Technique for computing the PDFs and CDFs of non-negative infinitely divisible random variables

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Goal

• Goal: To develop an efficient technique for computing numerically the PDF and CDF of non-negative infinitely divisible random variables.

• We want a high precision, for example $10^{-6}$.
The Fourier transform method

Let $\Pi$ be the Lévy-Khintchine measure.

Contrast with the Fourier transform method one needs to compute accurately two integrals with oscillating integrands:

$$\text{Chf}(\theta) = -\int_{0}^{\infty} (1 - e^{i\theta u})\Pi(du)$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\theta x} \text{Chf}(\theta) d\theta,$$

Our method: we only to compute integrals with exponentially decreasing integrands of the type:

$$\int_{0}^{\infty} u^n e^{-\lambda u} \Pi(du), \quad n \geq 1$$
Our technique uses the Lévy-Khintchine representation of the Laplace transform of the distribution.

It then uses the Post-Widder method for Laplace transform inversion and combines it with a convergence acceleration method to obtain accurate results.

We will demonstrate this technique on several examples including the stable distribution, mixtures thereof, and integrals with respect to non-negative Poisson and Lévy processes:

\[ I(g) = \int_{0}^{\infty} g(s) N(ds), \quad I_L(g) = \int_{0}^{\infty} g(s) L(ds). \]
Outline:

- Lévy-Khintchine formula for the characteristic function
- Post-Widder formula for the inversion of the Laplace transform
- Description of the 3 steps method
- Testing the method with known distributions (stable, chi-squared)
- Application of the method to unknown distributions (mixtures of stable, Poisson integrals, Lévy integrals)

● Software packages written in *Matlab* and *Mathematica* are available
The method applies to infinitely divisible random variables

\[ X > -a, \quad a \geq 0. \]

The distribution of \( X \) is said to be *infinitely divisible* if for any positive integer \( n \), we can find i.i.d. random variables \( X_{i,n} \), \( i = 1, 2, \ldots, n \) such that

\[ X \xrightarrow{d} X_{1,n} + X_{2,n} + \cdots + X_{n,n}. \]

We want to evaluate the PDF and CDF of \( X \). Since \( a \) only shifts the distribution, we will assume \( a = 0 \).
Since $X > 0$, we may consider the Laplace transform of its distribution:

$$\psi(\lambda) \equiv \mathbb{E} e^{-\lambda X} = e^{-\phi(\lambda)}, \quad \lambda > 0.$$ 

Its exponent $\phi(\lambda)$, called the *Laplace exponent*, can be expressed as

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda u}) \Pi(du).$$

Here, $\Pi$ is a measure on $(0, \infty)$, called the *Lévy-Khintchine measure*, which satisfies

$$\int_0^\infty (1 \land u) \Pi(du) < \infty.$$
We use a tilde to denote a Laplace transform:

$$\psi(\lambda) = E e^{-\lambda X} = \tilde{f}(\lambda) = \int_{0}^{\infty} e^{-\lambda x} f(x) dx,$$

$$\Psi(\lambda) \equiv \tilde{F}(\lambda) = \frac{\psi(\lambda)}{\lambda}.$$

Thus, obtaining the PDF $f$ and CDF $F$ is a matter of inverting a Laplace transform. Generally, this task is not easy. Typically it is done by complex integration of the Laplace transform, which can be difficult if the integrands are slowly-decaying, oscillatory functions. This causes many numerical integration methods to converge slowly. Here, we apply a different method of Laplace inversion known as the *Post-Widder (PW) method*. 
Suppose \( f \) continuous. Then for \( x > 0 \),

\[
f(x) = \lim_{k \to \infty} \frac{(-1)^{k-1}}{(k-1)!} \left( \frac{k}{x} \right)^k \tilde{f}(k-1) \left( \frac{k}{x} \right) = \lim_{k \to \infty} f_k(x)
\]

where \( \tilde{f}^{(k-1)}(k/x) \) denotes the \( (k-1)^{th} \) derivative of \( \tilde{f} \) evaluated at \( k/x \).

- Thus, instead of integrating the Laplace transform, we are taking arbitrarily high derivatives.
- The fact that \( f_k(x) \to f(x) \) as \( k \to \infty \) can be seen by approximating \( f(x) \) by \( \mathbb{E} f(\bar{Y}_k) \), where \( \bar{Y}_k \) is an average of i.i.d. gamma random variables with mean \( x \) and variance \( x^2 \) and then applying the law of large numbers.
But the convergence of the approximation $f_k(x)$ to the PDF $f(x)$ is slow in general.

