On Prandtl's (1945) turbulence model. Existence of weak solutions to the equations of unidirectional pipe-flow

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Abstract In 1945, L. Prandtl proposed a system of PDEs for modeling the developed turbulent flow of an incompressible fluid, where the unknown functions are the mean velocity $\mathbf{u} = (u_1, u_2, u_3)$, the mean pressure p and the mean turbulent kinetic energy k.

The unidirectional *stationary* flow through a pipe with cross section $\Omega \subset \mathbb{R}^2$ and length l is characterized by $\mathbf{u}(x) = (0, 0, u_3(x_1, x_2))$, $p(x) = -f_0 x_3$ and $k(x) = k(x_1, x_2)$ $(x = (x_1, x_2, x_3),\ (x_1, x_2) \in \Omega,\ x_3 \in \,]0, l[\,;\ f_0 = \text{const})$. With the notation $u := u_3$, $x' := (x_1, x_2)$, Prandtl's model takes the form

$$-\frac{1}{2}\operatorname{div}(\ell\sqrt{k}\,\nabla u) = f_0 \quad \text{in } \Omega, \tag{1}$$

$$-\operatorname{div}(\ell\sqrt{k}\,\nabla k) = \frac{1}{2}\,\ell\sqrt{k}\,|\nabla u|^2 - \frac{k\sqrt{k}}{\ell} \quad \text{in } \Omega, \tag{2}$$

where $\ell = \ell(x') > 0$ is the mixing length.

We consider (1), (2) on the assumption that either

$$\text{(i)}\quad \ell \equiv 1 \qquad \text{or} \qquad \text{(ii)}\quad \ell(x') = \operatorname{dist}(x', \partial\Omega), \quad x' \in \Omega.$$

We then prove the existence of a weak solution to (1), (2) under the boundary conditions

$$\text{case (i):}\quad u = 0, \quad \sqrt{k}\,\frac{\partial k}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

$$\text{case (ii):}\quad u = 0 \quad \text{on } \partial\Omega.$$