Continuous-time random walks (CTRWs)

This poster is based on work with Guido Germano, Mauro Politi, and René L. Schilling (see [Germano et al. (2009)]) and the relevant poster in this conference and, jointly with Francesco Mainardi, it will be presented at the FDA conference in Badajoz, Spain to celebrate Rudolf Gorenflo’s 80th birthday.

Let \( \{J_n(t)\}_{n \geq 0} \) be a sequence of independent and identically distributed (i.i.d.) positive random variables with the meaning of durations between events in a point process. They define a renewal process. Let

\[
T_n = \sum_{k=1}^{J_n} T_k
\]

be the epoch of the \( n \)-th event. Then the process \( N(t) \) counting the number of events up to time \( t \) is

\[
N(t) = \max \{n : T_n \leq t\},
\]

where it is assumed that the number of events occurring up to \( t \) is a finite, but arbitrary, non-negative integer.

Let \( \{Y_n(t)\}_{n \geq 0} \) be a sequence of i.i.d. random variables (also independent of the \( J_n \)) with the meaning of jumps taking place at each epoch, then the compound process \( X(t) \) defined by

\[
X(t) = X(N(t)) = \sum_{n=0}^{N(t)} Y_n(T_n),
\]

is called an uncorrelated continuous-time random walk (CTRW).

Let \( F_X(x) = P(0 \leq x) \) be the cumulative distribution function for the jumps \( Y_n \). Then, by purely probabilistic arguments, one can prove that the cumulative distribution function for \( X(t) \) is given by

\[
F_X(x) = \Phi(\phi_{\mu,\sigma^2}|x-\mu|^\alpha)^{-\frac{1}{\alpha}},
\]

where

\[
\Phi(\phi_{\mu,\sigma^2}|x-\mu|^\alpha) = \int_{0}^{\infty} e^{-\phi_{\mu,\sigma^2}s|\eta|^{\alpha}} d\eta
\]

and

\[
\sigma^2 = \frac{2}{\alpha} \int_{0}^{\infty} e^{\phi_{\mu,\sigma^2}s} ds
\]

is the error function. The process of this example is known as normal compound Poisson process (NCPP) and is a Lévy process (it has stationary and independent increments). As a final remark, if \( F_X(x) \) has a probability density function \( f_X(x) = dF_X/dx \), from equation (4) one can write for the density \( f_{X(N(t))}(x,t) \)

\[
f_{X(N(t))}(x,t) = \sum_{n=0}^{N(t)} f_Y(Y_n) f_{N(t)}(x|T_n)
\]

which is the \( n \)-fold convolution of \( f_Y \).

Diffusive limit for CTRWs

Assume that the duration \( J_n \) is rescaled by a real parameter \( \epsilon \) so that the rescaled duration \( J_{\epsilon} = 1/J_n \) and that the jump \( Y_n \) is rescaled by \( \epsilon \) getting \( Y_{\epsilon n} = T_{\epsilon} Y_n \). Let \( f_{\epsilon X}(x) \) denote the rescaled CTRW. Further assume that \( J_{\epsilon} \) has a density function \( f_{J_{\epsilon}}(x) \) whose Laplace transform has the following behaviour for \( \epsilon \rightarrow 0 \)

\[
f_{\epsilon X}(x) = \mathbb{E}[e^{\epsilon X}] = \int_{0}^{\infty} e^{\epsilon x} f_{\epsilon J}(x) dx = 1 - c_{\epsilon} x^{2} + o(x^2), \quad \epsilon \rightarrow 0,
\]

where \( c_{\epsilon} = 1 - c_{\epsilon} \alpha \).\( ^{1} \)

then, in the limit \( \epsilon \rightarrow 0 \), one can prove that \( f_{\epsilon X}(x) \) weakly converges to \( x(x) \) given by

\[
x(x) = \frac{1}{\alpha} \int_{0}^{\infty} e^{-x^{\frac{2}{\alpha}}} \frac{\alpha}{\sqrt{2\pi}} dx
\]

with

\[
\int_{0}^{\infty} e^{-x^{\frac{2}{\alpha}}} \frac{\alpha}{\sqrt{2\pi}} dx = \int_{0}^{\infty} e^{-x^{\frac{2}{\alpha}}} \frac{\alpha}{\sqrt{2\pi}} dx = \int_{0}^{\infty} e^{-x^{\frac{2}{\alpha}}} \frac{\alpha}{\sqrt{2\pi}} dx
\]

and coinciding with the exponential function for \( \alpha = 1 \). Notice that equation (12) is the Green function for the Cauchy problem of the fractional diffusion equation. The above results were discussed in [Scalas et al. (2006)], whereas in [Scalas (2006)], a simplified method was used to derive the probability density \( x(x) \) in the diffusive limit.

Quadratic variation and jumps

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