Multivariate supOU processes

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Based on joint work with Ole E. Barndorff-Nielsen
Motivation and Idea
Motivation

- Many time series exhibit long range dependence and jumps.
- In finance volatility is often considered to have long memory.
- Often the use of continuous-time models is desirable.
- Ornstein-Uhlenbeck (OU) type processes are a very fundamental time series model in continuous time. They are the continuous-time analogue of autoregressive processes of order one (AR(1)).
- Fractionally integrated processes driven by Lévy processes have long memory, but no jumps.
Stationary univariate Ornstein-Uhlenbeck processes

Let $L$ be a univariate Lévy process with $E(\ln^+(L_1)) < \infty$ and $a < 0$. Then the integrals

$$X_t = \int_{-\infty}^{t} e^{a(t-s)} dL_s$$

are well-defined and the process $X$ is a stationary Ornstein-Uhlenbeck type process.

Provided $\mathbb{V}ar(L_1) < \infty$, we have

$$\mathbb{C}ov(X_h, X_0) = e^{ah} \mathbb{V}ar(X_0),$$

hence a short memory process.
Finite superposition of OU type processes

- Let $X_1$ and $X_2$ be two independent stationary OU type processes of finite variance with mean reversion coefficients $a_1$ and $a_2$.
- $\pi_1, \pi_2 \geq 0$ and $\pi_1 + \pi_2 = 1$
- The process $X = \pi_1 X_1 + \pi_2 X_2$ is called a superposition of two OU type processes (supOU process).
- $\operatorname{Cov}(X_h, X_0) = \pi_1^2 e^{a_1 h} \operatorname{Var}(X_{1,0}) + \pi_2^2 e^{a_2 h} \operatorname{Var}(X_{2,0})$
- Assume w.l.o.g. $a_2 > a_1$.
  - For $h \to \infty$ we have $\operatorname{Cov}(X_h, X_0) \sim \pi_2^2 e^{a_2 h} \operatorname{Var}(X_{2,0})$.
  - Hence: Still a short memory process, asymptotic decay governed by slowest exponential decay rate
  - Initial decay of the autocovariance (i.e. close to zero) usually governed by faster exponential decay rate.
- The same holds for superpositions of $n$ independent OU processes.
Infinite superposition of OU type processes

- **Idea:** Adding up infinitely many OU type processes with “eventually arbitrarily slow exponential decay” (i.e. $a$ close to 0) may result in autocovariance with non-exponential decay.

- **Extension:** “Sum up” independent OU type processes with all possible mean-reversion speeds $a \in \mathbb{R}^-$ weighted by a probability measure $\pi$:

$$X_t = \int_{\mathbb{R}^-} \int_{-\infty}^{t} e^{a(t-s)} dL_s^{(a)} \pi(da)$$

$$\text{Cov}(X_h, X_0) = \int_{\mathbb{R}^-} \frac{e^{ah} \text{Var}(L_1)}{2a} \pi^2(da)$$

**Example:**

$\pi^2 = -CG(\alpha, \beta), \alpha > 1, C > 0$:

$$\text{Cov}(X_h, X_0) = \frac{C\beta^\alpha}{2(\alpha-1)} (\beta + h)^{1-\alpha} \text{Var}(L_1)$$

- **Need to address this in a rigorous mathematical manner.**

- **Intuitive idea:** Different “News” are forgotten at different exponential rates. Some news are forgotten very slowly $\Rightarrow$ Long-range dependence

- **Alternative to long memory via “fractional integration”.**
**Idea leading to rigorous definition in one dimension**

If $L$ is a pure jump Lévy process of finite variation, the stationary OU type process is given by:

$$X_t = \sum_{-\infty < s \leq t} e^{-a(t-s)} \Delta L_s$$

Hence: All jumps are forgotten exponentially fast at the same “speed” $a$.

**Idea to obtain long memory:**

Choose for each jump of the Lévy process a different speed randomly with distribution $\pi$.

