Symmetrization of Lévy processes and applications

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Isoperimetric inequalities

Classical Inequality

Among all figures of equal perimeter, the circle encloses the largest area.
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Equivalently: Among all regions of equal area, the disk has the smallest perimeter.
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Generalized Isoperimetric Inequalities

Among all regions of fixed measure, there is a large class of quantities that are maximized, or minimized, by the corresponding quantities for the ball: surface area, eigenvalues, capacities, exit times, etc.
Isoperimetric problem for exit times

Let $D$ be a domain in $\mathbb{R}^d$ with finite measure, and let $\tau_D$ the first exit time of $B_t$ from a domain $D$.

**Question**

Assuming same measure, which of the two figures have a largest survival time?
Isoperimetric problem for exit times

Let $D$ be a domain in $\mathbb{R}^d$ with finite measure, and let $\tau_D$ the first exit time of $B_t$ from a domain $D$.

**Question**

*Assuming same measure, which of the two figures have a largest survival time*

**Answer:** Obviously the ball
Define $D^* = B(0, R(D))$ the open ball centered at the origin "0" of same volume as $D$.

**Theorem**

*Let $D$ be a domain of finite area, then for all $z \in D$

$$P^z(\tau_D > t) \leq P^0(\tau_{D^*} > t)$$
Define $D^* = B(0, R(D))$ the open ball centered at the origin "0" of same volume as $D$.

**Theorem**

Let $D$ be a domain of finite area, then for all $z \in D$

$$P^z(\tau_D > t) \leq P^0(\tau_{D^*} > t)$$

"The isoperimetric theorem, deeply rooted in our experience and intuition so easy to conjecture, but not so easy to prove, is an inexhaustible source of inspiration." G. Pólya: Mathematics and Plausible Thinking.
Isoperimetric problem for exit times

Let $X_t$ be a symmetric $\alpha$-stable processes in $\mathbb{R}^d$, and $\tau_D^\alpha$ the first exit time of $X_t$ from $D$.

$$P^z (\tau_D^\alpha > t) = P^z (X_s \in D, 0 \leq s < t)$$

$$= \lim_{m \to \infty} P^z \left( X_{\frac{jt}{m}} \in D, j = 1, \ldots, m \right).$$
Isoperimetric problem for exit times

Let $X_t$ be a symmetric $\alpha$-stable processes in $\mathbb{R}^d$, and $\tau^\alpha_D$ the first exit time of $X_t$ from $D$.

$$P^z(\tau^\alpha_D > t) = P^z(X_s \in D, 0 \leq s < t)$$

$$= \lim_{m \to \infty} P^z\left(\frac{X_{jt}}{m} \in D, j = 1, \ldots, m\right).$$

Thus to prove

$$P^z(\tau^\alpha_D > t) \leq P^0(\tau^\alpha_{D^*} > t), \text{ for all } z \in \mathbb{R}^d.$$

It is enough to prove that

$$P^z\left(\frac{X_{t}}{m} \in D, \frac{X_{2t}}{m} \in D, \ldots, \frac{X_{mt}}{m} \in D\right)$$

$$\leq P^0\left(\frac{X_{t}}{m} \in D^*, \frac{X_{2t}}{m} \in D^*, \ldots, \frac{X_{mt}}{m} \in D^*\right).$$
Isoperimetric problem for exit times

Let $X_t$ be a symmetric $\alpha$-stable processes in $\mathbb{R}^d$, and $\tau_D^\alpha$ the first exit time of $X_t$ from $D$.

$$P_z^\alpha (\tau_D^\alpha > t) = P_z^\alpha (X_s \in D, 0 \leq s < t) = \lim_{m \to \infty} P_z^\alpha \left( X_{jt_m} \in D, j = 1, \ldots, m \right).$$

Thus to prove

$$P_z^\alpha (\tau_D^\alpha > t) \leq P_0^\alpha (\tau_D^{\alpha*} > t), \text{ for all } z \in \mathbb{R}^d.$$

