Invariance principles for local times of Lévy processes and random walks.

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1 Introduction

If a sequence of Lévy processes $X^{(n)}$ converges a.s. on the Skorokhod space to a limiting Lévy process $X$, does the corresponding sequence of local times at a fixed level of $X^{(n)}$ converge to the local time of $X$ at that level?

NOT IN GENERAL: in fact of course the local times of $X^{(n)}$ may not even exist.

However, in fluctuation theory of Lévy processes, it is the local times at extrema which play a major rôle, not the local times at fixed levels, so do these local times converge? A similar question can be posed about the local times at extrema of a sequence of normed random walks $S^{(n)}$ which converge to a Lévy process.

To our knowledge, the only results known are in GOT (Greenwood et al, 1982) and DLG (Duquesne and Le Gall, 2004).
GOT deals with the "classical" case where $S^{(n)}$ is got by norming a fixed random walk $S$, the assumption being that for some norming sequence $c_n$, $(S_{nt}^{(n)}/c_n, t \geq 0)$ converges in law to $X$, necessarily stable, and the conclusion is that a normed version of the bivariate ladder process of $S$ converges in law to the bivariate ladder process of $X$. Hence a normed version of the local time at the maximum of $S$ converges in law to the local time at the supremum of $X$. (A different proof of this result and a converse result can be found in Doney and Greenwood, 1993.)

DLG considers a more general scenario where each $S^{(n)}$ is got by norming a different random walk, but restricts itself to the case that each random walk is downwards skip-free, so that the limiting Lévy process is automatically spectrally positive. (This is because the result they prove is a tool for the study of the height process of the sequence of Galton-Watson processes related to $S^{(n)}$.)

Again convergence in law is assumed, and the conclusion is again convergence in law of a normed version of the local time.
Our aim was to prove similar results, making no assumptions.

The motivation was;

- It seems to be a natural and fundamental question.

- The proof in DLG is clever, but obscures what is really going on

- We were aiming to use these results to extend some limit theorems for conditioned processes.
2 Preliminaries

We write $S = (S_k, k = 0, 1, \cdots)$ for a random walk with $S_0 = 0$ and, for $k \geq 1$, $S_k = \sum_{1}^{k} Y_r$, where $Y_1, Y_2, \cdots$ are independent and identically distributed.

We define the local time at its maximum of any random walk $S$ by:

$$\Lambda_k = \#\{j \in \{1, \ldots, k\} : S_j = M_j := \max_{i \leq j} S_i\}.$$ 

The strict ascending ladder time process $T$ of $S$ is defined by $T_0 = 0$ and for all $k \geq 0$,

$$T_{k+1} = \min\{j > T_k : S_j > S_{T_k}\}$$

and the strict ascending ladder height process $Z$ is given by

$$Z_k = S(T_k).$$

The following result is due to Spitzer:

$$E(r^{T_1} e^{itZ_1}) = \exp - \sum_{1}^{\infty} \frac{r^n}{n} E(e^{itS_n} : S_n > 0).$$
Let $X$ be any oscillating Lévy process for which $0$ is regular for the open half-line $(0, \infty)$. Let $R := \overline{X} - \underline{X}$, where $\overline{X}_t = \sup_{0 \leq s \leq t} X_s$, and $L$ be a (continuous) local time for $R$ at $0$. We will assume throughout that

$$\mathbb{E} \left( \int_0^\infty e^{-t} \, dL_t \right) = 1. \tag{1}$$

The ascending bivariate ladder process $(\tau, H)$ is got by setting $\tau_t = L^{-1}(t)$ and $H_t = X(\tau_t), t \geq 0$. This process is a bivariate subordinator whose Laplace exponent $\kappa(\alpha, \beta)$ is given by

$$\exp \left( \int_0^\infty \int_0^\infty (e^{-t} - e^{-\alpha t - \beta x}) t^{-1} \mathbb{P}\{X_t \in dx\} \, dt \right).$$

We write $\delta^H$ and $\Pi^H$ for the drift and Lévy measure of $H$.

**Note** the regularity implies that if $\delta^H = 0$ then $\Pi^H((0, \infty)) = \infty$. 
We will say that $S^{(n)} \xrightarrow{\text{law}} X$ (resp. $\xrightarrow{\text{a.s.}} X$) towards $X$ if the sequence $(S^{(n)}_{[nt]}, t \geq 0)$ converges weakly (resp. almost surely) toward $X$ on the Skorokhod space of càdlàg paths.

Recall that a result of Skorokhod shows that convergence in finite dimensional distribution of the process $(S^{(n)}_{[nt]}, t \geq 0)$ implies its weak convergence.

**Note** We can include the "classical case" by setting $S^{(n)}_k = S_k/c_n$
3 Main results

Theorem 1 Suppose the sequence of r.w. $S^{(n)}$ converges in law toward $X$, a $L^p$ satisfying our basic assumptions. Then we have the following convergence in law:

\[
\left[\left(n^{-1}T^{(n)}_{[a_n]}, Z^{(n)}_{[a_n]}\right), t \geq 0 \right] \xrightarrow{\text{law}} (\tau, H),
\]

where

\[
a_n = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} e^{-k/n} \mathbb{P}(S^{(n)}_k > 0) \right). \tag{2}
\]

The proof consists of using the 2 versions of Fridstedt’s formula and (2) to show that the Laplace exponent of $(T, Z)$ converges to $\kappa$.

