A non-anticipative calculus for functionals of semimartingales

Rama Cont & David Fournié

Laboratoire de Probabilités et Modèles Aléatoires CNRS - Université de Paris VI-VII
and
Columbia University, New York

LEVY PROCESSES 2010
Functional representation of non-anticipative processes
Pathwise derivatives of functionals
A pathwise change of variable formula
Functional Ito formula
Martingale representation formula
Weak derivative and integration by parts formula
Functional equations for martingales

References

- R. Cont & D Fournié (2009) Functional Ito formula and stochastic integral representation of martingales, arxiv/math.PR.
Consider a cadlag $\mathbb{R}^d$-valued Ito-Lévy semimartingale on $(\Omega, \mathcal{B}, \mathcal{B}_t, \mathbb{P})$:

$$X(t) = \int_0^t \mu(u) du + \int_0^t \sigma(u).dW_u + \int_0^t \int_{\mathbb{R}^d} z ~ \tilde{J}_X(du dz)$$

$\mu$ integrable, $\sigma$ square integrable $\mathcal{B}_t$-adapted processes, $\tilde{J}_X(dtdz; \omega) = J_X(dtdz; \omega) - \nu(dt, dz; \omega)$ compensated jump measure

- Quadratic variation process $[X](t)$
- $D([0, T], \mathbb{R}^d)$ space of cadlag functions.
- $\mathcal{F}_t = \mathcal{F}_t^X$: natural filtration / history of $X$
- $C_0([0, T], \mathbb{R}^d)$ space of continuous paths.
Functional notation

For a path $\omega \in D([0, T], \mathbb{R}^d)$, denote by

- $\omega(t) \in \mathbb{R}^d$ the value of $\omega$ at $t$
- $\omega_t = \omega|_{[0,t]} = (\omega(u), 0 \leq u \leq t) \in D([0, t], \mathbb{R}^d)$ the restriction of $\omega$ to $[0, t]$.

We will also denote $\omega_{t-}$ the function on $[0, t]$ given by

$$\omega_{t-}(u) = \omega(u), \quad u < t \quad \omega_{t-}(t) = \omega(t-)$$

For a process $X$ we shall similarly denote

- $X(t)$ its value and
- $X_t = (X(u), 0 \leq u \leq t)$ its path on $[0, t]$.  

Properties of functionals of such processes—integration by parts formulae, sensitivity analysis, existence of densities—have been studied using the Malliavin calculus.

Its starting point is a path space $E$, and a Wiener/Poisson measure on $E$; but the notion of non-anticipativity with respect to a filtration $\mathcal{F}_t$ does not play a role in the construction.

We extend the Itô calculus in a non-anticipative manner to a large class of adapted path-dependent functionals defined on the space càdlàg paths.

Our construction builds on Föllmer’s (1979) pathwise approach to Itô calculus and a pathwise functional derivative proposed by B Dupire (2009).

The construction is not based on Gaussian properties of the Wiener process nor on regularity of semigroups and is carried out for a general càdlàg semimartingale.
We define a pathwise derivative $\nabla_\omega F$ for non-anticipative families $F = (F_t)_{t \in [0, T]}$ of functionals $F_t : (D([0, t], \mathbb{R}^d), \| \cdot \|_\infty) \to \mathbb{R}$.

Using this pathwise derivative, we derive a functional change of variable formula for processes of the type

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}) = F_t(X_t)$$

where $X$ is a cadlag semimartingale.

This pathwise derivative admits a closure $\nabla_X$ on the space of square integrable stochastic integrals w.r.t. $X$, which is shown to verify an integration by parts formula.

We derive a martingale representation formula and an integration by parts formula for stochastic integrals, in terms of $\nabla_X$. 

Rama Cont & David Fournié

Functional Ito calculus
I: Pathwise calculus for non-anticipative functionals.
II: An Ito formula for functionals of semimartingales.
III: Martingale representation formula. Weak derivative.
IV: Relation with Malliavin calculus.
V: Functional Kolmogorov equations.
A process $Y$ adapted to $\mathcal{F}_t$ may be represented as a family of functionals

$$Y(t, .) : \Omega = D([0, t], \mathbb{R}^d) \rightarrow \mathbb{R}$$

with the property that $Y(t, .)$ only depends on the path stopped at $t$: $Y(t, \omega) = Y(t, \omega(\cdot \wedge t))$ so

$$\omega\big|_{[0,t]} = \omega'\big|_{[0,t]} \Rightarrow Y(t, \omega) = Y(t, \omega')$$

Denoting $\omega_t = \omega\big|_{[0,t]}$, we can thus represent $Y$ as

$$Y(t, \omega) = F_t(\omega_t)$$

for some $F_t : D([0, t], \mathbb{R}^d) \rightarrow \mathbb{R}$

which is $\mathcal{F}_t$-measurable.
This motivates the following definition:

**Definition (Non-anticipative functional)**

A *non-anticipative functional* on the (canonical) path space $D([0, T], \mathbb{R}^d)$ is a family $F = (F_t)_{t \in [0, T]}$ where

$$F_t : D([0, t], \mathbb{R}^d) \to \mathbb{R}$$

is $\mathcal{F}_t$-measurable.