It is enough to prove that

$$P_z^\alpha \left( X_{t_m} \in D, X_{2t_m} \in D, \ldots, X_{mt_m} \in D \right) \leq P_0^\alpha \left( X_{t_m} \in D^*, X_{2t_m} \in D^*, \ldots, X_{mt_m} \in D^* \right).$$

**Main idea:** Reduce the problem to an inequality of finite dimensional distributions of $X_t$. 
Recall that $X_t$ has transition densities $p_\alpha(t, x, y)$ such that

$$p_\alpha(t, x, y) = f_\alpha(|x - y|),$$

with $f_X$ decreasing. We want

$$\int_D \cdots \int_D \prod_{j=1}^m p_\alpha \left( \frac{t}{m}, z_j - z_{j-1} \right) dz_1 \cdots dz_m$$

\[= P^{z_0} \left( X_{\frac{t}{m}} \in D, X_{\frac{2t}{m}} \in D, \ldots, X_{\frac{mt}{m}} \in D \right) \]

\[\leq P^0 \left( X_{\frac{t}{m}} \in D^*, X_{\frac{2t}{m}} \in D^*, \ldots, X_{\frac{mt}{m}} \in D^* \right) \]

\[= \int_{D^*} \cdots \int_{D^*} p_\alpha \left( \frac{t}{m}, z_1 \right) \prod_{j=2}^m p_\alpha \left( \frac{t}{m}, z_j - z_{j-1} \right) dz_1 \cdots dz_m.\]
Using rearrangement inequalities one can prove that,

\[
\int_D \cdots \int_D \prod_{j=1}^m p_\alpha \left( \frac{t}{m}, z_j - z_{j-1} \right) \, dz_1 \cdots dz_m
\]

\[
\leq \int_{D^*} \cdots \int_{D^*} \prod_{j=2}^m p_\alpha \left( \frac{t}{m}, z_j - z_{j-1} \right) \, dz_1 \cdots dz_m.
\]
Using rearrangement inequalities one can prove that,

\[ \int_D \cdots \int_D \prod_{j=1}^m \left( \frac{t}{m}, z_j - z_{j-1} \right) \, dz_1 \cdots dz_m \]

\[ \leq \int_{D^*} \cdots \int_{D^*} p_{\alpha} \left( \frac{t}{m}, z_1 \right) \prod_{j=2}^m p_{\alpha} \left( \frac{t}{m}, z_j - z_{j-1} \right) \, dz_1 \cdots dz_m. \]

**Theorem (Luttinger 73)**

Let \( f_j, 1 \leq j \leq r \) be nonnegative functions in \( \mathbb{R}^d \) and \( f_j^* \) be the symmetric decreasing rearrangement of \( f_j \). Then for any \( z_0 \in \mathbb{R}^d \) we have

\[ \int_D \cdots \int_D \prod_{j=1}^m f_j (z_j - z_{j-1}) \, dz_1 \cdots dz_m \leq \]

\[ \int_{D^*} \cdots \int_{D^*} f_1^* (z_1) \prod_{j=2}^m f_j^* (z_j - z_{j-1}) \, dz_1 \cdots dz_m. \]
Symmetric decreasing rearrangements

Given $f \geq 0$, its symmetric decreasing rearrangement $f^*$ is the function satisfying

$$f^*(x) = f^*(y), \text{ if } |x| = |y|,$$

$$f^*(x) \leq f^*(y), \text{ if } |x| \geq |y|,$$

and

$$m\{f > t\} = m\{f^* > t\}, \text{ same distribution}$$

for all $t \geq 0$. In particular

$$(\chi_D)^* = \chi_{D^*}.$$

If $m\{f > t\} < \infty$, then

$$f^*(x) = \int_0^\infty \chi_{\{|f| > t\}}(x) \, dt.$$
Theorem (Brascamp-Lieb-Luttinger)

Let \( p_j, 1 \leq j \leq r \) be nonnegative functions in \( \mathbb{R}^d \) and \( p_j^* \) be the symmetric decreasing rearrangement of \( p_j \). Let \( a_{jk} \) be a \( r \times m \) real matrix.

