$$\|U-U^I\|_{mSD}\leq Ch^k(\varepsilon^{1/2}+h^{1/2}) \tag{3.8}$$

Next we estimate U^I-u_h following an idea of Schieweck [2]. Notice $U^I\in V_h$ due to the fact that functions in V_h only satisfy the boundary condition at $x=1$. It follows

$$\begin{aligned}\tilde{\alpha}\parallel U^I-u_h\parallel_{mSD} &\leq a_h(U^I-u_h,U^I-u_h)=a_h(U^I-U,v_h)+a_h(U-u_h,v_h)\\ &= a_h(U^I-U,v_h)+a_h(U-u,v_h).\end{aligned} \tag{3.9}$$

Because U is smooth the first term can be estimated in the standard way of the SDFEM. In the second term we use that U satisfies the reduced equation:

$$a_h(U-u,v_h)=\varepsilon((U-u)',v_h')+(c(U-u)-b(U-u)',v_n)+\delta h(-\varepsilon U'',bv_h'). \tag{3.10}$$

Finally we use integration by parts:

$$-(b(U-u)',v_h)=b(0)(U-u)(0)v_h(0)+(U-u,bv_h')+(b'(U-u),v_h). \tag{3.11}$$

The definition of the modified SD-norm leads to the estimate

$$\mid a_h(U-u,v_h)\mid\leq C(\varepsilon^{3/2}+\varepsilon h^{1/2}+\varepsilon+\frac{\varepsilon^{3/2}}{h^{1/2}}+\varepsilon^{3/2})\|v_h\|_{mSD}. \tag{3.12}$$

Summarizing, (3.7), (3.8), the standard estimate for $a_h(U^I-U,v_h)$ and (3.12) result in
Theorem 2 If $\varepsilon\leq ch$, the error of the SDFEM for the given problem with weak exponential layers satisfies

$$\parallel u-u_h\parallel_{mSD}\leq C(\varepsilon+\frac{\varepsilon^{3/2}}{h^{1/2}}+\varepsilon^{1/2}h^{1/2}+h^{k+1/2}). \tag{3.13}$$

Remark 2 If we apply the same technique to the Galerkin method we obtain only $\dfrac{\varepsilon^{3/2}}{h}+h^k$ in the norm $\|\cdot\|_\varepsilon$ (which improves (3.2) only a little bit if ε is very small).

References

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ETNA, 32(2008), 1-16

II. Some properties of the one-point upwind scheme

1 Introduction

Let us similarly as in [3] study the singularly perturbed model problem

$$-\varepsilon u'' - u' = 0 \qquad \text{on} \;\; (0,1) \tag{1.1}$$

with the boundary conditions $u(0) = 0$, $u(1) = 1$ such that

$$u(x,\varepsilon) = \frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}$$

solves the problem. For $0 < \varepsilon \ll 1$, at $x = 0$ a boundary layer is located.

We want to compare central differencing and upwind differencing as well on an equidistant as on a Shishkin mesh and combine the positive properties of both schemes in a new scheme – the one-point upwind scheme.

2 One-point upwind on an equidistant mesh

Let u_i denote the finite difference approximation at $x_i = ih$ $(i = 0, 1, \ldots, N)$ of the given problem. Then, upwind differencing is characterized by

$$-\varepsilon \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - \frac{u_{i+1} - u_i}{h} = 0 \quad i = 1, \ldots, N-1; \;\; u_0 = 0, \;\; u_N = 1,$$

while central differencing reads

$$-\varepsilon \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - \frac{u_{i+1} - u_{i-1}}{2h} = 0 \quad i = 1, \ldots, N-1; \;\; u_0 = 0, \;\; u_N = 1.$$

Then it is well known, that central differencing leads to unrealistic oscillations which are for even N even unbounded:

$$\lim_{\varepsilon\to 0} u_j = \begin{cases} j/N & \text{for even } j \\ \infty & \text{for odd } j \end{cases} .$$

On the other hand, due to the maximum principle satisfied simple upwinding is free from oscillations. Now let us introduce the one-point upwind scheme:

$$\begin{aligned} &-\varepsilon\,\frac{u_0-2u_1+u_2}{h^2} - \frac{u_2-u_1}{h} = 0 \\ &-\varepsilon\,\frac{u_{i-1}-2u_i+u_{i+1}}{h^2} - \frac{u_{i+1}-u_{i-1}}{2h} = 0 \quad i=2,\ldots,N-2;\;\; u_0=0,\;\; u_N=1. \end{aligned} \tag{2.1}$$

