More about limits of nested subclasses of infinitely divisible distributions

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Outline

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2. Classes
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5. The limiting classes
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1 Preliminaries

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Notation

- $\mathcal{L}(X)$: the law of $\mathbb{R}^d$-valued random variable $X$
- $I(\mathbb{R}^d)$: the class of all infinitely divisible distributions on $\mathbb{R}^d$
- $I_{\log}^m(\mathbb{R}^d) = \{ \mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} (\log^+ |x|)^m \mu(dx) < \infty \}, \ m \in \mathbb{N}$
- $I_{\log}(\mathbb{R}^d) = I_{\log}^1(\mathbb{R}^d)$
- $\{ X_t^{(\mu)}, t \geq 0 \}$: the $\mathbb{R}^d$-valued Lévy process with $\mathcal{L}(X_1^{(\mu)}) = \mu$
- $S(\mathbb{R}^d)$: the class of all stable distributions on $\mathbb{R}^d$
- $SS(b, \mathbb{R}^d)$: the class of all semi-stable distributions with $b > 1$
  
  ($= \{ \mu \in I(\mathbb{R}^d) : \exists a > 1, \exists c \in \mathbb{R}^d \text{ s.t. } \hat{\mu}(z)^a = \hat{\mu}(bz)e^{i\langle c, z \rangle}, z \in \mathbb{R}^d \}$)

- $S(\mathbb{R}^d) = \bigcap_{b > 1} SS(b, \mathbb{R}^d)$
Proposition (Polar decomposition of Lévy measure $\nu$)

\[
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),
\]

where $\lambda$ is a measure on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$, $\{\nu_\xi : \xi \in S\}$ is a family of positive measures on $(0, \infty)$, $\lambda$ and $\{\nu_\xi\}$ are uniquely determined by multiplications of $c(\xi), c(\xi)^{-1}$. We call $\nu_\xi$ the radial component of $\nu$. 

$(A, \nu, \gamma) :$ Lévy-Khintchine triplet of $\mu \in I(\mathbb{R}^d)$
Stochastic integral mappings I

For a nonrandom measurable function $f : [0, \infty) \to \mathbb{R}$, let

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(t) \, dX_t^{(\mu)}\right), \quad \mu \in \mathcal{D}(\Phi_f).$$

- Domain: $\mathcal{D}(\Phi_f) = \left\{ \mu \in I(\mathbb{R}^d) : \int_0^\infty f(t) \, dX_t^{(\mu)} \text{ is definable} \right\}$.
- Range: $\mathcal{R}(\Phi_f) = \Phi_f(\mathcal{D}(\Phi_f))$.

For $m \in \mathbb{N}$, $\Phi_f^m$ is the $m$ times composite of $\Phi_f$ itself.
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Classes I

- $U(\mathbb{R}^d)$ (the Jurek class):

  $$\nu_\xi(dr) = \ell_\xi(r)dr,$$

  where $\ell_\xi(r)$ is decreasing in $r \in (0, \infty)$.

- $B(\mathbb{R}^d)$ (the Goldie–Steutel–Bondesson class):

  $$\nu_\xi(dr) = \ell_\xi(r)dr,$$

  where $\ell_\xi(r)$ is completely monotone in $r \in (0, \infty)$.

- $L(\mathbb{R}^d)$ (the class of selfdecomposable distributions):

  $$\nu_\xi(dr) = k_\xi(r)r^{-1}dr,$$

  where $k_\xi(r)$ is decreasing in $r \in (0, \infty)$. 
Classes II

- $T(\mathbb{R}^d)$ (the Thorin class):
  \[
  \nu_\xi(dr) = k_\xi(r)r^{-1}dr,
  \]
  where $k_\xi(r)$ is completely monotone in $r \in (0, \infty)$.

- $G(\mathbb{R}^d)$ (the class of generalized type $G$ distributions):
  \[
  \nu_\xi(dr) = g_\xi(r^2)dr,
  \]
  where $g_\xi(r)$ is completely monotone in $r \in (0, \infty)$.

