Lévy–Ornstein–Uhlenbeck processes in Hilbert spaces

Szymon Peszat

Institute of Mathematics Polish Academy of Sciences, Kraków
and
Faculty of Applied Mathematics, AGH University of Technology

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Let \( \mu \) be an invariant measure for a Markov family \( X = (X^x, \; x \in E) \) with the transition semigroup
\[
P_t \psi(x) = \mathbb{E} \psi(X^x(t)), \quad t \geq 0, \; x \in E.
\]
Then \((P_t)\) is a semigroup of contractions on any \( L^p(\mu) := L^p(E, \mathcal{B}(E), \mu)\)-space, \( p \in [1, +\infty] \).
In the talk $X$ is defined by the Ornstein–Uhlenbeck type equation

$$dX = AX\,dt + dL, \quad X(0) = x \in E,$$

where $(A, D(A))$ generates a $C_0$-semigroup $S$ on a Hilbert space $(E, \langle \cdot, \cdot \rangle_E)$. We will assume that $L$ is a Lévy process taking values in a Hilbert space $\tilde{E} \hookrightarrow E$. 
As an example consider $E = \tilde{E} = \mathbb{R}$, 

$$dX = -\gamma X dt + (\alpha \gamma)^{1/\alpha} dL,$$

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where $L$ is an $\alpha$-stable real-valued process. Then $\mu$ is $\alpha$-stable distribution.
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P_t = \Gamma(S^*_0(t)), \quad t \geq 0,
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The goal - formulate an analogous result for the Markov family defined by equation with Lévy noise. The new results will be in case of purely jump noise.
We will use the fact that $\mu$ is the distribution of

$$Y_\infty = \int_0^\infty S(t) dL(t)$$

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where $\mu_t$ is the distribution of

$$Y_t := \int_0^t S(s)dl(s).$$
Assume that $L(t) = BW(t)$, where $W$ is a cylindrical Wiener process in $E$,

$$W(t) = \sum_k W_k(t)e_k,$$

$(e_k)$ ONB of $E$, $W_k$ independent Wiener (standard) on $\mathbb{R}$, $B$ is a bounded linear operator on $E$. 
There is an invariant measure (on $E$) if and only if
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Moreover, the distribution of
\[ Y_\infty := \int_0^\infty S(s)B \, dW(s) \]
is an invariant measure. It is mean-zero, Gaussian with the covariance operator
\[ Q_\infty := \int_0^\infty S(s)BB^* S^*(s) \, ds. \]
For simplicity we assume that Ker $Q_\infty = \{0\}$. 
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$$
Q_\infty := \int_0^\infty S(s)BB^*S^*(s) \, ds.
$$

For simplicity we assume that Ker $Q_\infty = \{0\}$. Then $\mu$ is unique.
Recall that the Reproducing Hilbert Kernel Space of $\mu$ is the space $E_0 := \text{Range } Q_{\infty}^{1/2}$, equipped with the scalar product

$$\langle Q_{\infty}^{1/2}u, Q_{\infty}^{1/2}v \rangle_{E_0} = \langle u, v \rangle_E, \quad u, v \in E.$$
Given $h \in E_0$ define a linear functional

$$\psi_h(x) = \langle Q_{\infty}^{-1/2} h, x \rangle_E, \quad x \in E.$$ 

Then

$$\int_E \psi_h(x) \psi_u(x) \mu(dx) = \langle Q_{\infty} Q_{\infty}^{-1/2} h, Q_{\infty}^{-1/2} u \rangle_E = \langle h, u \rangle_E.$$ 

Since $E_0$ is dense in $E$, for any $h \in E$ there is a sequence $(h_n) \subset E_0$ converging in $E$ to $h$. 
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$$\int_E \psi_h(x) \psi_u(x) \mu(dx) = \langle h, u \rangle_E.$$
Let $\mathcal{P}_n$ be the closed subspace of $L^2(\mu)$ spanned by $p(\psi_{h_1}, \ldots, \psi_{h_k})$, $k \in \mathbb{N}$, and $p$ is a polynomial of order $\leq n$. Let $\mathcal{H}_0$ be the space of all constant functions, and let $\mathcal{H}_n$, $n \in \mathbb{N}$, be the orthogonal complement of $\mathcal{H}_{n-1}$ in $\mathcal{P}_n$. The Itô–Wiener chaos decomposition says that

$$L^2(\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$ 

Let $\text{Pr}_n$ be the orthogonal projection of $L^2(\mu)$ into $\mathcal{H}_n$. 
Let $R \in L(E, E)$. Define $\Gamma_n(R) : \mathcal{H}_n \mapsto \mathcal{H}_n$, $n = 0, 1 \ldots$, by

$$\Gamma_n(R) \Pr_n(\psi_{h_1} \ldots \psi_{h_n}) = \Pr_n(\psi_{Rh_1} \ldots \psi_{Rh_n}), \quad h_1, \ldots, h_n \in E.$$  