\[
\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{r} p_j \left( \sum_{k=1}^{m} a_{j,k} z_k \right) \, dz_1 \cdots dz_m \leq \\
\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{r} p_j^* \left( \sum_{k=1}^{m} a_{j,k} z_k \right) \, dz_1 \cdots dz_m.
\]

Thus, if \( X_t \) is a symmetric \( \alpha \)-stable processes in \( \mathbb{R}^d \), for all \( z \in D \)

\[
P^z \left( X_{\frac{t}{m}} \in D, X_{\frac{2t}{m}} \in D, \ldots, X_{\frac{mt}{m}} \in D \right) \leq \ P^0 \left( X_{\frac{t}{m}} \in D^*, X_{\frac{2t}{m}} \in D^*, \ldots, X_{\frac{mt}{m}} \in D^* \right).
\]

Proved by many, first appearance for Brownian Motion Aizenman and Simon 80.
Question

Under what conditions, if $X_t$ is a Lévy processes in $\mathbb{R}^d$,

$$P^z \left( X_{\frac{t}{m}} \in D, X_{\frac{2t}{m}} \in D, \ldots, X_{\frac{mt}{m}} \in D \right) \leq P^0 \left( X_{\frac{t}{m}} \in D^*, X_{\frac{2t}{m}} \in D^*, \ldots, X_{\frac{mt}{m}} \in D^* \right).$$

for all $z \in D$. 
Under what conditions, if $X_t$ is a Lévy processes in $\mathbb{R}^d$, 

$$P^z \left( X_{\frac{t}{m}} \in D, X_{\frac{2t}{m}} \in D, \ldots, X_{\frac{mt}{m}} \in D \right) \leq P^0 \left( X_{\frac{t}{m}} \in D^*, X_{\frac{2t}{m}} \in D^*, \ldots, X_{\frac{mt}{m}} \in D^* \right).$$

for all $z \in D$.

**Same proof:** If $X_t$ has transition densities such that 

$$p^X(t, x, y) = f_X(|x - y|),$$

with $f_X$ decreasing (Isotropic Unimodal). Then 

$$P^z \left( X_{\frac{t}{m}} \in D, X_{\frac{2t}{m}} \in D, \ldots, X_{\frac{mt}{m}} \in D \right) \leq P^0 \left( X_{\frac{t}{m}} \in D^*, X_{\frac{2t}{m}} \in D^*, \ldots, X_{\frac{mt}{m}} \in D^* \right).$$
How far can you extend this inequality

Not true in general. Using the same argument if \( X_t \) has transition densities \( p^X(t, x, y) \). We need

\[
\int_D \cdots \int_D \prod_{j=1}^m p^X\left(\frac{t}{m}, z_j - z_{j-1}\right) \, dz_1 \cdots dz_m
\]

\[
= P^{z_0}\left( X_{\frac{t}{m}} \in D, X_{\frac{2t}{m}} \in D, \ldots, X_{\frac{mt}{m}} \in D \right)
\]

\[
\leq \int_{D^*} \cdots \int_{D^*} \left[ p^X\left(\frac{t}{m}, z_1\right) \right]^* \prod_{j=2}^m \left[ p^X\left(\frac{t}{m}, z_j - z_{j-1}\right) \right]^* \, dz_1 \cdots dz_m.
\]

**Problem:** We cannot even ensure that

\[
[p(t, \cdot, y)]^*,
\]

is the transition density of a Lévy process.
Let $X_t$ be a Lévy process in $\mathbb{R}^d$ such that

$$E^x \left[ e^{-i\xi \cdot X_t} \right] = e^{-t\Psi(\xi) - i\xi \cdot x},$$

where

$$\Psi(\xi) = -i\langle b, \xi \rangle + \frac{1}{2} \langle A \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i\langle \xi, y \rangle \chi_B - e^{i\xi \cdot y} \right] \phi(y) \, dy.$$