- $M(\mathbb{R}^d)$ (the class $M$):
  \[
  \nu_\xi(dr) = g_\xi(r^2)r^{-1}dr,
  \]
  where $g_\xi(r)$ is completely monotone in $r \in (0, \infty)$. 
Classes III

- $B(\mathbb{R}^d) \cup L(\mathbb{R}^d) \cup G(\mathbb{R}^d) \subsetneq U(\mathbb{R}^d)$ and $T(\mathbb{R}^d) \subsetneq L(\mathbb{R}^d)$.
- Each class above $\supsetneq S(\mathbb{R}^d)$.
- $T(\mathbb{R}^d) \subsetneq B(\mathbb{R}^d) \subsetneq G(\mathbb{R}^d)$.

Remark

There is no relation between $B(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$, and also no clarion between $G(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$. Actually, the following are true.

\[
B(\mathbb{R}^d) \setminus L(\mathbb{R}^d) \neq \emptyset, \\
L(\mathbb{R}^d) \setminus B(\mathbb{R}^d) \neq \emptyset, \\
G(\mathbb{R}^d) \setminus L(\mathbb{R}^d) \neq \emptyset, \\
L(\mathbb{R}^d) \setminus G(\mathbb{R}^d) \neq \emptyset.
\]
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Stochastic integral mappings I

These six classes can be characterized as the ranges of stochastic integral mappings with respect to Lévy processes from $I(\mathbb{R}^d)$ (or $I_{\log}(\mathbb{R}^d)$) into $I(\mathbb{R}^d)$.

- $\mathcal{U}$-mapping (Jurek (1985)):
  \[
  \mathcal{U}(\mu) = \mathcal{L} \left( \int_0^1 t dX_t^{(\mu)} \right), \quad \mu \in I(\mathbb{R}^d).
  \]

- $\Upsilon$-mapping (Barndorff-Nielsen, M. and Sato (2006)):
  \[
  \Upsilon(\mu) = \mathcal{L} \left( \int_0^1 \log(t^{-1}) dX_t^{(\mu)} \right), \quad \mu \in I(\mathbb{R}^d).
  \]

- $\Phi$-mapping (Wolfe (1982) and others):
  \[
  \Phi(\mu) = \mathcal{L} \left( \int_0^\infty e^{-t} dX_t^{(\mu)} \right), \quad \mu \in I_{\log}(\mathbb{R}^d).
  \]
Stochastic integral mappings II

- **Ψ-mapping** (Barndorff-Nielsen, M. and Sato (2006)):
  Let $e(x) = \int_x^\infty u^{-1} e^{-u} du, x > 0$ and $e^*(t)$ its inverse function.

  $$\Psi(\mu) = \mathcal{L} \left( \int_0^\infty e^*(t) dX_t^{(\mu)} \right), \mu \in I_{\log}(\mathbb{R}^d).$$

- **G-mapping** (M. and Sato (2009)):
  Let $h(x) = \int_x^\infty e^{-u^2} du, x > 0$ and $h^*(t)$ its inverse function.

  $$G(\mu) = \mathcal{L} \left( \int_0^{\sqrt{\pi}/2} h^*(t) dX_t^{(\mu)} \right), \mu \in I(\mathbb{R}^d).$$

- **M-mapping** (Aoyama, M. and Rosiński (2008)):
  Let $m(x) = \int_x^\infty e^{-u^2} u^{-1} du, x > 0$ and $m^*(t)$ its inverse function.

  $$M(\mu) = \mathcal{L} \left( \int_0^\infty m^*(t) dX_t^{(\mu)} \right), \mu \in I_{\log}(\mathbb{R}^d).$$
The following are characterizations of six classes by mappings mentioned now.