We have $\|\Gamma_n(R)\|_{L(\mathcal{H}_n, \mathcal{H}_n)} = \|R\|_n^{L(E,E)}$. Hence for any linear contraction $R$ on $E$,

$$\Gamma(R) = \sum_{n=0}^{\infty} \Gamma_n(R) \Pr_n$$

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We have $\|\Gamma_n(R)\|_{L(\mathcal{H}_n, \mathcal{H}_n)} = \|R\|_{L(E,E)}^n$. Hence for any linear contraction $R$ on $E$,

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defines a contraction on $L^2(\mu)$. We call $\Gamma(R)$ the second quantized operator of $R$, and $\Gamma$ the second quantization operator.
In the Gaussian case the action of the second quantization operator is well understood. In fact, the following lemma gathers some basic properties of $\Gamma$. For its proof we refer the reader to Lemma 2 and Proposition 2 from Chojnowska-Goldys 1998, and to Chapter 1 of B. Simon book.
Theorem

- For any $p \geq 1$ and

$$q_0 = 1 + \frac{p - 1}{\|R\|_2^2_{L(E,E)},}$$

we have $\|\Gamma(R)\|_{L(L^p(\mu),L^{q_0}(\mu))} = 1$, and if $q > q_0$ then $\|\Gamma(R)\|_{L(L^p(\mu),L^q(\mu))} = \infty$.

- If $R$ is selfadjoint with a complete set of eigenvectors $(v_k)$, then $\Gamma(R)$ is also selfadjoint with the complete orthogonal set of eigenvectors

$$\prod_{j=1}^{\infty} \Pr_{a_j}(\psi_{v_j}^{a_j}):(a_j) \subset \mathbb{N} \cup \{0\} : \sum_j a_j < \infty.$$
Theorem (cont.)

- Let $p, q \geq 1$ and $R \neq 0$. Then $\Gamma(R) : L^p(\mu) \leftrightarrow L^q(\mu)$ is compact if and only if $R$ is a compact strict contraction and $q < q_0$, $q_0$ as above.

- The operator $\Gamma(R)$ is Hilbert–Schmidt on $L^2(\mu)$ if and only if $R$ is a strict Hilbert–Schmidt contraction. Moreover,

$$\|\Gamma(R)\|_{L(HS)(L^2(\mu), L^2(\mu))} = \frac{1}{\sqrt{\det(I - R^*R)}}.$$
Recall that

\[ P_t f(x) = \int_E f(S(t)x + y) \mu_t(dy), \]

where \( \mu_t \) is the distribution of \( Y_t = \int_0^t S(s)BdW(s) \). Clearly, \( \mu_t \) is mean-zero Gaussian with the covariance

\[ Q_t := \int_0^t S(s)BB^*S^*(s)ds. \]

It is convenient to formulate the following condition

\[ \text{Range } Q_t^{1/2} = \text{Range } Q_\infty^{1/2} = E_0. \quad (\star) \]
Lemma

For any $t \geq 0$, $S(t)E_0 \subset E_0$, and $S_0(t) = Q_\infty^{-1/2} S(t) Q_\infty^{1/2}$, $t \geq 0$, is a $C_0$-semigroup of contractions on $E$. Moreover, $\|S_0(t)\|_{L(E,E)} < 1$ if and only if ($\star$) holds.
Theorem

For any $t \geq 0$, $P_t = \Gamma(S_0^*(t))$ and $P_t^* = \Gamma(S_0(t))$. Moreover, the following statements hold:

- Let $t \geq 0$. If $(\star)$ holds, then for any $p, q \geq 1$, $\|P_t\|_{L(L^p(\mu), L^q(\mu))} = 1$ if and only if

$$\sqrt{\frac{p - 1}{q - 1}} \geq \|S_0(t)\|_{L(E,E)}.$$ 

Otherwise, $\|P_t\|_{L(L^p(\mu), L^q(\mu))} = \infty$.