Consider $X_t^*$ the Lévy process in $\mathbb{R}^d$ given by

$$E^x \left[ e^{-i\xi \cdot X_t^*} \right] = e^{-t\Psi^*(\xi) - i\xi \cdot x},$$

where

$$\Psi^*(\xi) = \frac{1}{2} \langle A^* \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 - e^{i\xi \cdot y} \right] \phi^*(y) \, dy.$$
Theorem (Bañuelos-Méndez 10)

Suppose $X_t$ is a Lévy process with Lévy measure absolutely continuous with respect to the Lebesgue measure and let $X_t^*$ be the symmetrization of $X_t$ constructed as above. Let $f_1, \ldots, f_m$ be nonnegative lower semicontinuous functions. Then for all $z \in \mathbb{R}^d$,

$$E^z \left[ \prod_{i=1}^{m} f_i(X_{t_i}) \chi_{D_i}(X_{t_i}) \right] \leq E^0 \left[ \prod_{i=1}^{m} f^*_i(X_{t_i}^*) \chi_{D_i^*}(X_{t_i}^*) \right],$$

for all $0 \leq t_1 \leq \ldots \leq t_m$. 

Suppose $X_t$ is a Lévy process with Lévy measure absolutely continuous with respect to the Lebesgue measure and let $X_t^*$ be the symmetrization of $X_t$ constructed as above. Let $f_1, \ldots, f_m$ be nonnegative lower semicontinuous functions. Then for all $z \in \mathbb{R}^d$,

$$E^z \left[ \prod_{i=1}^m f_i(X_{t_i}) \chi_{D_i}(X_{t_i}) \right] \leq E^0 \left[ \prod_{i=1}^m f_i^*(X_{t_i}^*) \chi_{D_i^*}(X_{t_i}^*) \right],$$

for all $0 \leq t_1 \leq \ldots \leq t_m$.

The proof is based on the fact that $X_t$ is the week limit of processes of the form

$$X^n_t = C^n_t + G^n_t.$$

where $C^n_t$ is a Compound Poisson Processes and $G^n_t$ is a non-singular Gaussian processes.
Let $\tau^X_D$ be the first exit time of $X_t$ from $D$, then

- If $V \geq 0$ and $f$ are continuous,

$$E^z \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right) ; \tau^X_D > t \right\} \leq E^z \left\{ f^*(X_t) \exp \left( - \int_0^t V^*(X_s) ds \right) ; \tau^X_D^* > t \right\}$$

where $V^*_* = - (-V)^*$. 

- If $\psi$ is a nonnegative increasing function, then

$$E^z \left[ \psi \left( \tau^X_D \right) \right] \leq E^0 \left[ \psi \left( \tau^X_D^* \right) \right],$$

In particular, for all $0 < p < \infty$,

$$E^z \left[ \left( \tau^X_D \right)^p \right] \leq E^0 \left[ \left( \tau^X_D^* \right)^p \right].$$
If \( p_D^X(t, x, y) \) is the transition density of \( X_t \) killed upon leaving \( D \), then

\[
\int_D f(y) p_D^X(t, x, y) \, dy \leq \int_{D^*} f^*(y) p_{D^*}^X(t, 0, y) \, dy. \tag{1}
\]

In particular for all \( x, y \in D \)

\[
p_D^X(t, x, y) \leq p_{D^*}^X(t, 0, 0),
\]

If \( X_t \) and \( X_t^* \) are transient then

\[
\int_D f(z) G_D^X(x, z) \, dz \leq \int_{D^*} f^*(z) G_{D^*}^X(0, z) \, dz. \tag{2}
\]

For all increasing convex function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \)

\[
\int_D \phi \left( p_D^X(t, x, y) \right) \, dy \leq \int_{D^*} \phi \left( p_{D^*}^X(t, 0, y) \right) \, dy
\]

\[
\int_D \phi \left( G_D^X(x, z) \right) \, dz \leq \int_{D^*} \phi \left( G_{D^*}^X(0, z) \right) \, dz.
\]
• Isoperimetric inequality for the trace

\[ \sum_{i=1}^{\infty} e^{-t\lambda_{D^*}^{k,X}} = \int_{D} p_{D}^{X}(t, z, z) dz \leq \int_{D^*} p_{D^*}^{X^*}(t, z, z) dz = \sum_{i=1}^{\infty} e^{-t\lambda_{D^*}^{k,X^*}}. \]