Asymptotically the smallest possible speeds will determine the dependence structure.

If $\pi$ has mass in any neighbourhood of zero, “dependence” should no longer decay exponentially and one should even by able to get power decays of the autocovariance function and long memory (non-integrable autocovariance function).
Some matrix notation

- $M_d(\mathbb{R})$: the real $d \times d$ matrices.
- $S_d$: the real symmetric $d \times d$ matrices.
- $S_d^+$: the positive semi-definite $d \times d$ matrices (covariance matrices) (a closed cone).
- $S_d^{++}$: the positive definite $d \times d$ matrices (an open cone).
- $A^{1/2}$: for $A \in S_d^+$ the unique positive semi-definite square root (functional calculus).
- $\text{tr}(A)$: The trace of a matrix $A$. 
Multivariate OU type processes
Theorem

Let \((L_t)_{t \in \mathbb{R}}\) be a \(d\)-dimensional Lévy process with \(E(\max(\log \|L_1\|, 0)) < \infty\) and \(A \in M_d(\mathbb{R})\) such that \(\sigma(A) \subset (-\infty, 0) + i\mathbb{R}\).

Then the stochastic differential equation of Ornstein-Uhlenbeck-type

\[dX_t = AX_{t-} dt + dL_t\]

has a unique stationary solution

\[X_t = \int_{-\infty}^{t} e^{A(t-s)} dL_s.\]
Multivariate supOU processes
**Lévy basis (i.d.i.s.r.m.)**

\[ M_d^- := \{ X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0) + i\mathbb{R} \} \]

\[ \mathcal{B}_b \left( M_d^- \times \mathbb{R} \right) : \text{the bounded Borel sets of } M_d^- \times \mathbb{R}. \]

**Definition**

A family \( \Lambda = \{ \Lambda(E) : E \in \mathcal{B}_b \left( M_d^- \times \mathbb{R} \right) \} \) of \( \mathbb{R}^d \)-valued random variables is called an \( \mathbb{R}^d \)-valued Lévy basis on \( M_d^- \times \mathbb{R} \) if:

1. the distribution of \( \Lambda(E) \) is infinitely divisible for all \( E \in \mathcal{B}_b \left( M_d^- \times \mathbb{R} \right) \),

2. for any \( n \in \mathbb{N} \) and pairwise disjoint sets \( E_1, \ldots, E_n \in \mathcal{B}_b \left( M_d^- \times \mathbb{R} \right) \) the random variables \( \Lambda(E_1), \ldots, \Lambda(E_n) \) are independent and

3. for any pairwise disjoint sets \( E_i \in \mathcal{B}_b \left( M_d^- \times \mathbb{R} \right) \) with \( i \in \mathbb{N} \) satisfying \( \bigcup_{n \in \mathbb{N}} E_n \in \mathcal{B}_b \left( M_d^- \times \mathbb{R} \right) \) the series \( \sum_{n=1}^{\infty} \Lambda(E_n) \) converges almost surely and \( \Lambda \left( \bigcup_{n \in \mathbb{N}} E_n \right) = \sum_{n=1}^{\infty} \Lambda(E_n) \).
Lévy basis (i.d.i.s.r.m.)

We consider only Lévy bases having characteristic function of the form

\[ E(\exp(iu^*\Lambda(E))) = \exp(\psi(u)\Pi(E)) \]

for all \( u \in \mathbb{R}^d \) and \( E \in \mathcal{B}_b\left(M_d^-(\mathbb{R}) \times \mathbb{R}\right) \),

where \( \Pi = \pi \times \lambda \) is the product of a probability measure \( \pi \) on \( M_d^-(\mathbb{R}) \) and the Lebesgue measure \( \lambda \) on \( \mathbb{R} \).