$F = (F_t)_{t \in [0, T]}$ may be viewed as a functional on the vector bundle $\bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d)$. 
Functional representation of predictable processes

An $\mathcal{F}_t$-predictable process $Y$ may be represented as a family of functionals

$$Y(t, .) : \Omega = D([0, T], \mathbb{R}^d) \mapsto \mathbb{R}$$

with the property

$$\omega|_{[0,t[} = \omega'|_{[0,t[} \Rightarrow Y(t, \omega) = Y(t, \omega')$$

We can thus represent $Y$ as

$$Y(t, \omega) = F_t(\omega_{t-}) \quad \text{for some} \quad F_t : D([0, t], \mathbb{R}^d) \rightarrow \mathbb{R}$$

where $\omega_{t-}(u) = \omega(u), u < t$ and $\omega_{t-}(t) = \omega(t-)$.  

So: an $\mathcal{F}_t$-predictable $Y$ can be represented as $Y(t, \omega) = F_t(\omega_{t-})$ for some non-anticipative functional $F$.

Ex: integral functionals $Y(t, \omega) = \int_0^t g(\omega(u))\rho(u)du$
Any process $Y$ adapted to $\mathcal{F}_t = \mathcal{F}_t^X$ may be represented as

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}) = F_t(X_t)$$

where the functional $F_t : D([0, t], \mathbb{R}^d) \to \mathbb{R}$ represents the dependence of $Y(t)$ on the path of $X$ on $[0, t]$. $F = (F_t)_{t \in [0, T]}$ may then be viewed as a functional on the vector bundle $\Upsilon = \bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d)$. 

Functional representation of non-anticipative processes
Pathwise derivatives of functionals
A pathwise change of variable formula
Functional Ito formula
Martingale representation formula
Weak derivative and integration by parts formula
Functional equations for martingales

Horizontal extension of a path

Rama Cont & David Fournié

Functional Ito calculus
Horizontal extension of a path

Let \( \omega \in D([0, T] \times \mathbb{R}^d) \), \( \omega_t \in D([0, T] \times \mathbb{R}^d) \) its restriction to \([0, t]\).

For \( h \geq 0 \), the \textit{horizontal} extension \( \omega_{t,h} \in D([0, t+h], \mathbb{R}^d) \) of \( \omega_t \) to \([0, t+h]\) is defined as

\[
\omega_{t,h}(u) = \omega(u) \quad u \in [0, t] ; \quad \omega_{t,h}(u) = \omega(t) \quad u \in ]t, t+h[.
\]
For $T \geq t' = t + h \geq t \geq 0$, $\omega \in D([0, t], \mathbb{R}^d) \times S^+_t$ and $\omega' \in D([0, t + h], \mathbb{R}^d) \times S^+_{t+h}$

$$d_\infty(\omega, \omega') = \sup_{u \in [0, t+h]} |\omega_{t,h}(u) - \omega'(u)| + h$$

On $D([0, t], \mathbb{R}^d)$, $d_\infty$ coincides with the supremum norm.
A non-anticipative functional \( F = (F_t)_{t \in [0, T]} \) is said to be **continuous at fixed times** if for all \( t \in [0, T[ \), \( F_t : D([0, t], \mathbb{R}^d) \rightarrow \mathbb{R} \) is continuous w.r.t. the supremum norm.

**Definition (Left-continuous functionals)**

Define \( \mathbb{C}^{0,0}_l \) as the set of non-anticipative functionals \( F = (F_t, t \in [0, T[) \) continuous w.r.t. \( \omega \) at fixed times and left-continuous in \( t \):

\[
\forall t \in [0, T], \quad \forall \epsilon > 0, \forall \omega \in D([0, t], \mathbb{R}^d),
\exists \eta > 0, \forall h \in [0, t], \forall \omega' \in \omega \in D([0, t - h], \mathbb{R}^d),
\quad d_\infty(\omega, \omega') < \eta \Rightarrow |F_t(\omega_t) - F_{t-h}(\omega')| < \epsilon
\]
Boundedness-preserving functionals

We call a functional “boundedness preserving” if it is bounded on each bounded set of paths:

**Definition (Boundedness-preserving functionals)**

Define $\mathcal{B}([0, T])$ as the set of non-anticipative functionals $F$ on $\Gamma([0, T])$ such that for every compact subset $K$ of $\mathbb{R}^d$, and $t_0 < T$

$$\exists C_{K, t_0} > 0, \quad \forall t \leq t_0, \quad \forall \omega \in D([0, t], K), \ |F_t(\omega_t)| \leq C_{K, t_0}$$
A non-anticipative functional $F = (F_t)$ applied to $X$ generates an $\mathcal{F}_t$–adapted process

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}) = F_t(X_t)$$

**Theorem**

Let $\omega \in D([0, T], \mathbb{R}^d)$.