**Proposition**

- $U(\mathbb{R}^d) = U(I(\mathbb{R}^d))$. (Jurek (1985))
- $B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d))$. (Barndorff-Nielsen, M. and Sato (2006))
- $L(\mathbb{R}^d) = \Phi(I_{\log}(\mathbb{R}^d))$. (Wolfe (1982) and others.)
- $T(\mathbb{R}^d) = \Psi(I_{\log}(\mathbb{R}^d))$. (Barndorff-Nielsen, M. and Sato (2006))
- $G(\mathbb{R}^d) = G(I(\mathbb{R}^d))$. (M. and Sato (2009))
- $M(\mathbb{R}^d) = \mathcal{M}(I_{\log}(\mathbb{R}^d))$. (Aoyama, M. and Rosiński (2008)))
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Put

\[ U_0(\mathbb{R}^d) = U(\mathbb{R}^d), \quad B_0(\mathbb{R}^d) = B(\mathbb{R}^d), \quad L_0(\mathbb{R}^d) = L(\mathbb{R}^d), \]
\[ T_0(\mathbb{R}^d) = T(\mathbb{R}^d), \quad G_0(\mathbb{R}^d) = G(\mathbb{R}^d), \quad M_0(\mathbb{R}^d) = M(\mathbb{R}^d). \]

In the following, the \( m \)-th power of the mapping means the \( m \) times compositions of the mapping.
Nested subclasses II

Definition

For $m = 0, 1, 2, \ldots$, define

1. $U_m(\mathbb{R}^d) = U^{m+1}(I(\mathbb{R}^d))$,
2. $B_m(\mathbb{R}^d) = \Upsilon^{m+1}(I(\mathbb{R}^d))$,
3. $L_m(\mathbb{R}^d) = \Phi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d))$,
4. $T_m(\mathbb{R}^d) = \Psi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d))$,
5. $G_m(\mathbb{R}^d) = G^{m+1}(I(\mathbb{R}^d))$,
6. $M_m(\mathbb{R}^d) = \mathcal{M}^{m+1}(I(\mathbb{R}^d))$.

and further $U_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} U_m(\mathbb{R}^d)$, $B_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} B_m(\mathbb{R}^d)$, $L_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} L_m(\mathbb{R}^d)$, $T_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} T_m(\mathbb{R}^d)$, $G_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} G_m(\mathbb{R}^d)$, $M_\infty(\mathbb{R}^d) = \bigcap_{m=0}^{\infty} M_m(\mathbb{R}^d)$.

The following is known.
Nested subclasses III

Proposition (Urbanik (1973), Sato (1980))

\[ L_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}. \]
(Here the closure is taken under weak convergence and convolution.)

Proposition (Jurek (2004))

\[ U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d). \]
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The limiting class I

**Theorem**

\[ U_\infty(\mathbb{R}^d) = B_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d) = T_\infty(\mathbb{R}^d) = G_\infty(\mathbb{R}^d) = M_\infty(\mathbb{R}^d) = S(\mathbb{R}^d). \]

For the proof, see Maejima and Sato, “The limits of nested subclasses of several classes of infinitely divisible distributions are identical with the closure of the class of stable distributions,” Probab. Theory Relat. Fields **145** (2009), 119–142, and for that \( M_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)} \), see Aoyama, Lindner and Maejima, “A new family of mappings of infinitely divisible distributions related to the Goldie-Steutel-Bondesson class,” Elect. J. Probab. **15** (2010), 1119–1142.
The next problem I

Is $S'({\mathbb{R}^d})$ the only class obtained as a limit of this type of the iterations of mappings?
Do other classes appear?
Smaller classes 1

Theorem (A characterization of $L_\infty(\mathbb{R}^d)$ (Sato (1980)))

$\mu \in L_\infty(\mathbb{R}^d)$ if and only if

$$\nu(B) = \int_{(0,2)} \Gamma^\mu(d\alpha) \int_S \lambda_\alpha(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where $\Gamma^\mu$ is a measure on $(0,2)$ satisfying the following.

$$\int_{(0,2)} \left( \frac{1}{\alpha} + \frac{1}{2 - \alpha} \right) \Gamma^\mu(d\alpha) < \infty.$$

Here $\Gamma^\mu$ is unique and so it can be one of characteristics of $\mu$.