Moreover,

\[ \psi(u) = iu^*\gamma - \frac{1}{2}u^*\sigma u + \int_{\mathbb{R}^d} (e^{iu^*x} - 1 - iu^*x1_{[0,1]\|x\|}) \nu(dx) \]

is the cumulant transform of an infinitely divisible distribution on \( \mathbb{R}^d \) with Lévy-Khintchine triplet \( (\gamma, \sigma, \nu) \).

\[
L_t = \Lambda(M_d^- \times (0, t]) \quad \text{and} \quad L_{-t} = \Lambda(M_d^- \times (-t, 0)) \quad \text{for} \quad t \in \mathbb{R}^+
\]

is a Lévy process with characteristic triplet \( (\gamma, \sigma, \nu) \) and it is called “the underlying Lévy process”.
Multivariate supOU processes

Theorem
Assume:

1. \( \int_{\|x\| > 1} \ln(\|x\|) \nu(dx) < \infty \)

2. There exist measurable functions \( \rho : M_d^- \to \mathbb{R}^- \setminus \{0\} \) and \( \kappa : M_d^- \to [1, \infty) \) such that:

\[
\|e^{As}\| \leq \kappa(A)e^{\rho(A)s} \quad \forall \ s \in \mathbb{R}^+, \ \pi-a.s., \quad -\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty.
\]

Then the process \((X_t)_{t \in \mathbb{R}}\) given by

\[
X_t = \int_{-\infty}^{t} \int_{M_d^-} e^{A(t-s)} \Lambda(dA, ds)
\]

is well-defined for all \( t \in \mathbb{R} \) and \( X \) is stationary.
Stationary distribution

The distribution of $X_t$ is infinitely divisible with characteristic function

$$E(\exp(iu^*X_t))) = \exp\left(iu^*\gamma_X - \frac{1}{2}u^*\sigma_X u + \int_{S_d} (e^{iu^*x} - 1 - iu^*x1_{[0,1]}(\|x\|))\nu_X(dx)\right),$$

$u \in \mathbb{R}^d$, where

$$\gamma_X = \int_{M_d} \int_0^\infty \left( e^{As}\gamma + \int_{\mathbb{R}^d} e^{As} x (1_{[0,1]}(\|e^{As}x\|) - 1_{[0,1]}(\|x\|))\nu(dx) \right) ds\pi(dA),$$

$$\sigma_X = \int_{M_d} \int_0^\infty e^{As}\sigma e^{A^*s} ds\pi(dA),$$

$$\nu_X(E) = \int_{M_d} \int_0^\infty \int_{\mathbb{R}^d} 1_E(e^{As}x)\nu(dx)ds\pi(dA) \quad \forall E \subseteq \mathcal{B}(\mathbb{R}^d).$$
Restricting $A$ to normal matrices

The condition

$$\|e^{As}\| \leq \kappa(A)e^{\rho(A)s} \forall s \in \mathbb{R}^+, \pi - \text{a.s.}, -\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty.$$ 

becomes:

- $-\int_{\mathbb{R}^-} \frac{1}{A} \pi(dA) < \infty$ in dimension 1 – the well-known necessary and sufficient existence condition for one-dimensional supOU processes. (cf. Barndorff-Nielsen (2001), Fasen and Klüppelberg (2007))

- For $\pi$ concentrated on the normal (especially symmetric) matrices:

  $$-\int_{M_d^-} \frac{1}{\max(\mathbb{R}(\sigma(A)))} \pi(dA) < \infty$$
Necessary conditions for the existence of supOU processes

\[ j(Z) = \min_{\|x\|=1} \|Zx\|, \ Z \in M_d(\mathbb{R}), \ \text{denotes the modulus of injectivity.} \]

**Proposition**

Assume there exist measurable functions \( \tau : M_d^- \to \mathbb{R}^+ \setminus \{0\} \) and \( \vartheta : M_d^- \to (0, 1] \) such that:

\[ j(e^{As}) \geq \vartheta(A)e^{-\tau(A)s} \ \forall \ s \in \mathbb{R}^+, \pi - \text{a.s.} \]