- If $F \in \mathbb{C}_l^{0,0}$, the path $t \mapsto F_t(\omega_{t-})$ is left-continuous and $Y(t) = F_t(X_t)$ defines an optional process.
- If $F \in \mathbb{C}_l^{0,0}$, $Y(t) = F_t(X_t)$ is a predictable process.
Horizontal derivative

Definition (Horizontal derivative)

We will say that the functional $F = (F_t)_{t \in [0, T]}$ on $\mathcal{Y}([0, T])$ is horizontally differentiable at $\omega \in D([0, t], \mathbb{R}^d)$ if

$$
\mathcal{D}_t F(\omega) = \lim_{h \to 0^+} \frac{F_{t+h}(\omega, h) - F_t(\omega)}{h}
$$

exists.

We call $\mathcal{D}_t F(\omega)$ the horizontal derivative of $F$ at $\omega$.

$\mathcal{D} F = (\mathcal{D}_t F)_{t \in [0, T]}$ defines a non-anticipative functional.

If $F_t(\omega_t) = f(t, \omega(t))$ with $f \in C^{1,1}([0, T] \times \mathbb{R}^d)$ then

$\mathcal{D}_t F(\omega) = \partial_t f(t, \omega(t))$. 
**Figure:** For $e \in \mathbb{R}^d$, the **vertical** perturbation $\omega_t^e$ of $\omega_t$ is the cadlag path obtained by shifting the endpoint:

$$\omega_t^e(u) = \omega(u) \text{ for } u < t \text{ and } \omega_t^e(t) = \omega(t) + e.$$
Definition (Dupire 2009)

A non-anticipative functional $F = (F_t)_{t \in [0, T]}$ is said to be \textit{vertically differentiable} at $\omega \in D([0, t]), \mathbb{R}^d$ if

$$\mathbb{R}^d \rightarrow \mathbb{R}$$

$$e \rightarrow F_t(\omega_t^e) = F_t(\omega_t + e1_{\{t\}})$$

is differentiable at 0. Its gradient at 0 is called the \textit{vertical derivative} of $F_t$ at $\omega$

$$\nabla_\omega F_t (\omega) = (\partial_i F_t(\omega_t), \ i = 1..d)$$

where

$$\partial_i F_t(\omega_t) = \lim_{h \to 0} \frac{F_t(\omega_t^h e_i) - F_t(\omega_t)}{h}$$
Vertical derivative of a non-anticipative functional

- $\nabla_t F_t (\omega) . e$ is simply a directional derivative of $F_t$ in the direction of the indicator function $1_{\{t\}} e$.

- If $F_t : D([0, t], \mathbb{R}^d) \mapsto \mathbb{R}$ has Fréchet derivative $D_x F_t$ then it has vertical derivative $\nabla_t F_t (\omega_t) = \langle D_x F_t, 1_{\{t\}} \rangle$. However, $F$ may be vertically differentiable without $F_t$ being Fréchet differentiable for any $t \in [0, T]$.

- Note that to compute $\nabla_t F_t (\omega)$ we need to compute $F$ outside $C_0$: even if $\omega \in C_0$, $\omega^h \notin C_0$.

- $\nabla_t F_t (\omega)$ is 'local' in the sense that it is computed for $t$ fixed and involves perturbing the endpoint of paths ending at $t$. 
Spaces of differentiable functionals

Definition (Spaces of differentiable functionals)

For $j, k \geq 1$ define $C_b^{j,k}([0, T])$ as the set of functionals $F \in C_l^{0,0}$ which are differentiable $j$ times horizontally and $k$ times vertically at all $\omega \in D([0, t], \mathbb{R}^d), t < T$, with

- horizontal derivatives $D_t^m F, m \leq j$ continuous on $(D([0, t], \mathbb{R}^d), \|\cdot\|_{\infty})$ for each $t \in [0, T]$
- left-continuous vertical derivatives: $\forall n \leq k, \nabla^\omega_n F \in C_l^{0,0}$
- $D_t^m F, \nabla^\omega_n F \in B([0, T])$.

