On a HJM approach for stock options
Setting Lévy in motion

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Outline

1. Introduction
2. The philosophy of HJM
3. Setting Lévy in motion
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Modelling stock options
Classical approach

- stock $S(t)$
- call options $C(t, T, K)$ with $C(T, T, K) = \max\{S(T) - K, 0\}$

Start with econometric model for the stock, e.g. geometric Brownian motion.

Use financial mathematics to derive option prices, e.g. arbitrage considerations.

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\[ C(0, T, K) = E_Q(\max\{ S(T) - K, 0 \}) \]
Choose parametric model for $S$ under $Q$.
Determine parameters such that observed option prices coincide with model prices (calibration).
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Modelling stock options

Second way out

- Treat vanilla options as primary assets (HJM-kind approach).
- some references:
Heath-Jarrow-Morton (HJM) in interest rate theory
Heath et al. (1992)

- money market account $S_0(t) = \exp(\int_0^t r(s)ds)$

- view bonds as primary assets: $B(t, T)$ with $B(T, T) = 1$

- reparametrisation: forward rates $f(t, T) = \frac{\partial}{\partial T} \log B(t, T)$

- dynamics: $df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t)$

- HJM drift condition: $\alpha(t, T) = \frac{\partial}{\partial T} \psi(\Sigma(t, T))$ with $\Sigma(t, T) := \int_t^T \sigma(t, s)ds$, $\psi(u) := \frac{u^2}{2}$

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What is essential about HJM ...

... if we want to transfer it to stock options etc.?

- Do not model the canonical reference asset in detail.
- Treat whole manifold (curve/surface) of liquid derivatives as primary assets.
- Do not model them immediately, but use convenient parametrisation instead (codebook).
- Key to convenient parametrisation: use family of simple models for the canonical reference asset, having the same dimension as the manifold under consideration.
- Set it in motion, i.e. design a stochastic model for this codebook (which is supposed to be deterministic in the simple model).
- Derive dynamics of the canonical reference asset from consistency condition.
- Derive drift part of the codebook dynamics from drift condition.
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Example 1: Interest rate theory
Heath et al. (1992)

- Canonical reference asset: money market account
  \[ dS_0(t) = S_0(t)r(t)dt \]
- Liquid derivatives: bonds \( B(t, T) \)
- Simple models: \( dS_0(t) = S_0(t)r(t)dt \) with deterministic short rate \( r(t), t \geq 0 \).
  - in this setup: bond prices \( B(t, T) = \exp(- \int_t^T r(s)ds) \).
  - this implies: \( r(T) = -\frac{\partial}{\partial T} \log B(t, T) \)
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Example 2: Credit derivative models
Bennani (2005)

- Canonical reference process: cumulative portfolio loss $L(t)$
- “Liquid” derivatives: $T$-forward loss $L(t, T) = E(L(T)|\mathcal{F}_t)$
- Simple models: $dL(t) = -(1 - L(t))x(t)dt$ with deterministic loss rate $x(t), t \geq 0$.
  - in this setup: bond prices $L(t, T) = 1 - \exp(-\int_t^T x(s)ds)$.
  - this implies: $x(T) = -\frac{\partial}{\partial T} \log(1 - L(t, T))$
- This inspires the codebook $X(t, T) = -\frac{\partial}{\partial T} \log(1 - L(t, T))$.
- Set it in motion: $df(t, T) = \alpha(t, T)dt + \beta(t, T)dW(t) + \text{jumps}$.
- Consistency condition?
- Drift condition: $\alpha(t, T) = -\frac{\partial}{\partial T} \psi(B(t, T))$ with $B(t, T) := \int_t^T \beta(t, s)ds, \quad \psi(u) := \frac{u^2}{2} + \text{jump terms}$
Example 3: Credit derivative models
Schönbucher (2005)

- Canonical reference process: cumulative portfolio loss $L(t)$
- Liquid credit derivatives: $C(t, T, K)$ with $C(T, T, K) = (L(T) - K)^+$
- Simple models: $L(t)$ time-inhomogeneous continuous-time Markov chain with simple transition matrix
- Codebook: instantaneous transition probabilities $a(t, T, x)$ of this Markov chain
- Set it in motion: $da(t, T, x) = \alpha(t, T, x)dt + \beta(t, T, x)dW(t)$.
- Consistency condition: $a(t, t, L(t)) = \text{jump intensity of } L$
- Drift condition: $\alpha(t, T, x) = \ldots$
Example 4: Calls with fixed strike \( K \)
Schönbucher (1999), Schweizer & Wissel (2008)