Our main result is:
Theorem 2 Let $X$ be as in Theorem 1, and assume now that

$$(S_{nt}^{(n)}, t \geq 0) \xrightarrow{a.s.} (X_t, t \geq 0).$$  \hspace{1cm} (3)$$

Then a normed version of $\Lambda^{(n)}$ converges uniformly in probability on compacts towards $L$. More specifically, for all $t \geq 0$ and $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P} \left( \sup_{s \in [0,t]} |a_n^{-1} \Lambda_{nt}^{(n)} - L_s| > \varepsilon \right) = 0,$$

where again $a_n$ is defined by (2).
Sketch of proof:

Case 1: $H$ is compound Poisson + drift $\delta$. In this case

$$\delta L_t = \lambda(\overline{X}_s : s \leq t).$$

We approximate $\overline{X}$ by discarding all its jumps bigger than $a$, getting $\overline{X}^a$. If $M^{n,a}$ is a corresponding approximation to $M^{(n)}$, the max process of $S^{(n)}$, then

$$\lim_{n \to \infty} M^{n,a}_{nt} a.s. = \overline{X}^a_t.$$

For $a$ small enough, all jumps have been discarded, and $\delta L_t = \overline{X}^a_t$, so it is enough to control

$$D^a_n := \sup_{[0,t]} |M^{n,a}_{ns} - \frac{\delta}{a_n} \Lambda^{(n)}_{[ns]}|.$$

We conclude by a straightforward second moment argument...
Case 2: $\Pi^H(0, \infty) = \infty$. Need a different approx to $L$, and we use

$$L_{t}^{a,b} = \#(s \leq t : \Delta \bar{X}_s \in (a, b]).$$

The point is that $(L_{\tau_t}^{a,b}, t \geq 0)$ is a Poisson point process with intensity $\pi^H(a, b) \to \infty$ as $a \downarrow 0$, so by a strong law

$$\frac{L_{\tau_t}^{a,b}}{\pi^H(a, b)} \xrightarrow{a.s.} t, \text{ and so } \frac{L_{t}^{a,b}}{\pi^H(a, b)} \xrightarrow{a.s.} L_{t}.$$

On the other hand, we can approximate $L_{t}^{a,b}$ in the obvious way by $\#(j \leq nt : \Delta M_j^{(n)} \in (a, b])$, and then the proof follows similar lines.
When 0 is regular for \((-\infty, 0)\), we define the local time at the minimum of \(X\) to be the local time at the maximum of \(-X\). Let us denote this process by \(\hat{L}\) and denote by \(\hat{\Lambda}^{(n)}\) the local at the maximum of the approximating sequence \(-S^{(n)}\). Another consequence of the Theorem 1 is the following result.

**Corollary 3** Under the same assumptions we have

\[
\left[\left(S^{(n)}_{[nt]}, \frac{1}{a_n}\Lambda^{(n)}_{[nt]}\right), \ t \geq 0\right] \xrightarrow{\text{(law)}} \left[(X_t, L_t), \ t \geq 0\right].
\]

If in addition 0 is regular for \((-\infty, 0)\) then

\[
\left[\left(S^{(n)}_{[nt]}, \frac{1}{a_n}\Lambda^{(n)}_{[nt]}, \frac{1}{\hat{a}_n}\hat{\Lambda}^{(n)}_{[nt]}\right), \ t \geq 0\right] \xrightarrow{\text{(law)}} \left[(X_t, L_t, \hat{L}_t), \ t \geq 0\right],
\]

where \(\hat{a}_n = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} e^{-k/n} \mathbb{P}(S^{(n)}_{k} < 0)\right)\).
Next we suppose that there is a sequence of Lévy processes $X^{(n)}$, all of which satisfy the same hypothesis as $X$. Call $L^{(n)}$ the version of the local time of $X^{(n)}$ at its maximum, as defined earlier, remembering the normalisation. Then, by considering the sequence of rws defined by $S_{j}^{(n,k)} = X_{j/k}^{(n)}$ it is easy to deduce:

**Theorem 4**  
Suppose that as $n$ tends to $\infty$,  

$$X^{(n)} \xrightarrow{\text{a.s.}} X,$$  

then the sequence of local times $L^{(n)}$ converges uniformly on compacts in probability toward $L$, i.e. for all $t > 0$ and $\varepsilon > 0$,  

$$\lim_{n \to \infty} \mathbb{P} \left( \sup_{s \in [0,t]} |L_{t}^{(n)} - L_{t}| > \varepsilon \right) = 0.$$  

**Note** we can formulate Lévy process versions of the other rw results.
4 Limit Theorems for Conditioned processes

Here is an example of our results: the $\uparrow$ stands for "conditioned to stay positive", and $\Lambda$ and $L$ are local times at the future minimum.

**Theorem 5** Suppose that some sequence of random walks $S^{(n)}$ converges almost surely toward $X$.

The sequence of processes $(S^{(n)}_{[nt]}, t \geq 0)$ converges almost surely toward $X^\uparrow$.

The sequence $[(S^{(n)}_{[nt]}, a_n^{-1}\Lambda^{(n)}_{[nt]}), t \geq 0]$ converges in probability toward $(X^\uparrow, L)$. 

Consequently, if some sequence of random walks $S^{(n)}$ converges weakly toward $X$, then the sequence $[(S^{(n)}_{[nt]}, a_n^{-1}\Lambda^{(n)}_{[nt]}), t \geq 0]$ converges weakly toward $(X^\uparrow, L)$. 
The main ingredients in the proof of this are Tanaka’s construction of $S^{(n)} \uparrow$ and $X \uparrow$, and time reversal.

We have a similar result for the meander, which informally is the process conditioned to stay positive for unit time. The proof uses the absolute continuity relation between this process and $X \uparrow$.