- For $p, q \geq 1$ and $t \geq 0$, the operator $P_t$ is compact from $L^p(\mu)$ into $L^q(\mu)$ if and only if $(\star)$ holds, $S_0(t)$ is compact and

$$q < 1 + \frac{p - 1}{\|S_0(t)\|^2_{L(E,E)}}.$$
The operator $P_t$ is Hilbert–Schmidt on $L^2(\mu)$ if and only if $S_0(t)$ is Hilbert–Schmidt on $E$ and $(\star)$ holds. In this case

$$\|P_t\|_{L(HS)(L^2(\mu),L^2(\mu))} = \frac{1}{\sqrt{\det(I - S_0(t)S_0^*(t))}}.$$
This part is based on G. Last and M. Penrose paper (PTRF), to appear.

Let \((E, \mathcal{B})\) be a measurable space, and let \(\Pi\) be a Poisson random measure on \(E\) with intensity measure \(\lambda\).

Let \(\mathbb{Z}_+(E)\) be the space of integers-valued \(\sigma\)-finite measures positive on \((E, \mathcal{B})\). Denote by \(\mathbb{P}_\Pi\) the law of \(\Pi\) in \(\mathbb{Z}_+(E)\). Let \(L^2(\mathbb{P}_\Pi)\) be the space of all measurable \(F: \mathbb{Z}_+(E) \mapsto \mathbb{R}\) such that

\[
\|F\|_{L^2(\mathbb{P}_\Pi)}^2 := \mathbb{E} F^2(\Pi) < \infty.
\]
Given $F : \mathbb{Z}_+(E) \mapsto \mathbb{R}$, and $y \in E$ write

$$D_y F(\xi) = F(\xi + \delta_y) - F(\xi), \quad \xi \in \mathbb{Z}_+(E).$$

Derivatives $D^n_{y_1 y_2 \ldots y_n} F$ of order $n$ are defined by induction. Note that

$$D^n_{y_1, \ldots, y_n} F(\xi) = \sum_{I \subset \{1, \ldots, n\}} (-1)^{n-|I|} F(\xi + \sum_{i \in I} \delta_{y_i}), \quad \xi \in \mathbb{Z}_+(E).$$
Set $T_0(F) = \mathbb{E}F(\Pi)$, and for $n \in \mathbb{N}$,

$$T^n F(y_1, \ldots, y_n) := \mathbb{E} D_{y_1 \ldots y_n} F(\Pi) = \int_{\mathbb{Z}_+(E)} D^n_{y_1 \ldots y_n} F(\xi) \mathbb{P}_\Pi(d\xi),$$

provided that the derivative on the right hand side in integrable with respect to $\mathbb{P}_\Pi$. 

Denote by $L^2_{(s)}(E^n, \lambda^n)$ the space of symmetric functions from $L^2(E^n, \lambda^n)$. We set $L^2_{(s)}(E^0, \lambda^0) = \mathbb{R}$.

**Theorem**

For any $F \in L^2(\mathbb{P}_\Pi)$ and for $\lambda^n$ almost all $y_1, \ldots, y_n$, $T^n F(y_1, \ldots, y_n)$ is well defined and $T^n F \in L^2_{(s)}(E^n, \lambda^n)$. Moreover, for any $F, G \in L^2(\mathbb{P}_\Pi)$,

$$\mathbb{E} F(\Pi) G(\Pi) = \mathbb{E} F(\Pi) \mathbb{E} G(\Pi) + \sum_{n=1}^{\infty} \frac{1}{n!} \langle T^n F, T^n G \rangle_{L^2(E^n, \lambda^n)}.$$
For $f \in L^2(E^n, \lambda^n)$ we denote by $I_n(f)$ the multiple Itô integral with respect to the compensated measure $\Pi - \lambda$. We set $I_0(f) = f$.

**Theorem**

Let $F \in L^2(\mathbb{P}_\Pi)$. Then

$$F(\Pi) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(T^n F).$$
Let $\mathcal{H}_0 = \mathbb{R}$, and let

$$\mathcal{H}_n := \{ l_n(f) : f \in L^2_{(s)}(E^n, \lambda^n) \}, \quad n \in \mathbb{N}. $$

By Theorems above, $\mathcal{H}_n$, $n \in \mathbb{N} \cup \{0\}$, are orthogonal closed subspaces of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, and the operators

$$L^2_{(s)}(E^n, \lambda^n) \ni f \mapsto \frac{1}{\sqrt{n!}} l_n(f) \in L^2(\Omega, \mathcal{F}, \mathbb{P}), \quad n \in \mathbb{N},$$

are linear and isometric. Let $\text{Pr}_n$ be the orthogonal projection of $L^2(\Omega, \sigma(\Pi), \mathbb{P})$ into $\mathcal{H}_n$. 