We can have $F \in C_b^{1,1}([0, T])$ while $F_t$ not Fréchet differentiable for any $t \in [0, T]$. 
Examples of regular functionals

Example (Cylindrical functionals)

For \( g \in C_0(\mathbb{R}^d) \),

\[
F_t(\omega_t) = [\omega(t) - \omega(t_n^-)] \quad 1_{t \geq t_n} \quad g(\omega(t_1^-), \omega(t_2^-), \ldots, \omega(t_n^-))
\]

is in \( C_{1,2} \)

More generally:

\[
F_t(\omega_t) = u(\omega(t) - \omega(t_n^-)) \quad 1_{t \geq t_n} \quad g(\omega(t_1^-), \omega(t_2^-), \ldots, \omega(t_n^-))
\]

where \( u \in C^2(\mathbb{R}^d, \mathbb{R}) \) with \( u(0) = 0 \).
Obstructions to regularity

Example (Jump of \( \omega \) at the current time)

\[ F_t(\omega_t) = \omega(t) - \omega(t-) \]

has regular pathwise derivatives:

\[ D_t F(\omega_t) = 0 \quad \nabla_\omega F_t(\omega_t) = 1 \]

But \( F \not\in \mathbb{C}^{0,0}_i \).

Example (Jump of \( \omega \) at a fixed time)

\[ F_t(\omega_t) = 1_{t \geq t_0}(\omega(t_0) - \omega(t_0-)) \]

\( F \in \mathbb{C}^{0,0} \) has horizontal and vertical derivatives at any order, but \( \nabla_\omega F_t(\omega_t) = 1_{t=t_0} \) fails to be left (or right) continuous.
Obstructions to regularity

Example (Maximum)

\[ F_t(\omega_t) = \sup_{s \leq t} \omega(s) \]
\[ F \in C^{0,0} \] but is not vertically differentiable on

\[ \{ \omega \in D([0, t], \mathbb{R}^d), \quad \omega(t) = \sup_{s \leq t} \omega(s) \}. \]
Derivatives of functionals defined on continuous paths

**Theorem**

*If\( F^1, F^2 \in \mathbb{C}^{1,1} \) coincide on continuous paths*

\[
\forall t < T, \quad \forall \omega \in C_0([0, t], \mathbb{R}^d), \quad F^1_t(\omega_t) = F^2_t(\omega_t)
\]

*then their pathwise derivatives also coincide:*

\[
\forall t < T, \quad \forall \omega \in C_0([0, t], \mathbb{R}^d), \quad \nabla_\omega F^1_t(\omega_t) = \nabla_\omega F^2_t(\omega_t), \quad D_t F^1_t(\omega_t) = D_t F^2_t(\omega_t)
\]
Quadratic variation for cadlag paths

Föllmer (1979): a path \( f \in D([0, T], \mathbb{R}) \) is said to have finite quadratic variation along a subdivision \( \pi_n = (0 = t_0^n < \ldots t_n^{k(n)} = T) \) if the measures:

\[
\xi^n = \sum_{i=0}^{k(n)-1} (f(t_{i+1}^n) - f(t_i^n))^2 \delta_{t_i^n}
\]

where \( \delta_t \) is the Dirac measure at \( t \), converge weakly to a Radon measure \( \xi \) on \([0, T]\) such that

\[
[f](t) = \xi([0, t]) = [f]^c(t) + \sum_{0 < s \leq t} (\Delta f(s))^2
\]

where \([f]^c\) is the continuous part of \([f]\). \([f]\) is called the quadratic variation of \( f \) along the sequence \((\pi_n)\).
Change of variable formula for cadlag paths

Let $\omega \in D([0, T] \times \mathbb{R}^d)$ where $\omega$ has finite quadratic variation along $(\pi_n)$ and

$$\sup_{t \in [0, T] - \pi_n} |\omega(t) - \omega(t-)| \to 0$$

Denote

$$\omega^n(t) = \sum_{i=0}^{k(n) - 1} \omega(t_{i+1}^-) 1_{[t_i, t_{i+1})}(t) + \omega(T) 1_{\{T\}}(t)$$

$$h^n_i = t^n_{i+1} - t^n_i$$
For any $F \in \mathbb{C}^{1,2}_{b}([0, T])$, the Föllmer integral, defined as

$$
\int_{0}^{T} \nabla_{\omega} F_{t}(\omega_{t-}) d^{\pi} \omega := \lim_{n \to \infty} \sum_{i=0}^{k(n)-1} \nabla_{\omega} F_{t_{i}^{n}}(\omega_{t_{i}^{n}-} \Delta \omega(t_{i}^{n}))(\omega(t_{i+1}^{n}) - \omega(t_{i}^{n}))
$$

exists and

$$
F_{T}(\omega_{T}) - F_{0}(\omega_{0}) = \int_{0}^{T} D_{t} F_{t}(\omega_{t-}) du
$$

$$
+ \int_{0}^{T} \frac{1}{2} \text{tr} \left( \nabla^{2}_{\omega} F_{t}(\omega_{t-}) d[\omega]^{c}(u) \right) + \int_{0}^{T} \nabla_{\omega} F_{t}(\omega_{t-}) d^{\pi} \omega
$$

$$
+ \sum_{u \in [0, T]} [F_{u}(\omega_{u}) - F_{u}(\omega_{u-}) - \nabla_{\omega} F_{u}(\omega_{u-}) \Delta \omega(u)]
$$
This pathwise formula implies a functional Ito formula for semimartingales:

