Finite variation of Lévy driven moving average-like processes

Jan Rosiński

University of Tennessee
Joint work with Andreas Basse-O’Connor

6th International Conference on Lévy Processes: Theory and Applications

Dresden, Germany, July 26 – 30, 2010
1. Mixed moving average-like processes
2. Zero-one law for locally finite total variation
3. Characterization of locally finite total variation
4. Stable mixed moving average-like processes
5. Semimartingale property
Let $X = (X_t)_{t \in \mathbb{R}}$ be an infinitely divisible mixed moving average-type process of the form

$$X_t = \int_{\mathbb{R} \times \mathcal{W}} (f(t - s, w) - f_0(-s, w)) M(ds, dw), \quad t \in \mathbb{R}. \quad (1)$$

$M$ is a homogeneous independently scattered random measure on $(\mathbb{R} \times \mathcal{W}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{A})$ with control measure $\text{Leb} \times m$, where $(\mathcal{W}, \mathcal{A}, m)$ is a $\sigma$-finite measure space.

The functions $f$ and $f_0$ are deterministic; when $f_0 = 0$ then (1) defines a mixed moving average and when $f_0 = f$ then $(X_t)_{t \in \mathbb{R}}$ is a generalized fractional process.
Moreover,
\[ \mathbb{E} e^{iuM((a,b] \times A)} = e^{(b-a)m(A)\psi(u)}, \]
where
\[ \psi(u) = -\frac{1}{2} \sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux) \rho(dx), \]
\( \sigma^2 \geq 0, \) and \( \rho \) is a Lévy measure on \( \mathbb{R} \) with
\[ \int x^2 \wedge |x| \rho(dx) < \infty. \]

Therefore, \( M \) is a mean zero random measure.
Let
\[ \Phi(u) = \int_{\mathbb{R}} (|ux|^2 \wedge |ux|) \rho(dx). \]

Process \((X_t)_{t \in \mathbb{R}}\) is defined to have finite mean, i.e., for all \(t \in \mathbb{R}\)
\[ \int_{\mathbb{R} \times \mathcal{W}} \Phi(f(t - s, w) - f_0(-s, w)) \, ds \, m(dw) < \infty. \]

It has the following properties
- stationary increments
- \(\mathbb{E}X_t = 0\)
- continuous in \(L^1\).
IDEA OF MIXING:

Suppose $m = \sum_{j=1}^{n} q_j \delta_{w_j}$, $q_j > 0$. Let $\{Z_j(t)\}_{\mathbb{R}}$ be i.i.d. (two-sided) Lévy processes with

$$\mathbb{E} e^{i u Z_j(t)} = e^{t \psi(u)}.$$ 

Put $f_j(t) = f(t, w_j)$ and $f_{0,j}(t) = f_0(t, w_j)$. Then the process $(X_t)_{t \in \mathbb{R}}$ in (1) has the same finite dim. distributions as

$$\sum_{j=1}^{n} \int_{\mathbb{R}} (f_j(t - s) - f_{0,j}(-s)) dZ_j(q_j s).$$
Gaussian m.a.’s blend together into a m.a.

Let $\rho = 0$, $\sigma^2 = 1$ and $X_t = \int_{\mathbb{R} \times W} f(t - s, w) M(ds, dw)$.

$$\text{Cov}(X_t, X_u) = \int_{W} \int_{\mathbb{R}} f(t - s, w) \overline{f(u - s, w)} \, dsm(dw)$$

$$= \frac{1}{2\pi} \int_{W} \int_{\mathbb{R}} e^{i(t-u)v} |\hat{f}(\cdot, w)(v)|^2 \, dvm(dw)$$

(Parseval’s theorem)

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(t-u)v} |g(v)|^2 \, dv$$

where

$$|g(v)|^2 = \int_{W} |\hat{f}(\cdot, w)(v)|^2 \, m(dw).$$
Put \( h(s) = \hat{g}(-s) \). Then we have

\[
\text{Cov}(X_t, X_u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(t-u)v} |\hat{h}(v)|^2 \, dv
\]

\[
= \int_{\mathbb{R}} h(t-s)h(u-s) \, ds
\]

Thus \((X_t)_{t \in \mathbb{R}}\) is the usual moving average

\[
(X_t)_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int_{\mathbb{R}} h(t-s)dW(s) \right\}_{t \in \mathbb{R}}
\]

\((W_t)_{t \in \mathbb{R}}\) is a two-sided standard BM.)
Mixed m.a.’s driven by lévy non-gaussian processes do not blend!