Corollary

One has

\[ L^2(\Omega, \sigma(\Pi), \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \]

and for any \( F \in L^2(\mathbb{P}_\Pi) \), \( \Pr_0 F(\Pi) = \mathbb{E} F(\Pi) \), and \( \Pr_n F(\Pi) = \frac{1}{n!} I_n(T^n F) \), \( n \in \mathbb{N} \).
Given $R \in L(E, E)$, and $f : E^n \mapsto \mathbb{R}$ write

$$\rho^n_R f(y_1, \ldots, y_n) = f(Ry_1, \ldots, Ry_n), \quad y_1, \ldots, y_n \in E.$$ 

**Lemma**

If $\rho_R = \rho^1_R$ is a contraction on $L^2(E, \lambda)$, then for any $n \in \mathbb{N}$, $\rho^n_R$ is a contraction on $L^2_{(s)}(E^n, \lambda^n)$. Moreover, $\|\rho^n_R\| \leq \|\rho_R\|^n$, where $\| \cdot \|$ stands for the operator norm on $L^2_{(s)}(E^n, \lambda^n)$ and on $L^2(E, \lambda)$. 
For any $R \in L(E, E)$ such that $\rho_R$ is a contraction on $L^2(E, \lambda)$ we can define the second quantized operator

$$\Gamma(R) : L^2(\mathbb{P}_\Pi) \mapsto L^2(\mathbb{P}_\Pi)$$

putting

$$\Gamma(R)F(\Pi) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\rho^n_R T_n F).$$

Obviously, $\Gamma(R)$ is a contraction on $L^2(\mathbb{P}_\Pi)$. 
Assume that:

(H.1) $E$ is densely and continuously imbedded into $\tilde{E}$.

(H.2) For any $t > 0$, $S(t)$ has an extension to a bounded linear map, denoted also by $S(t)$, from $\tilde{E}$ into $E$, and that $S$ is stable on $E$, that is $|S(t)x|_E \to 0$ as $t \uparrow +\infty$ for any $x \in E$.

(H.3) $L$ is a pure jump process, that is

$$\mathbb{E} e^{i\langle x, L(t) \rangle_{\tilde{E}}} = e^{-t\Psi(x)}, \quad x \in \tilde{E},$$

where the co-called Lévy exponent

$$\Psi(x) = \int_{\tilde{E}} \left( 1 - e^{i\langle x, y \rangle_{\tilde{E}}} + i\langle x, y \rangle_{\tilde{E}} \chi_{\{|y|_{\tilde{E}} \leq 1\}} \right) \nu(dy),$$

and $\nu$, called the Lévy measure of $L$, is a non-negative measure on $\tilde{E}$ satisfying $\int_{\tilde{E}} |x|_{\tilde{E}}^2 \wedge 1 \nu(dx) < \infty$. 

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\( (H.4) \)
\[
\int_0^\infty \int_{\tilde{E}} |S(s)y|^E \chi\{|y|_{\tilde{E}} \leq 1\} - \chi\{|S(s)y|^E \leq 1\} \nu(dy) < \infty.
\]

\( (H.5) \)
\[
\int_0^\infty \int_{\tilde{E}} |S(s)y|^2_E \wedge 1 \nu(dy) < \infty.
\]
For $t \in [0, +\infty]$, define

$$m_t := \int_0^t \int_{\tilde{E}} S(s)y \left( \chi_{\{|y|_{\tilde{E}} \leq 1\}} - \chi_{\{|S(s)y|_E \leq 1\}} \right) \nu(dy),$$

$$\nu_t := \int_0^t \nu \circ S(s)^{-1} ds.$$  

$$\Psi_t(x) := i\langle x, m_t \rangle_E + \int_E \left( 1 - e^{i\langle x, y \rangle_E} + i\langle x, y \rangle_E \chi_{\{|y|_E \leq 1\}} \right) \nu_t(dy).$$
Proposition