**Theorem (Functional change of variable formula)**

Let $F \in \mathbb{C}^{1,2}_b([0, T])$ be a “smooth” non-anticipative functional.

$$
F_t(X_t) - F_0(X_0) = \int_0^t \mathcal{D}_u F(X_u) du + \\
\int_0^t \nabla \omega F_u(X_u).dX(u) + \int_0^t \frac{1}{2} \text{tr} (t \nabla^2 \omega F_u(X_u) \ d[X](u)) + \\
\sum_{u \in [0, T]} [F_u(X_u) - F_u(X_{u-}) - \nabla \omega F_u(X_{u-}).\Delta X(u)] \quad a.s.
$$

In particular, $Y(t) = F_t(X_t)$ is a semimartingale.
If \( F_t(X_t) = f(t, X(t)) \) where \( f \in C^{1,2}([0, T] \times \mathbb{R}^d) \) this reduces to the standard Ito formula.

\( Y = F(X) \) depends on \( F \) and its derivatives only via their values on continuous paths: \( Y \) can be reconstructed from the second-order jet of \( F \) on \( \mathcal{C}_c = \bigcup_{t \in [0, T]} C_0([0, t], \mathbb{R}^d) \subset \mathcal{Y} \).
Sketch of proof

Consider first a cadlag piecewise constant process:

\[ X(t) = \sum_{k=1}^{n} 1_{[t_k, t_{k+1}[}(t) \phi_k \quad \phi_k \mathcal{F}_{t_k} \text{ measurable} \]

Each path of \( X \) is a sequence of horizontal, vertical moves and jumps:

\[ X_{t_{k+1}} = (X_{t_k}, h_k)^{\phi_{k+1} - \phi_k} \quad h_k = t_{k+1} - t_k \]

\[
F_{t_{k+1}}(X_{t_{k+1}}) - F_{t_k}(X_{t_k}) =
F_{t_{k+1}}(X_{t_{k+1}}) - F_{t_{k+1}}(X_{t_{k+1}^-}) \quad \text{jump}
\]

\[
+ F_{t_{k+1}}(X_{t_{k+1}^-}) - F_{t_{k+1}}(X_{t_k}, h_k) \quad \text{vertical move}
\]

\[
+ F_{t_{k+1}}(X_{t_k}, h_k) - F_{t_k}(X_{t_k}) \quad \text{horizontal move}
\]
Sketch of proof

Horizontal step: fundamental theorem of calculus for 

$$\phi(h) = F_{t_k+h}(X_{t_k}, h)$$

$$F_{t_{k+1}}(X_{t_k}, h_k) - F_{t_k}(X_{t_k})$$

$$= \phi(h_k) - \phi(0) = \int_{t_k}^{t_{k+1}} D_t F(X_t) \, dt$$

Vertical step: apply Ito formula to 

$$\psi(u) = F_{t_{k+1}}(X_{t_k}^u, h_k)$$

$$F_{t_{k+1}}(X_{t_{k+1}}) - F_{t_{k+1}}(X_{t_k}, h_k) = \psi(X(t_{k+1}) - X(t_k)) - \psi(0)$$

$$= \int_{t_k}^{t_{k+1}} \nabla \omega F_t(X_t) \, dX + \frac{1}{2} \text{tr}(\nabla^2 \omega F_t(X_t) d[X])$$
Sketch of proof

General case: approximate $X$ by a sequence of simple predictable processes $nX$ with $nX(0) = X(0)$. For each $n \geq 1$,

$$F_T(nX_T) - F_0(X_0) = \int_0^T D_t F(nX_t) \, dt + \int_0^T \nabla \omega F(nX_t) \cdot dX$$

$$\sum_{\Delta nX(t) \neq 0} F(X_t) - F(X_{t-}) - \Delta X(t) \cdot \nabla \omega F(X_{t-}) + \frac{1}{2} \int_0^T \text{tr} \nabla^2 \omega F(nX_t) d[X]$$

The $C_b^{1,2}$ assumption on $F$ implies that all derivatives involved in the expression are left continuous in $d_\infty$ metric, which allows to control their convergence as $n \to \infty$ using dominated convergence theorem for stochastic integrals.
Extension: locally regular functional

If \((\tau_n)_{n \geq 0}\) is an increasing sequence of \textbf{optional} times with \(\tau_n \to T\) a.s. and \(F^n \in \mathbb{C}^{1,2}_b([0, T[)\) such that

\[
F_t(\omega) = \sum_{n \geq 1} 1_{[\tau_n(\omega), \tau_{n+1}(\omega)]}(t) \quad F^n_t(\omega)
\]

then \(Y(t) = F_t(X_t)\) is a semimartingale and

\[
F_t(X_t) - F_0(X_0) = \int_0^t D_u F(X_u) du + \int_0^t \nabla \omega F_u(X_u) \cdot dX(u) + \int_0^t \frac{1}{2} \text{tr} \left( t \nabla^2 \omega F_u(X_u) d[X](u) \right) + \sum_{u \in ]0, T]} [F_u(X_u) - F_u(X_{u-}) - \nabla \omega F_u(X_{u-}) \cdot \Delta X(u)] \quad \text{a.s.}
\]