For example, let

\[ f_1(t) = \begin{cases} \sin(\pi t) & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad f_2(t) = \begin{cases} 1 - |1 - 2t| & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

and let \( \{Z_j(t)\}_{\mathbb{R}} \) be i.i.d. symmetric \( \alpha \)-stable Lévy processes.

Then there is no function \( f_3 : \mathbb{R} \mapsto \mathbb{R} \) such that

\[
\left\{ \int f_1(t - s) \, dZ_1(s) + \int f_2(t - s) \, dZ_2(s) \right\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int f_3(t - s) \, dZ_3(s) \right\}_{t \in \mathbb{R}}
\]

2. Zero-one law for locally finite total variation

\[ \|X\|_{TV[a,b]} := \sup_{a=t_0 < \cdots < t_n=b} \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}| \]

where the supremum is understood as the least modulo \( \mathbb{P} \) upper bound (a random variable in \([0, \infty])\).

Since our \((X_t)_{t \in \mathbb{R}}\) is stochastically continuous,

\[ \|X\|_{TV[a,b]} = \sup_{n \in \mathbb{N}} \sum_{k=1}^{2^n} |X_{r_{n,k}} - X_{r_{n,k-1}}| \]

\(r_{n,k} = a + (b - a)k2^{-n}\).

If \((X_t)_{t \in \mathbb{R}}\) is separable (e.g., it has right or left continuous paths), then the total variation can be evaluated pathwise

\[ \omega \mapsto \|X(\omega)\|_{TV[a,b]} . \]
Theorem

If \( \|f\|_{TV[-N,N]} < \infty \) m-a.e. for all \( N \in \mathbb{N} \), then

\[
P \left\{ \|X\|_{TV[-N,N]} < \infty \quad \forall N \in \mathbb{N} \right\} = 0 \text{ or } 1.
\]

Otherwise, this probability is zero.


3. Characterization of locally finite total variation

Theorem

Assume that $M$ does not have a Gaussian component ($\sigma^2 = 0$).

(A) Suppose for $m$-a.a. $w \in \mathcal{W}$, the map $s \mapsto f(s, w)$ is absolutely continuous and its derivative $\dot{f}(\cdot, w)$ satisfies

$$C_f := \int_{\mathbb{R} \times \mathcal{W}} \int_{\mathbb{R}} (|\dot{f}(s, w)|^2 \wedge |\dot{f}(s, w)|) \rho(dx) \, ds \, m(dw) < \infty.$$  

(2)

Then $(X_t)_{t \in \mathbb{R}}$ has absolutely continuous sample paths a.s. whose total variation is integrable on each finite interval. Moreover, a.s.

$$\frac{dX(t)}{dt} = \int_{\mathbb{R} \times \mathcal{W}} \dot{f}(t - s, w) \, M(ds, dw), \quad t \in \mathbb{R},$$

where on the right hand side is a stationary process with paths in $L^1$ on each finite interval.
Theorem (continue)

(B) Suppose that \((X_t)_{t \in \mathbb{R}}\) has finite total variation a.s. on each finite interval. Assume additionally that \(\int_{|x| \leq 1} |x| \rho(dx) = \infty\). Then for \(m\)-a.a. \(w \in W\), the map \(s \mapsto f(s, w)\) is absolutely continuous and its derivative \(\dot{f}(\cdot, w)\) satisfies

\[
k^*(w) = \sup_{s \in \mathbb{R}} \int_s^{s+1} |\dot{f}(u, w)| \, du < \infty.
\]

