Fine Properties of Processes
Given as Solutions of Lévy Driven SDEs

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Motivation & Overview
Every \( R^d \)-valued Lévy process \((X_t)_{t \geq 0}\) is a semimartingale. If one allows the process to start in every \( x \in R^d \) it becomes as well a (universal) Markov process which is homogeneous in space and time. In the sequel we assume that the lifetime of the process is infinite. It is a well known fact that the characteristic function of a Lévy process can be written as

\[
\phi(t) = e^{it\cdot x - \frac{1}{2}\sigma^2 |t|^2},
\]

where \( \psi \) is continuous and negative definite, i.e. it admits a Lévy-Khinchine-representation

\[
\psi(t) = \sum_{\nu \in \nu} \nu |t|^p / p! - \frac{1}{2} \sigma^2 |t|^2.
\]

The Symbol of a Process
We now introduce a probabilistic formula for the symbol. It turns out that in the known cases this is nothing else than the symbol from above.

Definition (Symbol), Let \( X_t \) be an \( R^d \)-valued Markov semimartingale. Fix a starting point \( x \) and define \( \sigma = \sigma_t \) to be the first exit time from a compact neighborhood \( K \) of \( x \):

\[
\sigma = \sigma_t \equiv \inf \{ t \geq 0 : X_t \notin K \}.
\]

We call the function \( \sigma \) a Lévy-Khinchine-representation of the generator \( A \).

\[
\psi(t) = \sum_{\nu \in \nu} \nu |t|^p / p! - \frac{1}{2} \sigma^2 |t|^2.
\]

where \( \ell \) is a vector in \( R^d \). \( Q \) is a positive semi-definite matrix and \( N \) is the so-called Lévy measure on \( \{ 0 \} \) with the property

\[
\int_{\{ 0 \}} (1 + |y|^2) N(dy) < \infty.
\]

We call \( \ell, Q, N \) the Lévy triplet of the process. It depends on the choice of the truncation function \( \chi \).

The characteristic exponent (or symbol) \( \psi \) can be used to deduce path properties of the process.

It is possible to associate a semigroup of operators \((T_t)_{t \geq 0}\) on \( C_c(R^d) \) (the continuous functions vanishing at infinity) with the process \((X_t)_{t \geq 0}\) via

\[
T_t \phi(x) = \mathbb{E}^x [\phi(X_t)],
\]

and to calculate the (strong) generator \((A, D(A))\) of this semigroup:

\[
A u(x) = \lim_{t \downarrow 0} \frac{T_t u - u}{t} \quad u \in D(A)
\]

where

\[
D(A) = \{ u \in C_c(R^d) : \lim_{t \downarrow 0} \frac{T_t u - u}{t} \text{ exists in } \| \cdot \|_{L^p} \}
\]

is the domain of the operator. The symbol appears again in the Fourier representation of the generator \( A \):

\[
\mathbb{E}^x [e^{i \langle \xi, X_t \rangle} d^t] = \int_{R^d} e^{i \langle \xi, x \rangle} \psi(t, \xi) d^t
\]

This last statement remains true for all (rich) Feller processes: A time homogeneous Markov process is called Feller if its corresponding semigroup of operators \((T_t)_{t \geq 0}\) fulfills the following properties:

\( \bullet \) \( T_t \circ C_c \circ \text{compacts} \rightarrow C_c \circ \text{compacts} \) for all \( t \geq 0 \)

\( \bullet \) \( \lim_{t \downarrow 0} \| T_t u - u \|_p = 0 \) for all \( u \in C_c(R^d) \)

We say that the process is ‘rich’, if the test functions \( C_c(R^d) \) are contained in the domain of the generator. The generator restricted to \( C_c(R^d) \) can be written as

\[
A u(x) = \int_{R^d} e^{i \langle \xi, x \rangle} \psi(t, \xi) d^t
\]

where \( p \) is locally bounded and for fixed \( x \) a continuous negative definite function in \( \xi \). We say: Like the process the symbol becomes ‘state space dependent’. It turns out that the symbol \( \psi \) still contains a lot of informations about the process (e.g. conservativeness, \( \gamma \)-variation, Hausdorff dimension of the paths...).

In [9] we have shown that every rich Feller process \( X \) is an Itô process in the sense of [1], i.e. a Markov semimartingale with local characteristics

\[
B_t(\omega) = \int_0^t f(X_s(\omega)) \, ds
\]

\[
C_t(\omega) = \int_0^t Q(X_s(\omega)) \, ds
\]

\[
\nu(\omega, dx, dt) = N(X_t(\omega), dx) \, dt
\]

with respect to a fixed truncation function \( \chi \). We have used a stochastic formula in order to get general the notion of the symbol from Feller to Itô processes.

Indices & Fine Properties
Using the symbol of a Feller Process it is possible to introduce so called indices which are generalizations of the Blumenthal-Getoor index \( \beta \). This was done in [8]. These indices can be used to obtain results on the global behavior and the paths of the process. We have shown in [8] that the index \( \beta \) can be characterized in the following way: for every \( x \in R^d \) such that \( \xi \equiv x \) into \( p(\cdot, \xi) \) is not identically zero, the index \( \beta \) is given by

\[
\beta = \lim_{\rho \to 0} \sup_{x \in R^d} \frac{\log |p(y, \rho)|}{\log |y|}.
\]

Using this characterization of the index we have proved a result on the strong \( \gamma \)-variation:

Theorem. Let \( X^t = (X^t_s)_{s \geq 0} \) be the solution of the SDE

\[
dX_t^x = \Phi(X_t^x) \, d\xi_t^x
\]

where \( \Phi \) is a Lévy process with characteristic function \( \psi \) and index \( \beta \).

Finally we obtained some results on the ‘smoothness’ of the sample paths of the solution. Since we deal with càdlàg-functions, in general, the right scale of function spaces are (polygonally weighted) Besov spaces (cf. [7])

\[
E^x_t / L^p((1 + t^{-\gamma})^{2/\beta} dt)
\]

References
For details see the following papers and monographs and the references given therein.


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