This result covers in particular functionals involving exit times.
Tangent process: intrinsic definition

**Definition (Vertical derivative of a process)**

Define \( C_{1,2}^b(X) \) the set of processes \( Y \) which admit a functional representation in \( C_{1,2}^b \) in terms of \( X \):

\[
C_{1,2}^b(X) = \{ Y, \exists F \in C_{1,2}^b([0,T]), \ Y(t) = F_t(X_t) \text{ a.s.} \}
\]

If \( \det(\sigma.\sigma^*) > 0 \) a.s. then for \( Y \in C_{1,2}^b(X) \), the process:

\[
\nabla_X Y(t) = \nabla_\omega F_t(X_t)
\]

is uniquely defined up to an evanescent set, independently of the choice of \( F \in C_{1,2}^b \). We call \( \nabla_X Y \) the **vertical derivative** of \( Y \) with respect to \( X \).
In particular when $X$ is a standard Brownian motion, $A = I_d$:

**Definition**

Let $W$ be a standard $d$-dimensional Brownian motion. For any $Y \in \mathcal{C}^{1,2}_b(W)$ with representation $Y(t) = F_t(W_t)$, the predictable process

$$\nabla_W Y(t) = \nabla_\omega F_t(W_t)$$

is uniquely defined up to an evanescent set, independently of the choice of the representation $F \in \mathcal{C}^{1,2}_b$. 
Consider now the case where

\[ X(t) = \int_0^t \sigma(t) \, dW(t) + \int_0^t \int z \tilde{J}_X \, (dtdz) \]

is a martingale. Consider an \( \mathcal{F}_T \)-measurable functional \( H = H(X(t), t \in [0, T]) = H(X_T) \) with \( E[|H|^2] < \infty \) and define the martingale \( Y(t) = E[H|\mathcal{F}_t] \). It is well known (Ito, Jacod-Yor) that

\[ Y(T) = E[Y(T)] + \int_0^T \phi \, dX + \int_0^T \int_{\mathbb{R}^d} \psi(t, z) \tilde{J}_X \, (dtdz) \]

for some predictable process \( \phi \) and predictable random function \( \psi \) measurable with respect to the \( \mathbb{P} \)-completed filtration \( \sigma(\mathcal{N} \vee \mathcal{F}_t) \). Many applications –stochastic control, solution of BSDEs, hedging formulas in mathematical finance– involve the computation of \( \phi, \psi \).
Let $Y(t) = E[H|\mathcal{F}_t]$ where $H = H(X(t), t \in [0, T])$ with $E[|H|^2] < \infty$.

**Theorem**

If $Y \in C_b^{1,2}(X)$ then

$$Y(T) = E[Y(T)] + \int_0^T \nabla_X Y(t) dX(t) + \int_0^T \int_{\mathbb{R}^d} \left( F_t(X_{t-} + z1_{\{t\}}) - F_t(X_{t-}) \right) \tilde{J}_X(dt dz)$$

In particular $\phi, \psi$ have regular, $\mathcal{F}_t$–adapted versions (no need to complete).
Pathwise computation of martingale representations

Consider for example the case where $X$ is solution of a stochastic differential equation. Then we can numerically simulate $X$. Let $nX$ be the solution of a $n$-step Euler scheme and $\hat{Y}_n$ a Monte Carlo estimator of $Y$ obtained using $nX$.

- Compute the Monte Carlo estimator $\hat{Y}_n(t, nX_t^h(\omega))$.
- Bump the endpoint by $h$.
- Recompute the Monte Carlo estimator $\hat{Y}_n(t, nX_t^h(\omega))$ (with the same simulated paths).
- An estimator of the integrand is given by

$$\hat{\phi}_n(t, \omega) := \frac{\hat{Y}_n(t, nX_t^h(\omega)) - \hat{Y}_n(t, nX_t^h(\omega))}{h}$$
Pathwise computation of martingale representations

\[ \hat{\phi}_n(t, \omega) \sim \frac{\hat{Y}_n(t, nX^h_t(\omega)) - \hat{Y}_n(t, nX^h_t(\omega))}{h} \]

For a general $C^{1,2}_b(S)$ + some mild regularity assumptions

\[ \forall 1/2 > \epsilon > 0, n^{1/2-\epsilon} |\hat{\phi}_n(t) - \phi(t)| \to 0 \quad \mathbb{P} - a.s. \]