Suppose there are \(u_0 \geq 0\) and \(K_0 > 0\) such that

\[
u \int_u^{\infty} \rho(|x| > v) \, dv \leq K_0 \int_{|x| \leq u} x^2 \rho(dx), \quad u > u_0. \tag{3}
\]

If \(u_0 = 0\), or if \(u_0 > 0\) and \(\text{esssup}_W k^*(w) < \infty\), then (2) holds and so \(\mathbb{E}\|X\|_{TV[0,1]} < \infty\). If only \(u_0 > 0\), then

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} (|x\dot{f}(s, w)|^2 \wedge |x\dot{f}(s, w)|) \rho(dx) \, ds < \infty \quad m - a.e.
\]
Remark

Condition (3) holds when $M$ has finite variance, i.e.,
\[ \int_{\mathbb{R}} x^2 \rho(dx) < \infty. \]
Indeed, put $a = \int_{\mathbb{R}} x^2 \rho(dx) < \infty$ and let $u_0 > 0$ be such that $b = \int_{|x| \leq u_0} x^2 \rho(dx) > 0$. Then

\[ u \int_u^\infty \rho(|x| > v) \, dv \leq u \int_u^\infty v^{-2} a \, dv = a, \]

implying (3) with $K_0 = a/b$. 
We need

**Theorem (M.B. Marcus, JR (2001))**

*Let $X$ be a mean zero infinitely divisible random vector in a separable Hilbert space $E$, with Lévy measure $\nu$ and no Gaussian component. Let $\ell = \ell(\nu)$ be a unique solution of the equation*

$$
\int_E \|\ell^{-1}x\|^2 \wedge \|\ell^{-1}x\| \nu(dx) = 1.
$$

*Then*

$$(0.25) \ell(\nu) \leq \mathbb{E}\|X\| \leq (2.125) \ell(\nu).$$

*If $\nu$ is symmetric, the the upper bound constant can be decreased to 1.25.*

Actually, we use a corollary to this theorem.

**Corollary**

*Under the above assumptions,*

$$
\frac{1}{4} \min\{c(\nu), c(\nu)^{1/2}\} \leq \mathbb{E}\|X\| \leq \frac{5}{4} \max\{c(\nu), c(\nu)^{1/2}\}
$$

*where $c(\nu) = \int_E \|x\|^2 \wedge \|x\| \nu(dx)$.*
By the stationarity of increments,

\[ \mathbb{E}\|X\|_{TV[0,1]} = \sup_{n \in \mathbb{N}} \sum_{k=1}^{n} \mathbb{E}\left|X_{k2^{-n}} - X_{(k-1)2^{-n}}\right| = \sup_{n \in \mathbb{N}} \mathbb{E}\left|2^n(X_{2^{-n}} - X_0)\right|. \]

Recall

\[ \Phi(u) = \int_{\mathbb{R}} (|ux|^2 \wedge |ux|) \rho(dx). \]

\( \Phi \) is symmetric, strictly increasing, and comparable with a convex function \( \tilde{\Phi} \) given by

\[ \tilde{\Phi}(u) = \int_{\mathbb{R}} (|ux|^2 \mathbf{1}_{|ux| \leq 1} + (2|ux| - 1) \mathbf{1}_{|ux| > 1}) \rho(dx). \]

Notice that \( \tilde{\Phi}(u)/2 \leq \Phi(u) \leq \tilde{\Phi}(u). \)
By the above Corollary

\[ \frac{1}{4} \min \{ I_n, I_n^{1/2} \} \leq \mathbb{E} |2^n(X_{2^{-n}} - X_0)| \leq \frac{5}{4} \min \{ I_n, I_n^{1/2} \}, \]

where

\[ I_n = \int_{\mathbb{R} \times W} \Phi \left( 2^n[f(2^{-n} - s, w) - f(-s, w)] \right) \, ds \, m(dw). \]