Then necessary conditions for the supOU integral to exist are:

\[
\int_{\vartheta(A) \geq \epsilon} \frac{1}{\tau(A)} \pi(dA) < \infty, \ \text{for any} \ \epsilon \in (0, 1]
\]

such that \( \nu(\{\|x\| > 1/\epsilon\}) > 0, \pi(\{\vartheta(A) \geq \epsilon\}) > 0, \)

\[
\int_{M_d^-} \frac{\vartheta(A)^2}{\tau(A)} \pi(dA) < \infty, \ \text{provided} \ j(\Sigma) > 0 \ \text{or} \ \nu(\{\|x\| \leq 1\}) > 0, \ \text{and}
\]

\[
\int_{\|x\|>1} \ln(\|x\|) \nu(dx) < \infty.
\]
“SDE representation” and path properties

**Theorem**

Assume that \( \int_{\|x\| \leq 1} \|x\| \nu(dx) < \infty \). Provided

\[
- \int_{M_d^-} \frac{(\|A\| \vee 1)\kappa(A)}{\rho(A)} \pi(dA) < \infty \quad \text{and} \quad \int_{M_d^-} \|A\|\kappa(A)\pi(dA) < \infty
\]

it holds that

\[ X_t = X_0 + \int_0^t Z_u du + L_t \]

where \( L_t = \Lambda(M_d^- \times (0, t]) \) is a Lévy process of finite variation and

\[ Z_u = \int_{M_d^-} \int_{-\infty}^u Ae^{A(u-s)} \Lambda(dA, ds) \]

for all \( u \in \mathbb{R} \) with the integral existing \( \omega \)-wise.

⇒ The paths of \( X \) are càdlàg and of finite variation on compacts.

Multivariate SupOU Processes

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Existence of moments

Theorem

1. If \( \int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \) for \( r \in (0, 2) \), then \( E(\|X_t\|^r) < \infty \).

2. If \( r \in (2, \infty) \) and

\[
\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty, \quad \int_{M_d^-} \frac{\kappa(A)^r}{\rho(A)} \pi(dA) < \infty,
\]

then \( E(\|X_t\|^r) < \infty \).

3. Necessary conditions for \( X \) to have a finite \( r \)-th moment are

\[
\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \text{ in general and } \int_{\vartheta(A) \geq \varepsilon} \frac{\vartheta(A)^r}{\tau(A)} \pi(dA) < \infty, \text{ for any } \varepsilon \text{ such that } \nu(\{\|x\| > 1/\varepsilon\}) > 0 \text{ and } \pi(\{\vartheta(A) \geq \varepsilon\}) > 0.
\]
The extremal properties can be analysed using the notion of multivariate regular variation.

If $\Lambda$ is regularly varying with index $\alpha$, then so is the supOU process $X$.

This follows from a general result for mixed moving average processes, see the poster of Martin Moser outside for more details.
Second order moment structure

If $\int_{||x||>1} ||x||^2 \nu(dx) < \infty$, then $E(||X_0||^2) < \infty$ and we have

$$E(X_0) = - \int_{M_d^-} A^{-1} \left( \gamma + \int_{|x|>1} x \nu(dx) \right) \pi(dA)$$

$$\text{Var}(X_0) = - \int_{M_d^-} \left( \mathcal{A}(A))^{-1} \left( \sigma + \left( \int_{\mathbb{R}^d} xx^* \nu(dx) \right) \right) \right) \pi(dA)$$

$$\text{Cov}(X_h, X_0) = - \int_{M_d^-} e^{Ah} \left( \mathcal{A}(A))^{-1} \left( \sigma + \left( \int_{\mathbb{R}^d} xx^* \nu(dx) \right) \right) \right) \pi(dA) \text{ for } h \in \mathbb{R}^+.$$

with $\mathcal{A}(A) : M_d(\mathbb{R}) \rightarrow M_d(\mathbb{R})$, $X \mapsto AX + XA^*$. Moreover, it holds that

$$\lim_{h \rightarrow \infty} \text{Cov}(X_h, X_0) = 0.$$
An example with long memory

\( \pi \): the distribution of \( RB \) with a diagonalisable \( B \in M_d^\sim \) and \( R \) a real \( \Gamma(\alpha,\beta) \)-distributed random variable with \( \alpha > 1, \beta \in \mathbb{R}^+ \setminus \{0\} \).