Analogously, the ℓ-point upwind scheme is characterized by the application of simple upwinding in the first ℓ points and central differencing in the remaining points. Introducing $q = \dfrac{2\varepsilon}{h}$ let us assume $q<1$. Using q the ℓ-point upwind scheme can be written as

$$\begin{aligned} &-\frac{q}{2}\,u_{i-1} + (1+q)u_i - (1+\frac{q}{2})\,u_{i+1} = 0 \quad i=1,\ldots,\ell \\ &(1-q)u_{i-1} + 2qu_i - (1+q)u_{i+1} = 0 \quad i=\ell+1,\ldots,N-1;\;\; u_0=0,\;\; u_N=1. \end{aligned} \tag{2.2}$$

Lemma 1: In the ℓ-point upwind scheme the oscillations are bounded and of order $O(q^\ell)$ for $q\to 0$.

Proof: Introducing $\tilde r = \dfrac{q/2}{1+q/2}$ and $r = \dfrac{q-1}{q+1}$, the solution of the difference equation (2.2) is

$$u_i = \begin{cases} u_{\ell+1}\dfrac{1-\tilde r^i}{1-\tilde r^{\ell+1}} & \text{for } i\le \ell+1 \\ 1+C(r^i-r^N) & \text{for } i\ge \ell \end{cases} .$$

Comparing both expressions for $i=\ell$, $i=\ell+1$ leads to the unknowns $u_{\ell+1}$ and C. C determines the character of the oscillations. From

$$C\left[r^\ell - r^N - \frac{1-\tilde r^\ell}{1-\tilde r^{\ell+1}}(r^{\ell+1}-r^N)\right] = \frac{\tilde r^\ell - \tilde r^{\ell+1}}{1-\tilde r^{\ell+1}}$$

we see that C is bounded and of order $O(q^\ell)$ for $q\to 0$.

Lemma 1 tells us that even one-point upwinding already helps a lot: the unbounded oscillations of central differencing are replaced by small oscillations proportional to q which tend to zero on a fixed mesh for $\varepsilon\to 0$.

In the next step we shall study the discrete Green's function related to the one-point upwind scheme (2.1). That means, we consider

$$c_iG_{i-1}(j) + b_iG_i(j) + a_i\,G_{i+1}(j) = \delta_{ij}, \quad G_0(j) = G_N(j) = 0 \tag{2.3}$$

$$\text{with} \quad \begin{array}{ll} c_1 = -q/2, \;\; b_1 = 1+q, \;\; a_1 = -(1+q/2) & \text{and} \\ c_i = 1-q, \;\; b_i = 2q, \;\; a_i = -(1+q) & \text{for} \quad i = 2, \ldots, N-1. \end{array}$$

Then, $G_i(j)$ is explicitly given by

$$G_i(j) = \begin{cases} \dfrac{-f_j}{f_0} \, \dfrac{g_i}{g_1} \, \dfrac{1}{c_j} \displaystyle\prod_{\ell=1}^{j-1} \dfrac{a_\ell}{c_\ell} & \text{for} \quad i \le j \\ \dfrac{g_j}{g_N} \, \dfrac{f_i}{f_{N-1}} \, \dfrac{1}{a_j} \displaystyle\prod_{\ell=j+1}^{N-1} \dfrac{c_\ell}{a_\ell} & \text{for} \quad i \ge j. \end{cases} \tag{2.4}$$

Here, f_i and g_i solve the homogeneous difference equation with $f_N = 0$, $f_{N-1} = 1$ and $g_0 = 0$, $g_1 = 1$, respectively.

Lemma 2: $G_i(j)$ is uniformly bounded with respect to q.

The proof is again an elementary calculation based on (2.4) and the explicit solution of the difference equations for f_i and g_i.

Remark the fundamental difference between central differencing and one-point upwinding. For central differencing,

$$\frac{f_j}{f_0} = \frac{r^j - r^N}{1 - r^N} \quad \overset{N \text{ even}}{\underset{\varepsilon \to 0}{\overset{j \;\text{odd}}{\longrightarrow}}} \quad \infty$$

and thus for central differencing $G_i(j)$ is not bounded. However, for one-point upwinding we have, for instance for $j = 1$,

$$G_1(1) = \frac{f_1}{f_0} \, \frac{2}{q},$$

further

$$\frac{q}{2} \, f_0 = (1+q) \, f_1 - (1 + \frac{q}{2}) \, f_2.$$

Thus, $q \, f_0$ is bounded away from 0 and, consequently, $G_1(1)$ uniformly bounded.