Definition

For $A \in \mathcal{B}((0,2))$, $L_\infty^A(\mathbb{R}^d) := \{ \mu \in L_\infty(\mathbb{R}^d) : \Gamma^\mu ((0,2) \setminus A) = 0 \}$. 
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Smaller classes II

- Mappings $\Phi_\alpha, \alpha < 2$. 
  Let

\[
\Phi_\alpha(\mu) = \begin{cases} 
\mathcal{L} \left( \int_0^{-1/\alpha} (1 + \alpha t)^{-1/\alpha} dX_t^{(\mu)} \right), & \text{when } \alpha < 0, \\
\mathcal{L} \left( \int_0^\infty e^{-t} dX_t^{(\mu)} \right), & \text{when } \alpha = 0, \\
\mathcal{L} \left( \int_0^\infty (1 + \alpha t)^{-1/\alpha} dX_t^{(\mu)} \right), & \text{when } 0 < \alpha < 2. 
\end{cases}
\]
Smaller classes III

- Mappings $\Psi_{\alpha,\beta}$, $\alpha < 2$, $\beta > 0$:

Let

$$t = G_{\alpha,\beta}(s) = \int_{s}^{\infty} u^{-\alpha-1} e^{-u^\beta} du, \quad s \geq 0,$$

and let $s = G_{\alpha,\beta}^*(t)$ be its inverse function. Define

$$\Psi_{\alpha,\beta}(\mu) = \mathcal{L} \left( \int_{0}^{G_{\alpha,\beta}(0)} G_{\alpha,\beta}^*(t)dX_{t}^{(\mu)} \right),$$

where

$$G_{\alpha,\beta}(0) = \begin{cases} \beta^{-1}\Gamma(-\alpha\beta^{-1}), & \text{when } \alpha < 0, \\ \infty, & \text{when } \alpha \geq 0. \end{cases}$$

These mappings are introduced first by Sato (2006) for $\beta = 1$ and later by M. and Nakahara (2009) for general $\beta > 0$. 
Due to Sato (2006), M. and Nakahara (2009), we have the domains $\mathcal{D}(\Phi_\alpha)$ and $\mathcal{D}(\Psi_{\alpha,\beta})$ as follows. Let $\beta > 0$.

$$\mathcal{D}(\Phi_\alpha) = \mathcal{D}(\Psi_{\alpha,\beta}) = \begin{cases} I(\mathbb{R}^d), & \text{when } \alpha < 0, \\ I_{\log}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ I_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ I_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ I_0^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2, \end{cases}$$

where

$$I_\alpha(\mathbb{R}^d) = \left\{ \mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \right\}, \quad \text{for } \alpha > 0,$$

$$I_0^0(\mathbb{R}^d) = \left\{ \mu \in I_\alpha(\mathbb{R}^d) : \int_{\mathbb{R}^d} x \mu(dx) = 0 \right\}, \quad \text{for } \alpha \geq 1,$$

$$I_1^*(\mathbb{R}^d) = \left\{ \mu = \mu(A,\nu,\gamma) \in I_0^0(\mathbb{R}^d) : \lim_{T \to \infty} \int_1^T t^{-1} dt \int_{|x| > t} x \nu(dx) \text{ exists in } \mathbb{R}^d \right\}.$$
Smaller classes V

- \( \mathcal{U} = \Phi_{-1} \), \( \Upsilon = \Psi_{-1,1} \), \( \Phi = \Phi_0 \), \( \Psi = \Psi_{0,1} \), \( \mathcal{G} = \Psi_{-1,2} \), \( \mathcal{M} = \Psi_{0,2} \).