• Rayleigh-Faber-Krahn Inequality

\[ \lambda_{1}^{1,X^*} \leq \lambda_{1}^{1,X}. \]

• The Gamma function, for all \( s > \frac{d}{\alpha} \)

\[ \sum_{i=1}^{\infty} \frac{1}{\left(\lambda_{D}^{k,X}\right)^{s}} \leq \sum_{i=1}^{\infty} \frac{1}{\left(\lambda_{D^*}^{k,X^*}\right)^{s}}. \]
Consider a Lévy process of the form

\[ X_t = C_t + G_t, \]

where

- \( G_t \) is a Gaussian process with covariance matrix \( A \), and mean \( m \).
- \( C_t \) is an independent compound Poisson process with characteristic function

\[ E \left( e^{i \xi \cdot C_t} \right) = \exp \left\{ -c \int_{\mathbb{R}^d} \left[ 1 - \exp(i \xi \cdot y) \right] \phi(y) \, dy \right\}, \]

Then

\[ X_t^* = C_t^* + G_t^*, \]

where

- \( G_t^* \) is a Gaussian process with covariance matrix \( A^* = (\text{det} A)^{1/d} I_d \).
- \( C_t^* \) is an independent compound Poisson process with characteristic function

\[ E \left( e^{i \xi \cdot C_t^*} \right) = \exp \left\{ -c \int_{\mathbb{R}^d} \left[ 1 - \exp(i \xi \cdot y) \right] \phi^*(y) \, dy \right\}. \]
Let $B$ be a Borel subset of $\mathbb{R}^d$, then

$$E^x \left[ C_t + G_t \in B \right] = \int_B f_{A,m}(t, u - x - x_0) \, d\mu_t(x_0) \, du,$$

where $\mu_t$ is the distribution of $C_t$, and $f_{A,m}$ is the density of $G_t$. 

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Symmetrization & Lévy processes  
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Let $B$ be a Borel subset of $\mathbb{R}^d$, then

$$E^x [ C_t + G_t \in B ] = \int_B f_{A,m}(t, u - x - x_0) \, d\mu_t(x_0) \, du,$$

where $\mu_t$ is the distribution of $C_t$, and $f_{A,m}$ is the density of $G_t$. One can prove that

$$\int_B f_{A,m}(t, u - x - x_0) \, d\mu_t(x_0) \, du = P[N_t = 0] \int_B f_{A,m}(t, u) \, du$$

$$+ \sum_{k=1}^{\infty} P[N_t = k] \int_B f_{A,m}(t, u - x - x_0) \, q_k(t, x_0) \, dx_0 \, du$$

where $N_t$ is a Poisson process and

$$\int_B f_{A,m}(t, u - x - x_0) \, q_k(t, x_0) \, dx_0 \, du =$$

$$\int_B \int \cdots \int f_{A,m}(t, u - x - x_0) \prod_{i=1}^{k} \phi(x_i - x_{i-1}) dx_1 \cdots dx_k \, dx_0 \, du.$$
Then using B-L-L

\[
\int_B \int \cdots \int f_{A,m}(t, x - x_0) \prod_{i=1}^{k} \phi(x_i - x_{i-1}) \, dx_1 \cdots dx_k \, dx_0
\]

\[
\leq \int_{B^*} \int \cdots \int f_{A,m}^*(t, x_0) \prod_{i=1}^{k} \phi^*(x_i - x_{i-1}) \, dx_1 \cdots dx_k \, dx_0
\]

\[
= \int_{B^*} \int \cdots \int f_{A^*,0}(t, x_0) \prod_{i=1}^{k} \phi^*(x_i - x_{i-1}) \, dx_1 \cdots dx_k \, dx_0.
\]
Then using B-L-L

\[
\int_B \int \cdots \int f_{A,m}(t, x - x_0) \prod_{i=1}^{k} \phi(x_i - x_{i-1}) \, dx_1 \cdots dx_k \, dx_0
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\]

\[
= \int_{B^*} \int \cdots \int f_{A^*,0}(t, x_0) \prod_{i=1}^{k} \phi^*(x_i - x_{i-1}) \, dx_1 \cdots dx_k \, dx_0.
\]