This rate is attained for a time-step size $h = cn^{-1/4+\epsilon/2}$. 
\[ \mathcal{I}^2(X) = \left\{ \int_0^T \phi \, dX, \quad \phi \text{ } \mathcal{F}_t \text{ adapted}, \quad E\left[ \int_0^T \|\phi(t)\|^2 d[X](t) \right] < \infty \right\} = \left\{ \int_0^T \phi \, dX, \phi \in L^2(X) \right\} \]

**Theorem (Non-anticipative integration by parts formula)**

Let \( Y \in C_{b}^{1,2}(X) \) be a \((\mathbb{P}, (\mathcal{F}_t))\)-martingale with \( Y(0) = 0 \) and \( \phi \) an \( \mathcal{F}_t \)-adapted process with \( E\left[ \int_0^T \|\phi(t)\|^2 d[X](t) \right] < \infty \). Then

\[
E \left( Y(T) \int_0^T \phi \, dX \right) = E \left( \int_0^T \nabla_X Y \cdot \phi \, d[X] \right)
\]

This allows to extend the functional Ito formula to the closure of \( C_{b}^{1,2}(X) \cap \mathcal{I}^2(X) \) wrt to the norm

\[
E \|Y(T)\|^2 = E \left[ \int_0^T \|\nabla_X Y(t)\|^2 d[X](t) \right]
\]
A “Martingale Sobolev space”

**Definition (Martingale Sobolev space)**

Define $\mathcal{W}^{1,2}(X)$ as the closure in $\mathcal{I}^2(X)$ of $D(X) = C_b^{1,2}(X) \cap \mathcal{I}^2(X)$.

**Lemma**

$\{\nabla_X Y, Y \in D(X)\}$ is dense in $\mathcal{L}^2(X)$ and

$$
\mathcal{W}^{1,2}(X) = \left\{ \int_0^T \phi \, dX, \quad E \int_0^T \|\phi\|^2 \, d[X] < \infty \right\}.
$$

So $\mathcal{W}^{1,2}(X)$ = all square-integrable integrals with respect to $X$. 
Weak derivative for stochastic integrals

**Theorem (Weak derivative on **$\mathcal{W}^{1,2}(X)$**))**

The vertical derivative $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$. Its closure defines a bijective isometry

$$\nabla_X : \mathcal{W}^{1,2}(X) \mapsto \mathcal{L}^2(X)$$

$$\int_0^T \phi. dX \mapsto \phi$$

characterized by the following integration by parts formula: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y$ is the unique element of $\mathcal{L}^2(X)$ such that

$$\forall Z \in D(X), \quad E[Y(T)Z(T)] = E \left[ \int_0^T \nabla_X Y(t) \nabla_X Z(t) d[X](t) \right].$$
We have thus shown that the (weak) vertical derivative $\nabla_X$ acts as the inverse of the Ito stochastic integral on $\mathcal{W}^{1,2}(X)$:

$$Y(t) = \int_0^t \phi \, dX$$

then

$$\nabla_X Y = \phi$$

i.e. $\nabla_X$ acts as a “stochastic derivative” (Zabczyk 1978).
Computation of the weak derivative

For $D(X) = C_b^{1,2}(X) \cap \mathcal{I}^2(X)$, the weak derivative may be computed \textbf{pathwise}:

$$\forall Y \in C_b^{1,2}(X), \quad \nabla_X Y(t, \omega) = \lim_{h \to 0} \frac{Y(t, X_t(\omega)) - Y(t, X_t(\omega))}{h}$$

For $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y = \lim_n \nabla_X Y^n$ where $Y^n \in D(X)$ with

$$E\|Y^n(T) - Y(T)\|_2^2 \to 0$$

If $\hat{Y}^n$ is an estimator of $Y$ computed from a discretization scheme for $X$ then one may use the following estimator for $\phi = \nabla_X Y$:

$$\nabla_X Y(t, X_t(\omega)) = \frac{\hat{Y}^n(t, X_t^h(n)(\omega)) - \hat{Y}^n(t, X_t(\omega))}{h(n)}$$

where $h(n) \sim cn^{-\alpha}$ is chosen according to the “smoothness” of the functional $Y$. 

Rama Cont & David Fournié
Consider the case where $X = W$. Then for $Y \in \mathcal{W}^{1,2}(W)$

$$Y(T) = E[Y(T)] + \int_0^T \nabla_\mathcal{W} Y(t) dW(t)$$

If $H = Y(T)$ is Malliavin-differentiable e.g. $H = Y(T) \in \mathcal{D}^{1,1}$ then the Clark-Haussmann-Ocone formula implies

$$Y(T) = E[Y(T)] + \int_0^T pE[d_t^H | \mathcal{F}_t] dW(t)$$

where $\mathcal{D}$ is the Malliavin derivative.
Theorem (Intertwining formula)