This is a general feature of the PW formula since the errors $\epsilon_k(f; x) \equiv f(x) - f_k(x)$ have a power series expansion:

$$\epsilon_k(f; x) = \sum_{m=1}^{\infty} \frac{a_m(x)}{k^m} = O(1/k).$$

The PW method alone is thus inadequate for computing the inverse Laplace transform to a high level of precision.
Example illustrating the slow convergence

Suppose \( f \) is the PDF of an inverse Gaussian distribution whose Laplace transform is given by \( \tilde{f}(\lambda) = \exp(-\lambda^{-1/2}) \). We plot the exact formula for \( f \) (top) as well as \( f_k \) for \( k = 1, 10 & 50 \). Notice that even with 49 derivatives of \( \tilde{f} \), the approximation is still poor (barely \( 10^{-1} \) accuracy instead of the desired \( 10^{-6} \)).
To obtain high precision, it is necessary to couple the PW approximations with a convergence acceleration method which extrapolates the limit based on a finite collection of terms in the sequence.
Our method produces the best results if the PDF and/or CDF are smooth functions for $x > 0$.

The method still produces useful results in the non-smooth case, however it may fail to converge near points where the PDF or CDF lacks smoothness.

This method should not be used for distributions which contain atoms, such as the Poisson distribution, or even distributions whose density lacks smoothness, such as a compound Poisson distributions with bounded jump distribution. For such distributions we developed another method based on the Kolmogorov-Feller forward equation.
The 3 steps of the method

The method for computing the PDF $f$ at $x > 0$ involves three steps:

1. Compute the derivatives of the Laplace exponent $\phi(\lambda)$.
2. For some sequence $k_1 < k_2 < \cdots < k_N$, compute the approximations $f_{k_i}(x)$.
3. Use the points $(k_1, f_{k_1}(x)), \ldots, (k_N, f_{k_N}(x))$ to extrapolate $\lim_{k \to \infty} f_k(x)$.

For the CDF, same method, but replace $f$ with $F$. 
Step 1: Obtaining the derivatives of the Laplace exponent $\phi(\lambda)$

These can be computed directly, since the LK formula implies that for any $\lambda > 0$,

$$
\phi^{(n)}(\lambda) = \begin{cases} 
\int_{0}^{\infty} (1 - e^{-\lambda u}) \Pi(du), & n = 0 \\
(-1)^{n+1} \int_{0}^{\infty} u^{n} e^{-\lambda u} \Pi(du), & n \geq 1 
\end{cases}
$$

- For the examples considered here, these integrals have a closed form expression and can be computed easily.
- If a closed form is not available, many numerical integration methods are effective as these integrands are non-negative, exponentially decaying functions. Use a small relative error tolerance, as $\phi^{(n)}(\lambda)$ can become extremely small for large $\lambda$. 
Step 2: Obtaining the derivatives of 
\( \psi(\lambda) = \tilde{f}(\lambda) = \exp(-\phi(\lambda)) \)

Since \( \psi'(\lambda) = -\phi'(\lambda)\psi(\lambda) \), Leibnitz’s formula implies that for any \( n \geq 1 \), the \( k^{th} \) derivative of \( \psi \) is given by

\[
\psi^{(k)}(\lambda) = - \sum_{j=0}^{k-1} \binom{k-1}{j} \psi^{(j)}(\lambda) \phi^{(k-j)}(\lambda).
\]

Notice that for \( \lambda \) fixed, the values of \( \psi(\lambda), \psi'(\lambda), \ldots, \psi^{(k)}(\lambda) \) can be computed recursively if one has first computed \( \phi^{(j)}(\lambda) \) for \( j = 1, 2, \ldots, k \).
Step 3: Extrapolation

Since \( \tilde{f}^{(k-1)} = \psi^{(k-1)} \), we can compute

\[
 f_k(x) = \frac{(-1)^{k-1}}{(k-1)!} \left( \frac{k}{x} \right)^k \tilde{f}^{(k-1)} \left( \frac{k}{x} \right).
\]

The PW formula says \( f(x) = \lim_{k \to \infty} f_k(x) \).

- We now want to approximate the limit as \( k \to \infty \).
- It is convenient to set

\[
 h = \frac{1}{k}
\]

and consider what happens as

\[
 h \to 0.
\]

(We shall use both notations \( h \) and \( k \).)
Step 3a: Polynomial interpolation/extrapolation

Choose non-consecutive $k_1 < k_2 < \cdots < k_n$ and recall that $h = k^{-1}$. We have the points (functions of $h = k^{-1}$):

$$(h_1, f_{k_1}(x)), (h_2, f_{k_2}(x)), \ldots, (h_N, f_{k_N}(x)).$$

- Interpolate these points by a polynomial in $h$ of degree $N - 1$:

$$P_N(h) = \sum_{i=1}^{N} c_i(h)f_{k_i}(x) \text{ where } c_i(h) = \prod_{j \neq i} \frac{h - h_j}{h_i - h_j}.$$  

- Extrapolate this polynomial to $h = 0$ (corresponding to $k = \infty$).