We conclude

\[
E^x \{ G_{t_1} + C_{t_1} \in D, \ldots, G_{t_m} + C_{t_m} \in D \} \leq
\]

\[
E^0 \{ G_{t_1}^* + C_{t_1}^* \in D^*, \ldots, G_{t_m}^* + C_{t_m}^* \in D^* \}.
\]

Then take a sequence \( X^n_t = C^n_t + G^n_t \) that converges weakly to \( X_t \).
If $X_t$ has transition densities such that

$$p^X(t, x, y) = f_X(|x - y|),$$

with $f_X$ decreasing. There are similar results for exit times of Lévy processes $X_t$ from convex domains with fixed inner inradius $r_D$. Assume the biggest ball inside $D$ is centered at 0.
If $X_t$ has transition densities such that

$$p^X(t, x, y) = f_X(|x - y|),$$

with $f_X$ decreasing. There are similar results for exit times of Lévy processes $X_t$ from convex domains with fixed inner inradius $r_D$. Assume the biggest ball inside $D$ is centered at 0.

**Classical results:**

Using the maximum principle for the Laplacian it is proved that

$$\lambda^B_{(-r_D, r_D)} \leq \lambda^B_D, \quad (\text{Hersh } n = 2\ 60, \text{ Protter } n \geq 2\ 81)$$

where $B$ is Brownian motion.
Extremal for convex domains of fixed inradius for B.M.

Consider the infinite slab

\[ S(D) = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : -r_D < x_1 < r_D \right\}. \]

Equivalently

\[ \lambda^B_{(-r_D, r_D)} = \lambda^B_S(D). \]

Besides Bañuelos-Kröger 97 prove that

\[ P^x(\tau^B_D > t) \leq P^0(\tau^B_S(D) > t) = P^0(\tau^B_{(-r_D, r_D)} > t). \]
Consider the infinite slab

\[ S(D) = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : -r_D < x_1 < r_D \right\}. \]

Equivalently

\[ \lambda^B_{(-r_D, r_D)} = \lambda^B_{S(D)}. \]

Besides Bañuelos-Kröger 97 prove that

\[ P^x(\tau^B_D > t) \leq P^0(\tau^B_{S(D)} > t) = P^0(\tau^B_{(-r_D, r_D)} > t). \]

In a similar way we can reduced this problem to finite dimensional distributions. To obtained a similar result for \( X_t \), it is enought to prove that

\[ P^x \left( X_{\frac{t}{m}} \in D, X_{\frac{2t}{m}} \in D, \ldots, X_{\frac{mt}{m}} \in D \right) \leq P^0 \left( X_{\frac{t}{m}} \in S(D), X_{\frac{2t}{m}} \in S(D), \ldots, X_{\frac{mt}{m}} \in S(D) \right). \]
**Theorem** Let $D$ be a convex domain in $\mathbb{R}^d$ of finite inradius $r_D$ and let $S(D)$ be the infinite slab. Let $p_1, \ldots, p_m$ be nonnegative nonincreasing radially symmetric functions on $\mathbb{R}^d$. Then for any $z_0 \in \mathbb{R}^d$ we have

\[
\int_D \cdots \int_D \prod_{j=1}^m p_j(z_j - z_{j-1}) \, dz_1 \cdots dz_m \leq \\
\int_{S(D)} \cdots \int_{S(D)} p_1(z_1) \prod_{j=2}^m p_j(z_j - z_{j-1}) \, dz_1 \cdots dz_m.
\]  