Let $Y$ be a $(\mathbb{P}, (\mathcal{F}_t^W)_{t \in [0,T]})$ martingale. If $Y \in C^{1,2}(W)$ and $Y(T) = H \in D^{1,2}$ then

$$E[\mathbb{D}_t H | \mathcal{F}_t] = (\nabla W Y)(t) \quad dt \times d\mathbb{P} - a.e.$$  

i.e. the conditional expectation operator intertwines $\nabla W$ and $\mathbb{D}$:

$$E[\mathbb{D}_t H | \mathcal{F}_t] = \nabla W (E[H|\mathcal{F}_t]) \quad dt \times d\mathbb{P} - a.e.$$
The following diagram is commutative, in the sense of $dt \times d\mathbb{P}$ almost everywhere equality:

$$
\begin{align*}
\mathcal{W}^{1,2}(\mathcal{W}) & \xrightarrow{\nabla} \mathcal{L}^{2}(\mathcal{W}) \\
\uparrow(E[.|\mathcal{F}_t])_{t \in [0,T]} & \xrightarrow{\mathbb{D}} (E[.|\mathcal{F}_t])_{t \in [0,T]} \\
D^{1,2} & \rightarrow L^{2}([0, T] \times \Omega)
\end{align*}
$$

Note however that $\nabla X$ may be constructed for any Itô-Lévy martingale $X$ and its construction does not involve Gaussian properties of $X$. 

Rama Cont & David Fournié

Functional Ito calculus
Consider now a $\mathbb{R}^d$-valued Lévy process $L$ with characteristic triplet $(b, A, \nu)$

Denote $\text{supp}(L)$ the topological support of the law of $L$ in $(D([0, T], \mathbb{R}^d), \|\cdot\|_\infty)$:

$$\text{supp}(L) = \{\omega, \text{any neighborhood } V \text{ of } \omega, \mathbb{P}(L \in V) > 0\}$$

The topological support may be characterized in terms of $(b, A, \nu)$ see e.g. Th Simon (2002)
Theorem (Harmonic functionals of a Lévy process)

Let $F \in \mathbb{C}^{1,2}_b$. Then $Y(t) = F_t(L_t)$ is a local martingale if and only if $F$ satisfies the following functional equation:

$$
\int_{\mathbb{R}^d} \left( F_t(\omega_{t-} + z 1_{\{t\}}) - F_t(\omega_{t-}) - 1_{|z| \leq 1} z \nabla \omega F_t(X_{t-}) \right) \nu(dz) + D_t F(\omega_t) + b \cdot \nabla \omega F_t(\omega_t) + \frac{1}{2} \text{tr} \left[ \nabla^2 \omega F_t(\omega_t) A \right] = 0
$$

for $\omega \in \text{supp}(X)$.

We call such functionals $X$–harmonic functionals.
In particular when $X = W$ is a $d$-dimensional Wiener process, we obtain a characterization of ‘regular’ Brownian local martingales as solutions of a functional PDE:

**Theorem**

Let $F \in \mathbb{C}^{1,2}_b$. Then $Y(t) = F_t(W_t)$ is a local martingale on $[0, T]$ if and only if

$$
\forall t \in [0, T], \quad \omega \in C_0([0, T], \mathbb{R}^d),
\quad \mathcal{D}_t F(\omega_t) + \frac{1}{2} \text{tr} \left( \nabla^2 \omega F(\omega_t) \right) = 0.
$$

Rama Cont & David Fournie
Theorem (Uniqueness of solutions)

Let $h$ be a continuous functional on $(D([0, T]), \| \cdot \|_\infty)$. Then any $F \in \mathbb{C}^{1,2}_b$ verifying $\forall \omega \in D([0, T])$,

$$
\int_{\mathbb{R}^d} \left( F_t(\omega_{t-} + z1_{\{t\}}) - F_t(\omega_{t-}) - 1_{|z|\leq 1} z.\nabla \omega F_t(X_{t-}) \right) \nu(dz) + \\
\mathcal{D}_t F(\omega_t) + b\nabla \omega F_t(\omega_t) + \frac{1}{2} \text{tr} [\nabla^2 \omega F(\omega_t) . A] = 0 \\
F_T(\omega) = h(\omega), \quad E[ \sup_{t \in [0,T]} |F_t(X_t)|] < \infty
$$

is uniquely defined on the topological support of $X$: if $F^1, F^2 \in \mathbb{C}^{1,2}_b([0, T])$ are two solutions then

$$
\forall \omega \in \text{supp}(X), \quad \forall t \in [0, T], \quad F^1_t(\omega_t) = F^2_t(\omega_t).
$$
Extensions and applications

- The results can be extended to functionals of cadlag semimartingales and Dirichlet processes with cadlag paths and functionals of the quadratic variation process (JFA 2010).
- All results (except global uniqueness for Functional PDE) can be localized using stopping times: important for applying to functionals involving stopped processes/exit times.
- Application: pathwise maximum principle for non-Markovian control problems.
- Infinite-dimensional extension to functionals of a Banach-valued process.
- Application: hedging of path-dependent derivatives.
References