For the autocovariance function for positive lags \( h \) one obtains

\[
\text{Cov}(X_h, X_0) = -\frac{\beta^\alpha}{\alpha - 1} (\beta I_d - Bh)^{1-\alpha} B^{-1} \left( \Sigma + \int_{\mathbb{R}^d} xx^* \nu(dx) \right)
\]

with \( B : M_d(\mathbb{R}) \to M_d(\mathbb{R}), X \mapsto BX + XB^* \).

\( \Rightarrow \) power decay in the autocovariance function

\( \Rightarrow \) For \( \alpha \in (1, 2) \): long memory.
Application: A Multivariate Stochastic Volatility Model
Positive semi-definite OU type processes

Theorem
Let \((L_t)_{t \in \mathbb{R}}\) be a matrix subordinator with \(E(\max(\log \|L_1\|, 0)) < \infty\) and \(A \in \mathcal{M}_d(\mathbb{R})\) such that \(\sigma(A) \subset (-\infty, 0) + i\mathbb{R}\). Then the stochastic differential equation of Ornstein-Uhlenbeck-type

\[ d\Sigma_t = (A \Sigma_t + \Sigma_t A^T)dt + dL_t \]

has a unique stationary solution

\[ \Sigma_t = \int_{-\infty}^t e^{A(t-s)} dL_s e^{A^T(t-s)} \]

or, in vector representation,

\[ \text{vec}(\Sigma_t) = \int_{-\infty}^t e^{(I_d \otimes A + A \otimes I_d)(t-s)} d\text{vec}(L_s). \]

Moreover, \(\Sigma_t \in \mathbb{S}_d^+\) for all \(t \in \mathbb{R}\).
Positive semi-definite supOU processes

Theorem

Assume:

- $\int_{\|x\|>1} \ln(\|x\|) \nu(dx) < \infty$
- There exist measurable functions $\rho : M_d^- \to \mathbb{R}^{-}\{0\}$ and $\kappa : M_d^- \to [1, \infty)$ such that:

$$\|e^{As}\| \leq \kappa(A)e^{\rho(A)s} \forall s \in \mathbb{R}^+, \pi-a.s., -\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty.$$

Then the process $(\Sigma_t)_{t \in \mathbb{R}}$ given by

$$\Sigma_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds)e^{A^T(t-s)}$$

is well-defined for all $t \in \mathbb{R}$ and $\omega \in \Omega$ and $\Sigma$ is stationary. $\Sigma_t \in S_d^+$ for all $t \in \mathbb{R}$. 
A Stochastic volatility model

Let $\Sigma$ be a positive semi-definite supOU process with càdlàg paths.

Then

$$Y_t = Y_0 + \int_0^t (\mu + \beta \Sigma_s) \, ds + \int_0^t \Sigma_s^{1/2} \, dW_s + \rho dL_t$$

with $\mu \in \mathbb{R}^d$, $\beta, \rho : \mathbb{S}_d \to \mathbb{R}^d$ linear and

$$L_t = \int_{M_d^-} \int_0^t \Lambda(dA, ds)$$

the underlying Lévy process is a well-defined $d$-dimensional stochastic volatility model.
The autocovariance structure of the log-returns and the integrated volatility

Assume $\mu, \beta, \rho = 0$, take $\Delta > 0$ and set

$$Y_n = Y_{n\Delta} - Y_{(n-1)\Delta} \quad (1)$$

$$V_n = \int_{(n-1)\Delta}^{n\Delta} \Sigma_s ds. \quad (2)$$

Then

$$\text{Cov}(Y_1 Y_1^T, Y_{h+1} Y_{h+1}^T) = \text{Cov}(V_1, V_{h+1})$$

for all $h \in \mathbb{N}$. 