\((X^x, x \in E)\) is well defined Markov family on \(E\), with a unique invariant measure \(\mu\). Moreover, \(X^x(t) = S(t)x + Y_t\), where \(Y_t = \int_0^t S(t-s)dL(s)\), the integral converges \(\mathbb{P}\)-a.s. in \(E\), and its distribution \(\mu_t\) is infinitely divisible with the Lévy exponent \(\Psi_t\) and Lévy measure \(\nu_t\). Finally \(\mu\) is the distribution of

\[ Y_\infty := \int_0^\infty S(s)dL(s), \]

it is infinitely divisible with the Lévy exponent \(\Psi_\infty\) and Lévy measure \(\nu_\infty\).
Fron now on $\mu = \mu_\infty$, $\lambda = \nu_\infty$, and $Y_\infty$ and $\Pi$ are related by the LK decomposition formula

$$Y_\infty = m + \int_E x\overline{\Pi}(dx),$$

$$\overline{\Pi}(dx) = \Pi(dx) - \chi_{\{|x| \leq 1\}}\lambda(dx).$$
The main result can be illustrated by the following diagram.

\[
\begin{align*}
L^2(\mu) & \xrightarrow{P_t} L^2(\mu) \\
\downarrow j & \quad \downarrow j \\
L^2(\mathbb{P}_\Pi) & \xrightarrow{\Gamma(S(t))} L^2(\mathbb{P}_\Pi) \\
\tau & \quad \tau \\
\bigoplus_{n=0}^{\infty} L^2_{(s)}(E^n, \lambda^n) & \xrightarrow{\bigoplus_{n=0}^{\infty} \rho^n_{S(t)}} \bigoplus_{n=0}^{\infty} L^2_{(s)}(E^n, \lambda^n).
\end{align*}
\]
Theorem

Under assumptions (H.1) to (H.5), for any $t > 0$, $\rho_{S(t)}$ is a contraction on $L^2(E, \lambda)$ and (1) holds with

$$jf(\xi) := f(m + \int_E x\xi(dx)), \quad f \in L^2(\mu), \; \xi \in \mathbb{Z}_+(E),$$

and

$$\tau := \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} T^n.$$
Proof

Let us fix $t > 0$. The fact that $\rho_{S(t)}$ is a contraction on $L^2(E, \lambda)$ follows from the fact that $\lambda = \nu_\infty$. For we have

$$
\int_E \|\rho_{S(t)} f(x)\|_E^2 \lambda(dx) = \int_0^\infty \int_E \|f(S(t)x)\|_E^2 \nu \circ S^{-1}(s) ds(dx)
$$

$$
= \int_0^\infty \|f(S(t+s)x)\|_E^2 \nu(dx)
$$

$$
\leq \int_0^\infty \|f(S(s)x)\|_E^2 \nu(dx)
$$

$$
= \int_E \|f(x)\|_E^2 \lambda(dx).
$$
To see that (1) it is enough to show that for all $n \in \mathbb{N}$ and $f \in L^2(\mu)$, we have

$$T^n(jP_t f) = \rho^n_{S(t)} T^n(jf).$$

To do this fix an $f \in L^2(\mu)$. Given a $t \in [0, +\infty]$ denote by $\tilde{Y}_t$ a copy of $Y_t$, independent of $Y_s$, $s \in [0, +\infty]$, and by $\tilde{\mathbb{E}}$ the expectation with respect to $\tilde{Y}_\infty$. Note that for any $t$, the random variables $S(t) \tilde{Y}_\infty + Y_t$ and $S(t) Y_\infty + \tilde{Y}_t$ have the law $\mu$. 
\[ T^n(jP_t f)(y_1, \ldots, y_n) = \mathbb{E} \sum_l (-1)^{n-|l|} P_t f(\sum_{i \in l} y_i + Y_\infty) \]
\[ = \sum_l (-1)^{n-|l|} \mathbb{E} \tilde{\mathbb{E}} f(\sum_{i \in l} S(t)y_i + S(t)Y_\infty + \tilde{Y}_t) \]
\[ = \sum_l (-1)^{n-|l|} \mathbb{E} f(\sum_{i \in l} S(t)y_i + Y_\infty) \]
\[ = \sum_l (-1)^{n-|l|} \mathbb{E} jf(\sum_{i \in l} \delta_{S(t)y_i} + \Pi) \]
\[ = \mathbb{E} D_{S(t)y_1, \ldots, S(t)y_n} jf(\Pi) \]
\[ = T^n(jf)(S(t)y_1, \ldots, S(t)y_n). \]
Let \( (P_t) \) be the transition semigroup of the following Ornstein–Uhlenbeck process on \( E = \mathbb{R} \),

\[
dX = -\gamma X dt + dW,
\]

where \( \gamma > 0 \). Then \( S(t) = e^{-\gamma t} \), and

\[
P_t f(x) = \int_{\mathbb{R}} f(S(t)x + y) \mu_t(dy),
\]

where \( \mu_t \) is the distribution of

\[
\int_0^t S(t-s) dW(s) = \int_0^t e^{-\gamma(t-s)} dW(s).
\]