( Bañuelos-Latała-Méndez 00 $d = 2$, Méndez 02 $d > 2$)
**Theorem** Let $D$ be a convex domain in $\mathbb{R}^d$ of finite inradius $r_D$ and let $S(D)$ be the infinite slab. Let $p_1, \ldots, p_m$ be nonnegative nonincreasing radially symmetric functions on $\mathbb{R}^d$. Then for any $z_0 \in \mathbb{R}^d$ we have

$$\int_D \cdots \int_D \prod_{j=1}^m p_j(z_j - z_{j-1}) \, dz_1 \cdots dz_m \leq \int_{S(D)} \cdots \int_{S(D)} p_1(z_1) \prod_{j=2}^m p_j(z_j - z_{j-1}) \, dz_1 \cdots dz_m.$$  \hspace{1cm} (4)

( Bañuelos-Latała-Méndez 00 $d = 2$, Méndez 02 $d > 2$)

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**Key in the proof:**

1. $D$ can be assume to be a polyhedron
2. Any nonnegative radially symmetric nonincreasing function $f$ can be expressed in the form

$$f(z) = \int_0^\infty I_{B(0,r)}(z) \, d\mu(r)$$

for some nonnegative measure on $(0, \infty]$
Consequences

- For all $z \in D$
  
  \[ E^z \left\{ \tau^X_D > t \right\} \leq E^0 \left\{ \tau^X_{S(D)} > t \right\}. \]

  Thus

  \[ \lambda^X_{S(D)} \leq \lambda^X_D. \]

- If \( \psi \) is a nonnegative increasing function, then

  \[ E^z \left[ \psi \left( \tau^X_D \right) \right] \leq E^0 \left[ \psi \left( \tau^X_{S(D)} \right) \right], \]

  In particular, for all $0 < p < \infty$,

  \[ E^z \left[ \left( \tau^X_D \right)^p \right] \leq E^0 \left[ \left( \tau^X_{S(D)} \right)^p \right]. \]
Inequalities for the heat content and the torsional rigidity

\[ \int_D \int_D p^X_D(t, z, w) \, dz \, dw \leq \int_{S(D)} \int_{S(D)} p^X_{S(D)}(t, z, w) \, dz \, dw, \]

\[ \int_D \int_D G^X_D(z, w) \, dz \, dw \leq \int_{S(D)} \int_{S(D)} G^X_{S(D)}(z, w) \, dz \, dw. \]

For all increasing function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \)

\[ \int_D \phi \left( p^X_D(t, 0, y) \right) \, dy \leq \int_{S(D)} \phi \left( p^X_{S(D)}(t, 0, y) \right) \, dy \]

\[ \int_D \phi \left( G^X_D(0, z) \right) \, dz \leq \int_{S(D)} \phi \left( G^X_{S(D)}(0, z) \right) \, dz. \]
Theorem (Méndez 02) Let $D$ be a convex domain in $\mathbb{R}^2$ of finite inradius $r_D$ and of diameter $d_D$ (which may be infinite). Let $p_1, \ldots, p_m, q_1, \ldots, q_m$ be nonnegative nonincreasing radially symmetric functions on $\mathbb{R}^2$. Then for any $z_0 \in H_{i,j}$ we have that

$$\int_{D} \cdots \int_{D} \prod_{j=1}^{m} q_j(z_j)p_j(z_j - z_{j-1}) \, dz_1 \cdots dz_m \leq$$

$$\int_{C(D)} \cdots \int_{C(D)} q_j(z_1)p_j(z_1) \prod_{j=2}^{m} q_j(z_j)p_j(z_j - z_{j-1}) \, dz_1 \cdots dz_m,$$

where $C(D) = (-r_D, r_D) \times (-d_D + r_D, -r_D + d_D)$.

Question

Are there higher dimensional analogues? For instance can we replace $S(D)$ by

$$C(D) = (-r_D, r_D) \times (-d_D + r_D, r_D - d_D)^{d-1}.$$