Let us now study the boundary value problem

$$-\varepsilon u'' - u' = s(x), \qquad u(0) = u(1) = 0 \tag{2.5}$$

and its one-point upwind discretization.

Lemma 3: If the boundary value problem (2.5) admits a smooth solution, then its one-point upwind discretization satisfies

$$|u(x_i) - u_i| \le C h^2.$$

Proof: The discrete Green's function allows with $s_j = s(x_j)$ the following representation of the discrete solution:

$$u_i = \sum_j hG_i(j)s_j \,.$$

Therefore the discretization error satisfies

$$u_i - u(x_i) = \sum_j hG_i(j)\psi_j,$$

here ψ_j denotes the consistency error in x_j. $|G_i(j)|$ is uniformly bounded. For smooth solutions ψ the consistency error is of order 2 with exception of exactly one mesh point where upwinding is used. Thus,

$$|u_i - u(x_i)| \leq C(h^2 + h \cdot h) = C^* h^2$$

and Lemma 3 is proved.

We expect that the statement of Lemma 3 holds true in the class of boundary value problems

$$-\varepsilon u'' - b(x)u' + c(x)u = s(x), \qquad u(0) = u(1) = 0 \tag{2.6}$$

with $b(x) \geq b_0 > 0$.

3 One-point upwinding on a Shishkin mesh

The solution of problem (1.1) and, in general, of problem (2.6) has a layer at $x = 0$. Therefore, it makes sense to use a fine mesh in the layer region. As an example for a layer adapted mesh we take a Shishkin mesh (see [4] for a survey).

Let N be an even integer. Set

$$\tau = \min\left(2\varepsilon \ln N, 1/2\right).$$

Divide each of the intervals $[0, \tau]$ and $[\tau, 1]$ into $N/2$ equidistant subintervals. In practice one usually has $\tau \ll \frac{1}{2}$, so the mesh is fine on $[0, \tau]$ and coarse on $[\tau, 1]$.

From [3] we know that the central difference scheme for problem (1.1) on a Shishkin mesh has the following properties: The solution in the boundary layer has no oscillations, outside the layer region the approximate solution oscillates, but not unboundedly. Moreover, these oscillations decay to zero as N increases. Later Andreev and Kopteva proved for the discretization of (2.6) the error estimate [1]

$$|u(x_i) - u_i| \leq C(N^{-1} \ln N)^2. \tag{3.1}$$

If already the central scheme on a Shishkin mesh is nice: why the one-point upwind idea should still improve the situation?

The answer is intuitively relatively clear. Let us consider the central scheme for problem (1.1)

$$-\frac{2\varepsilon}{h_i+h_{i+1}}\left(\frac{u_{i+1}-u_i}{h_{i+1}}-\frac{u_i-u_{i-1}}{h_i}\right)-\frac{u_{i+1}-u_{i-1}}{h_i+h_{i+1}}=0\,.$$

In the transition point $x_{N/2}$ the dominating term $\frac{\varepsilon}{h_i}(u_{N/2}-u_{N/2-1})$ tells us $u_{N/2} \approx u_{N/2-1}$.But in the next point $x_{N/2+1}$ – the first point outside the layer region –the term $(u_{N/2+2}-u_{N/2})/(2H)$ dominates leading to $u_{N/2+2} \approx u_{N/2}$. This leads to an oscillation of $u_{N/2}$, $u_{N/2+1}$ and $u_{N/2+2}$ which causes, in general, the maximal error of the scheme. (see Figure 1)