Multivariate SupOU Processes

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The integrated volatility

Proposition

If

$$\int_{M_d^-} \kappa(A)^2 \pi(dA) < \infty,$$

the paths of $\Sigma$ are locally uniformly bounded in $t$ for every $\omega \in \Omega$.

Furthermore, $\Sigma_t^+ = \int_0^t \Sigma_s ds$ exists for all $t \in \mathbb{R}^+$ and

$$\Sigma_t^+ = \int_{M_d^-} \int_{-\infty}^t (A(A))^{-1} \left( e^{A(t-s)} \wedge (dA, ds) e^{A^T(t-s)} \right) - \int_{M_d^-} \int_{-\infty}^0 (A(A))^{-1} \left( e^{-As} \wedge (dA, ds) e^{-A^T s} \right)$$

$$- \int_{M_d^-} \int_0^t (A(A))^{-1} \wedge (dA, ds)$$

with $A(A) : \mathbb{S}_d \to \mathbb{S}_d, X \mapsto AX + XA^T$.
Second order structure of $V$ and $YY^T$

**Theorem**

Let $\mu = \beta = \rho = 0$ and assume $\Sigma \in L^2$. Then $(V_n)_{n \in \mathbb{N}}$ is stationary and square-integrable with

$$E(V_1) = -\Delta \int_{M_d} A(A)^{-1} \left( \gamma_0 + \int_{S_d} x\nu(dx) \right) \pi(dA),$$

$$\text{Var}(\text{vec}(V_1)) = r^{++}(\Delta) + r^{++}(\Delta)^*,$$

$$\text{Cov}(\text{vec}(V_{h+1}), \text{vec}(V_1)) = r^{++}(h\Delta + \Delta) - 2r^{++}(h\Delta) + r^{++}(h\Delta - \Delta)$$

$$= -\int_{M_d} g(A, h)(A(A))^{-1} \left( \int_{S_d} \text{vec}(x)\text{vec}(x)^*\nu(dx) \right) \pi(dA), \quad h \in \mathbb{N},$$

with $r^{++}(t) = \int_0^t \int_0^u \text{Cov}(\text{vec}(\Sigma_s), \text{vec}(\Sigma_0))dsdu$ and

$$g(A, h) = (A \otimes I_d + I_d \otimes A)^{-2} \cdot \left( e^{(A \otimes I_d + I_d \otimes A)(h\Delta + \Delta)} - 2e^{(A \otimes I_d + I_d \otimes A)h\Delta} + e^{(A \otimes I_d + I_d \otimes A)(h\Delta - \Delta)} \right).$$

It holds that $\lim_{h \to \infty} \text{Cov}(\text{vec}(V_{h+1}), \text{vec}(V_1)) = 0$. 

Multivariate SupOU Processes

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Second order structure of $V$ and $YY^T$ II

Theorem (Continued)

Likewise the log-price increments $(Y_n)_{n \in \mathbb{N}}$ as well as their “squares” $(Y_n Y_n^T)_{n \in \mathbb{N}}$ are stationary and square-integrable with

\[
E(Y_1) = 0, \quad \text{Var}(Y_1) = E(V_1),
\]
\[
\text{Cov}(Y_{h+1}, Y_1) = 0 \quad \forall \ h \in \mathbb{N},
\]
\[
E(Y_1 Y_1^T) = E(V_1),
\]
\[
\text{Var}(\text{vec}(Y_1 Y_1^T)) = (I_d^2 + Q + PQ)(r^{++}(\Delta) + r^{++}(\Delta)^T) + (I_d^2 + P)(E(V_1) \otimes E(V_1))
\]
\[
\text{Cov}\left(\text{vec}(Y_{h+1} Y_{h+1}^T), \text{vec}(Y_1 Y_1^T)\right) = \text{Cov}(\text{vec}(V_{h+1}), \text{vec}(V_1)) \quad \text{for} \ h \in \mathbb{N}
\]

where $P, Q$ are certain linear operators.
Long memory in the SV model?