Thus

\[ \mathbb{E} \| X \|_{TV[0,1]} < \infty \iff \sup_n I_n < \infty. \]
Jensen's inequality

\[ I_n \leq \int_{\mathbb{R} \times W} \tilde{\Phi} \left( 2^n [f(2^{-n} - s, w) - f(-s, w)] \right) \, ds \, m(dw) \]

\[ = \int_{\mathbb{R} \times W} \tilde{\Phi} \left( 2^n \int_{-s}^{2^{-n}-s} \dot{f}(t, w) \, dt \right) \, ds \, m(dw) \]

\[ \leq \int_{\mathbb{R} \times W} 2^n \int_{-s}^{2^{-n}-s} \tilde{\Phi} \left( \dot{f}(t, w) \right) \, dt \, ds \, m(dw) \]

\[ = \int_{\mathbb{R} \times W} \tilde{\Phi} \left( \dot{f}(t, w) \right) \, dt \, m(dw) \leq 2C_f. \]

Hence \((X_t)_{t \in \mathbb{R}}\) has integrable total variation on \([0, 1]\). By the stationarity of increments, it has integrable total variation on each finite interval.

\[ \square \]
Theorem

Let $M$ be a Gaussian random measure ($\rho = 0$). Then $(X_t)_{t \in \mathbb{R}}$ has finite total variation a.s. on each finite interval if and only if for $m$-a.a. $w \in W$, the map $s \mapsto f(s, w)$ is absolutely continuous and its derivative $\dot{f}(\cdot, w)$ satisfies

$$\int_W \int_{\mathbb{R}} |\dot{f}(s, w)|^2 \, ds \, m(dw) < \infty.$$
4. Stable mixed moving average-like processes

If

\[ \rho(dx) = \begin{cases} 
  c_1 |x|^{-1-\alpha} \, dx, & x < 0, \\
  c_2 x^{-1-\alpha} \, dx, & x > 0,
\end{cases} \]

then condition (3) holds with \( u_0 = 0 \) (\( \alpha \in (1, 2) \)). Indeed,

\[
u \int_0^\infty \rho(|x| > v) \, dv = \frac{c_1 + c_2}{\alpha(\alpha - 1)} u^{2-\alpha}
= \frac{2 - \alpha}{\alpha(\alpha - 1)} \int_{|x| \leq u} x^2 \, \rho(dx).
\]
Corollary

Let $M$ be a $\alpha$-stable random measure with $\alpha \in (1, 2)$. Then $(X_t)_{t \in \mathbb{R}}$ has finite total variation a.s. on each finite interval if and only if for $m$-a.a. $w \in W$, the map $s \mapsto f(s, w)$ is absolutely continuous and its derivative $\dot{f}(\cdot, w)$ satisfies

$$\int_W \int_{\mathbb{R}} |\dot{f}(s, w)|^\alpha ds \, m(dw) < \infty.$$

(see the Gaussian case)
Remark

Above computation shows that condition (3) holds for tempered-like distributions. In such cases we have the necessary and sufficient conditions for the finite variation, as a corollary to the main theorem.
5. Semimartingale property

Theorem

Assume \((Z_t)_{t \in \mathbb{R}}\) is square integrable, centered Lévy process of unbounded variation. Consider

\[ X_t = \int_{-\infty}^{t} f(t - s) \, dZ_s, \]

where \(f(s) = 0\) for \(s < 0\). Then \((X)_{t \geq 0}\) is an \(\mathcal{F}_t = \sigma(Z_s : -\infty < s \leq t)\)-semimartingale if and only if \(f\) is absolutely continuous with a density \(\dot{f}\) such that

\[ \int_{0}^{\infty} \int_{\mathbb{R}} \left( |\dot{f}(s)x|^2 \wedge |\dot{f}(s)x| \right) \rho(dx)ds < \infty, \]

and if \(\sigma^2 > 0\), \(\dot{f} \in L^2\).
Moreover, when \((X)_{t \geq 0}\) is an \((\mathcal{F}_t)\)-semimartingale it has the following decomposition

\[
X_t = X_0 + f(0) Z_t + \int_0^t \left( \int_{-\infty}^u f(u - s) dZ_s \right) du, \quad t \geq 0.
\]


These arguments are difficult to extend to the mixed case. This is a current work.
Related works:


Thank you!