Clearly \( \mu_t \in \mathcal{N}(0, \sigma_t^2) \), where

\[
\sigma_t^2 = \int_0^t e^{-2\gamma(t-s)} ds = \frac{1}{2\gamma} \left( 1 - e^{-2\gamma t} \right).
\]
Let $\mu$ be the invariant measure. Then $\mu \in \mathcal{N}(0, \sigma^2)$, where $\sigma^2 = \frac{1}{2\gamma}$. Therefore

$$P_t f(x) = \int_{\mathbb{R}} f(e^{-\gamma t}x + \sqrt{1 - e^{-2\gamma t}}y) \mu(dy)$$

$$= \int_{\mathbb{R}} f(S_t x + \sqrt{1 - S_t^2}1y) \mu(dy).$$
Let $\psi_h(x) = 2\gamma hx$. Given a $z \in \mathbb{R}: |z| < 1$ we denote by $\Gamma_n(z)$ an operator on $\mathcal{H}_n$ by

$$\Gamma_n(z) \Pr_n(\psi_{h_1} \ldots \psi_{h_n}) = \Pr_n(\psi_{zh_1} \ldots \psi_{zh_n}).$$

Finally $\Gamma(z)$ is a linear operator $L^2(\mu)$ given by

$$\Gamma(z) F = \sum_{n=0}^{\infty} \Gamma_n(z) \Pr_n(F).$$
Let

\[ E_h(x) = e^{\psi_h(x) - \gamma h^2}. \]

We have

\[ P_t E_h(x) = \frac{\sqrt{2\gamma}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{2\gamma h x e^{-\gamma t} - \alpha h^2 + 2\gamma h \sqrt{1 - e^{-2\gamma t}} y - \gamma y^2} dy \]

\[ = e^{2\gamma he^{-\gamma t} x - \gamma h^2 + \gamma h^2 (1 - e^{-\gamma t})} \]

\[ = E_{e^{-\gamma t} h}(x). \]

On the other hand

\[ E_h = \sum_{n=0}^{\infty} \frac{1}{n!} \Pr_n(\psi_h^n). \]

Thus \( P_t E_h = \Gamma(e^{-\gamma t}) E_h. \)
Consider the following OU equation on $\mathbb{R}$,
\[ dX = -\frac{1}{2}Xdt + dW. \]

Then $\mu = N(0, 1)$. Let $P^W$ be the Wiener measure on $C([0, +\infty))$, and let $\mathcal{W}$ be the Wiener process with values in $C([0, +\infty))$ such that $\mathcal{W}(1)$ has the law $P^W$. Then $P^W$ is invariant measure to the OU equation on $C([0, +\infty))$;
\[ d\mathcal{X} = -\frac{1}{2}\mathcal{X}dt + d\mathcal{W}. \]

Let $(P_t)$ and $P_t$ be the corresponding transition semigroup, and let
\[ j: L^2(\mu) \hookrightarrow L^2(P^W) \]
be isometric imbedding given by
\[ jf(w) = f(w(t)), \quad f \in L^2(\mu), \ w \in C([0, +\infty)). \]
Theorem

The following diagram commutes

\[ L^2(\mu) \xrightarrow{P_t} L^2(\mu) \]
\[ j \downarrow \quad j \downarrow \]
\[ L^2(\mathcal{P}^W) \xrightarrow{\mathcal{P}_t} L^2(\mathcal{P}^W). \]
For $f \in L^2(\mu)$ we have

$$Q_t(jf)(w) = \mathbb{E} f \left( e^{-\frac{1}{2}t} w(1) + \sqrt{1 - e^{-t}} \mathcal{W}(1)(1) \right)$$

$$= \int_\mathbb{R} f \left( e^{-\frac{1}{2}t} x + \sqrt{1 - e^{-t}} y \right) \mu(dy)$$

$$= P_t f(w(1)).$$
References