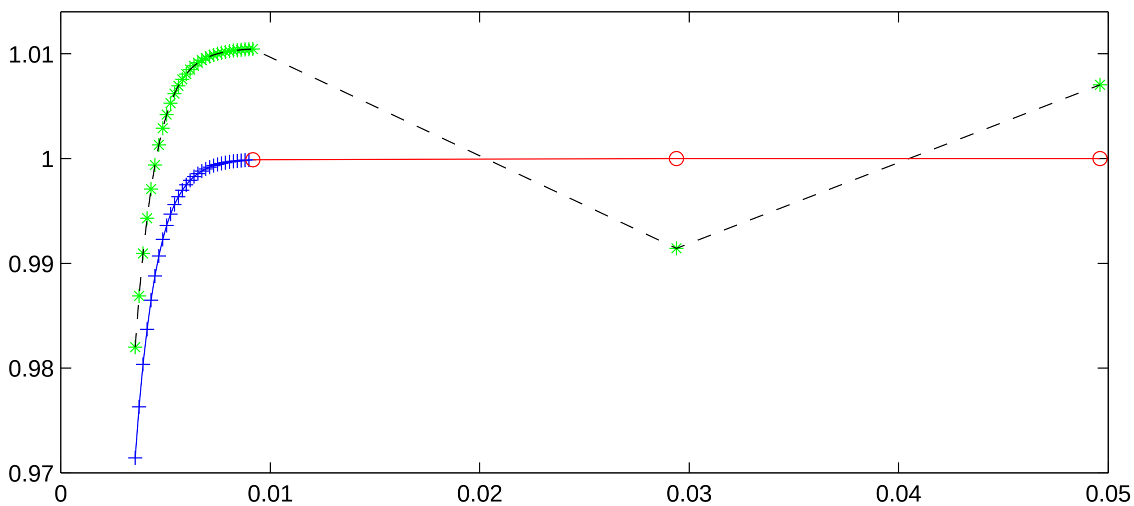

Figure 1: linear interpolate of the exact solution and the central scheme in $u_{N/2}$, $u_{N/2+1}$, $u_{N/2+2}$

Because the approximation $u_{N/2+2} \approx u_{N/2+1}$ makes much more sense, we replace the central scheme in $x_{N/2+1}$ by the upwind scheme. Therefore, one-point upwinding on a layer-adapted mesh reads: (here H is the mesh width on $[\tau, 1]$).

$$\text{(3.2)} \qquad \frac{-2\varepsilon}{h_i+h_{i+1}}\left(\frac{u_{i+1}-u_i}{h_{i+1}}-\frac{u_i-u_{i-1}}{h_i}\right)-\frac{u_{i+1}-u_{i-1}}{h_i+h_{i+1}}=0$$

. for $i = 1,\ldots,N/2$ and $i = N/2+2,\ldots,N-1$

$$\frac{-2\varepsilon}{h_i+h_{i+1}}\left(\frac{u_{i+1}-u_i}{h_{i+1}}-\frac{u_i-u_{i-1}}{h_i}\right)-\frac{u_{i+1}-u_i}{H}=0 \text{ for } i=N/2+1; u_0=0, u_N=1.$$

Table 2 shows the maximal error for one-point upwinding compared with central differencing:

N	one-point	central
14	0.015938	0.015974
28	0.006425	0.007762
56	0.002358	0.002681
112	0.000824	0.000887
224	0.000275	0.000286
448	0.000088	0.000091
896	0.000028	0.000028

($\varepsilon = 10^{-4}$)

N	one-point	central
14	0.015933	0.015934
28	0.006423	0.007814
56	0.002357	0.000737
112	0.000824	0.000923
224	0.000274	0.000299
448	0.000088	0.000094
896	0.000028	0.000029

($\varepsilon = 10^{-8}$)

What about convergence? The numerical rates observed indicate that one-point-upwinding converges with the same rate as central differencing corresponding to (3.1).

Andreev and Kopteva proved (3.1) based on the property that the discrete Green's function for central differencing on a Shishkin mesh is uniformly bounded. Inspecting the corresponding proof in [1] one realizes that the modification of the scheme in one point to upwinding does not influences the proof at all. Therefore it makes no sense to repeat the proof leading to the fact that on a Shishkin mesh as well the discrete Green's function of central differencing as one-point upwinding is uniformly bounded.

Based on the representation

$$u_i - u(x_i) = \sum_j \frac{h_i + h_{i+1}}{2} G_i(j)\psi_j$$

the boundedness of G on a Shishkin mesh leads to:

Theorem 1: The application of the one-point upwind scheme on a Shishkin mesh to (2.6) results in

$$|u(x_i) - u_i| \leq C(N^{-1}\ln N)^2.$$

Inspecting Table 2 one could argue that one gains not so much using one-point upwind. But reducing oscillations could also help to reduce the numerical problems to solve the generated discrete problems in $2D$, see the discussion in [2].

The investigation of local upwinding in $2D$ is an promising project for the future.

References

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