- Canonical reference asset: stock \( dS(t) = S(t)\sigma(t)dB(t) \)
- Liquid derivatives: calls \( C(t, T) \) with maturity \( T \)
- Simple models: \( dS(t) = S(t)\sigma(t)dB(t) \) with deterministic volatility \( \sigma(t), t \geq 0 \).
  - in this setup: call prices \( C(t, T) = BS_{S(t),K}(\int_t^T \sigma^2(s)ds) \) from Black-Scholes formula
  - conversely \( \sigma^2(T) = \frac{\partial}{\partial T}(BS^{-1}_{S(t),K}(C(t, T))) \)
- This inspires the codebook: \( X(t, T) = \frac{\partial}{\partial T}(BS^{-1}_{S(t),K}(C(t, T))) \)
- Set it in motion: \( dX(t, T) = \alpha(t, T)dt + \beta(t, T)dW(t) \)
- Consistency condition: \( \sigma^2(t) = X(t, t) + \ldots \)
- Drift condition: \( \alpha(t, T) = \ldots \)
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- Liquid derivatives: calls $C(t, T)$ with maturity $T$
- Simple models: $dS(t) = S(t)\sigma(t)dB(t)$ with deterministic volatility $\sigma(t)$, $t \geq 0$.
  - in this setup: call prices $C(t, T) = BS_{S(t),K}(\int_t^T \sigma^2(s)ds)$ from Black-Scholes formula
  - conversely $\sigma^2(T) = \frac{\partial}{\partial T}(BS_{S(t),K}^{-1}(C(t, T)))$
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1 Introduction

2 The philosophy of HJM

3 Setting Lévy in motion
Calls/puts with all maturities/strikes

Setting Lévy in motion

- Canonical reference asset: stock
  \[ S(t) = \exp(X(t)), \]
  where \( X \) semimartingale with “local exponent” \( \psi_X(t, u) \)
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Setting Lévy in motion

- Simple models: exponential time-inhomogeneous Lévy process
  \( S(t) = S(0) \exp(X(t)) \), where \( E(e^{iuX(t)}) = \exp(\int_0^t \psi(s,u) \, ds) \).

  - in this setup: call prices
    \[ C(t, T, K) = K \mathcal{F}f(T, \log \frac{K}{S(t)}) \]
    with \( f(u) = \frac{1-\exp(\int_t^T \psi(t,s,u) \, ds)}{u^2 + iu} \) and \( \mathcal{F}f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} f(u) \, du \)

  - conversely
    \[ \psi(T, u) = \frac{\partial}{\partial T} \log(1 - (u^2 + iu) \mathcal{F}g(u)) \]
    with \( g(x) = \frac{C(t, T, S(t) e^x)}{K} \) and \( \mathcal{F}f(u) = \int_{-\infty}^{\infty} e^{iux} f(x) \, dx \)
Calls/puts with all maturities/strikes

Setting Lévy in motion

- Simple models: exponential time-inhomogeneous Lévy process
  \( S(t) = S(0) \exp(X(t)) \), where \( E(e^{iuX(t)}) = \exp(\int_0^t \psi(s, u)ds) \).

  - in this setup: call prices
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    C(t, T, K) = K \overline{F}f(T, \log \frac{K}{S(t)})
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    with \( f(u) = \frac{1 - \exp(\int_t^T \psi(t, s, u)ds)}{u^2 + iu} \) and \( \overline{F}f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} f(u)du \)

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Setting Lévy in motion

- This inspires the codebook:

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- Set it in motion: \( d\psi(t, T, u) = \alpha(t, T, u)dt + \beta(t, T, u)dM(t) \)

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Example 1: Time-inhomogeneous Lévy processes
the trivial example

- \[ S(t) = \exp(X(t)) \]
- \[ d\psi(t, T, u) = \alpha(t, T, u)dt + \beta(t, T, u)dM(t) \]
- consider
  \[ \beta(t, T, u) \equiv 0 \]

\[ \sim \psi(t, T, u) \text{ deterministic, constant in } t, \]
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Example 2: A simple dynamic codebook model
a simple nontrivial example

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$$\beta(t, T, u) = \frac{-iu + u^2}{2}e^{-\lambda(T-t)}$$

$\leadsto$ stock return process of the form

$$d\log(S(t)) = dL(t) - \frac{1}{2}v(t)dt + \sqrt{v(t)}dW(t) + \rho dM(t),$$

$$dv(t) = -\lambda v(t)dt + dM(t),$$

i.e. slight generalisation of the Barndorff-Nielsen & Shephard (2001) stochastic volatility model
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Conclusion

- HJM-kind approach provides general framework for option surface modelling
- concrete models etc.? \(\sim\) econometrics
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