Theorem

(i) If $\text{Cov}(\text{vec}(\Sigma_h), \text{vec}(\Sigma_0))_{ij} \sim Ch^{-\alpha}$ for $h \to \infty$ with $\alpha > 0$ and $C \in \mathbb{R}\{0\}$, then

$$\text{Cov}(\text{vec}(V_{h+1}), \text{vec}(V_1))_{ij} \sim C\Delta^{2-\alpha}h^{-\alpha} \text{ for } h \to \infty.$$  \hfill (3)

(ii) If $\text{Cov}(\text{vec}(\Sigma_h), \text{vec}(\Sigma_0))_{ij} \sim Ce^{-\alpha h}$ with $\alpha > 0$ and $C \in \mathbb{R}\{0\}$, then

$$\liminf_{h \to \infty} \left| \frac{\text{Cov}(\text{vec}(V_{h+1}), \text{vec}(V_1))_{ij}}{C\Delta^{2}e^{-\alpha(h\Delta+\Delta)}} \right| \geq 1, \quad \text{(4)}$$

$$\limsup_{h \to \infty} \left| \frac{\text{Cov}(\text{vec}(V_{h+1}), \text{vec}(V_1))_{ij}}{C\Delta^{2}e^{-\alpha(h\Delta-\Delta)}} \right| \leq 1. \quad \text{(5)}$$
An example with long memory revisited

$\pi$: the distribution of $RB$ with a diagonalisable $B \in M_d^-$ and $R$ a real $\Gamma(\alpha, \beta)$-distributed random variable with $\alpha > 1, \beta \in \mathbb{R}^+ \setminus \{0\}$.

Then the squared returns have a polynomially decaying autocovariance

$$\text{Cov}(Y_1 Y_1^T, Y_{h+1} Y_{h+1}^T)_{ij} \sim C_{ij} h^{1-\alpha}$$

and long memory if $\alpha \in (1, 2)$.

This is far from obvious from the explicit formulae (with $\mathcal{B} = (B \otimes I_d + I_d \otimes B)$):

$$\Gamma_h = \mathcal{B}^{-2} \left( (\beta l_d^2 - \mathcal{B}(h\Delta + \Delta))^{3-\alpha} - 2(\beta l_d^2 - \mathcal{B}h\Delta)^{3-\alpha} + (\beta l_d^2 - \mathcal{B}(h\Delta - \Delta))^{3-\alpha} \right) (2 - \alpha)(3 - \alpha), \quad \alpha \neq 2, 3$$

$$\Gamma_h = \mathcal{B}^{-2} \left( (\beta l_d^2 - \mathcal{B}(h\Delta + \Delta)) \log(\beta l_d^2 - \mathcal{B}(h\Delta + \Delta)) 
- 2(\beta l_d^2 - \mathcal{B}h\Delta) \log(\beta l_d^2 - \mathcal{B}h\Delta) + (\beta l_d^2 - \mathcal{B}(h\Delta - \Delta)) \log(\beta l_d^2 - \mathcal{B}(h\Delta - \Delta)) \right), \quad \alpha = 2$$

$$\Gamma_h = \mathcal{B}^{-2} \left( \log(\beta l_d^2 - \mathcal{B}(h\Delta + \Delta)) - 2\log(\beta l_d^2 - \mathcal{B}h\Delta) + \log(\beta l_d^2 - \mathcal{B}(h\Delta - \Delta)) \right) (2 - \alpha), \quad \alpha = 3$$
Regular variation

- If $\Lambda$ is regularly varying with index $\alpha$, then so are the supOU volatility process $\Sigma$, the integrated volatility $V$ and the log price increments $Y$.

- This follows again from general results for mixed moving average processes, see the poster of Martin Moser outside for more details.
Thank you very much for your attention!