**Theorem (Sato (2010), M. and Ueda (2010))**

\[
\bigcap_{m=1}^{\infty} \mathcal{R}(\Phi^m_{\alpha}) = \bigcap_{m=1}^{\infty} \mathcal{R}(\Psi^m_{\alpha,1})
\]

\[
= \begin{cases} 
L_{\infty}(\mathbb{R}^d), & \text{for } \alpha \in (-\infty, 0], \\
L^{(\alpha,2)}_{\infty}(\mathbb{R}^d), & \text{for } \alpha \in (0, 1), \\
\left\{ \mu = \mu(A, \nu, \gamma) \in L^{(1,2)}_{\infty}(\mathbb{R}^d) : \right. \\
\lim_{\varepsilon \downarrow 0} \int_{(1,2)} \frac{B(3-\beta, \beta+1)}{\beta - 1} \left( \frac{1}{\Gamma(\beta - 1)} \int_{\varepsilon}^{\infty} t^{\beta-2} e^{-t} dt \right) \Gamma^\mu(\beta) \int S \xi \chi_{\mu}^\beta(d\xi) = -\gamma \right\}, & \text{for } \alpha = 1, \\
\left\{ \mu \in L^{(\alpha,2)}_{\infty}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x \mu(dx) = 0 \right\}, & \text{for } \alpha \in (1, 2). 
\end{cases}
\]
Theorem (M. and Ueda (2010))

For $\beta > 0$,

$$\bigcap_{m=1}^{\infty} \mathcal{R}(\Psi_{\alpha,\beta}^m) = \begin{cases} 
L_{\infty}(\mathbb{R}^d), & \text{for } \alpha \in (-\infty, 0], \\
L_{\infty}^{(\alpha,2)}(\mathbb{R}^d), & \text{for } \alpha \in (0, 1), \\
\left\{ \mu \in L_{\infty}^{(\alpha,2)}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x \mu(dx) = 0 \right\}, & \text{for } \alpha \in (1, 2) \setminus \{ 1 + n\beta : n \in \mathbb{N} \}. 
\end{cases}$$
Remark

For $\Phi_f = \Phi_\alpha$, $\Psi_{\alpha, 1}$, $\alpha < 2$, and $\Psi_{\alpha, \beta}$, $\beta > 0$, $\alpha \in (-\infty, 2) \setminus \{1 + n\beta : n \in \mathbb{Z}_+\}$, it holds that

$$\bigcap_{m=1}^{\infty} \mathcal{K}(\Phi_f^m) = \mathcal{K}(\Phi_f) \cap \overline{S(\mathbb{R}^d)}.$$

In this sense, the iterated limits of $\Phi_f$'s above are essentially the same as $\overline{S(\mathbb{R}^d)}$. 

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Infinitely divisible distributions

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Different classes $\overline{SS(\mathbb{R}^d)}$

Let $b > 1$. Define a mapping $\Phi_b$ by

$$\Phi_b(\mu) = \mathcal{L} \left( \int_0^\infty b^{-[t]} dX_t(\mu) \right), \quad \mu \in \mathcal{D}(\Phi_b) = I_{\log}(\mathbb{R}^d),$$

where $[t]$ denotes the largest integer not greater than $t \in \mathbb{R}$.

**Theorem (M. and Ueda (2009))**

$$\Phi_b \left( I_{\log}(\mathbb{R}^d) \right) = L(b, \mathbb{R}^d),$$

which is the class of all semi-selfdecomposable distributions with span $b$ on $\mathbb{R}^d$.

**Theorem (M. and Ueda (2009))**

$$\lim_{m \to \infty} \Phi^m_b \left( I_{\log^m}(\mathbb{R}^d) \right) = \overline{SS(b, \mathbb{R}^d)},$$

which is a bigger class than $\overline{S(\mathbb{R}^d)}$. 

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Infinitely divisible distributions

Conference on Lévy Processes
Thank you very much for your attention.
M. Maejima and G. Nakahara.
A note on new classes of infinitely divisible distributions on $\mathbb{R}^d$.

M. Maejima and Y. Ueda.
Nested subclasses of the class of $\alpha$-selfdecomposable distributions.

M. Maejima and Y. Ueda.
Stochastic integral characterizations of semi-selfdecomposable distributions and related Ornstein-Uhlenbeck type processes.

M. Maejima and Y. Ueda.
Compositions of mappings of infinitely divisible distributions with applications to finding the limits of some nested subclasses.

K. Sato.
Two families of improper stochastic integrals with respect to Lévy processes.

K. Sato.
Description of limits of ranges of iterations of stochastic integral mappings of infinitely divisible distributions