- We obtain the approximation:

$$f(x) \approx P_N(0) = \sum_{i=1}^{N} c_i(0)f_{k_i}(x).$$
Step 3b: Improving the approximation

- Construct a second estimate of \( f(x) \), \( \tilde{P}_N(0) \), with the property that

\[
\lim_{N \to \infty} \frac{P_N(0) - f(x)}{\tilde{P}_N(0) - f(x)} = -1.
\]

Thus, asymptotically, \( \tilde{P}_N(0) \) is as good an approximation to \( f(x) \) as \( P_N(0) \), except that it approaches \( f(x) \) from the opposite direction.

- Use as approximation:

\[
f(x) \approx \frac{P_N(0) + \tilde{P}_N(0)}{2}
\]
Approximation error

**Theorem**

Suppose $f(x) \neq 0$ and let $P_N(0)$ and $\tilde{P}_N(0)$ be approximations of $f(x)$ obtained using the polynomial extrapolation. Then the absolute and relative errors converge to 0 as $N \to \infty$:

\[
\left| \frac{P_N(0) + \tilde{P}_N(0)}{2} - f(x) \right| \leq |P_N(0) - \tilde{P}_N(0)| \to 0,
\]

\[
\left| \frac{P_N(0) + \tilde{P}_N(0)}{2 f(x)} - f(x) \right| \leq \frac{|P_N(0) - \tilde{P}_N(0)|}{P_N(0)} \to 0.
\]
Typical choices of $k_i$

To obtain a relative error of $10^{-6}$, typically use

$$k_i = 10i, \ i = 1, \cdots, N \quad \text{with} \quad N = 6.$$
Right skewed normalized stable distribution with $\alpha = 1/2$

Chi-squared distribution with one degree of freedom

Inverse Gaussian distribution

For example, we compute the PDF of a $(1/2)$-stable distribution at $x = 1$ and obtain (programming in Mathematica)

$$f(1) = 0.219695644733861$$

This is exact to 15 decimal places and the computation took a fraction of a second.

An alternative method for this example is given in Nolan (1997).
Chi-squared distribution with one degree of freedom

For the Chi-squared distribution with one degree of freedom, one has

$$\psi_{\chi^2}(\lambda) = (1 + 2\lambda)^{-1/2}$$

$$\phi_{\chi^2}(\lambda) = -\log \psi_{\chi^2}(\lambda) = \frac{1}{2} \log(1 + 2\lambda) = \int_0^\infty \left(1 - e^{-\lambda u}\right) \left[\frac{e^{-u/2}}{2u}\right] du.$$  

$$\phi^{(n)}_{\chi^2}(\lambda) = \frac{(-1)^{n+1}(n - 1)!}{2} \left(\frac{1}{2} + \lambda\right)^{-n}, \quad n \geq 1.$$  

Lets apply our method and compare the result with the true PDF:

$$f_{\chi^2}(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2},$$
The chi-squared distribution: PDF

(a) Relative error for PDF of $\chi^2$ distribution

Figure: Plot of the relative error versus the truncation level $N$ for the PDF of the $\chi^2$ distribution at three values of $x$. The relative error is the largest at $x = 20$ when the PDF is very close to 0.
The chi-squared distribution: CDF

(a) Relative error for CDF of $\chi^2$ distribution

Figure: Plot of the relative error versus the truncation level $N$ for the CDF of the $\chi^2$ distribution at three values of $x$. 

Relative Error

$N$

$0.001$

$10^{-7}$

$10^{-11}$

$10^{-15}$

$10^{-19}$

$5$

$10$

$15$

$0.00001$

$x = 1$

$x = 20$

$x = 0.00001$
### Table for the chi-squared distribution

<table>
<thead>
<tr>
<th>Chi-squared</th>
<th>( f_{\chi^2}(x) )</th>
<th>( N_6 )</th>
<th>( N_{15} )</th>
<th>( F_{\chi^2}(x) )</th>
<th>( N_6 )</th>
<th>( N_{15} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( 10^{-5} )</td>
<td>( 10^{-1} )</td>
<td>1</td>
<td>10</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 1.26 \times 10^2 )</td>
<td>1.20</td>
<td>0.242</td>
<td>( 8.50 \times 10^{-4} )</td>
<td>4.05 \times 10^{-6}</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 2.52 \times 10^{-3} )</td>
<td>0.248</td>
<td>0.683</td>
<td>0.998</td>
<td>1 ( - 7.7 \times 10^{-6} )</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Using \( k_i = 10i \), this table shows the value of \( N \) required to obtain a relative error of \( 10^{-6} \) \((N_6)\) and \(10^{-15} \)(\(N_{15}\)) at the given values of \( x \), using polynomial interpolation. Notice that it is harder to reduce the relative error when the functions are very small.
Distributions not known in closed form

- Stable distribution with \( 0 < \alpha < 1 \)
  \[
  S_{\alpha}(\cos(\pi \alpha/2)^{1/\alpha}, 1, 0), \quad \phi(\lambda) = \lambda^\alpha
  \]

- Mixtures of stable distributions: Laplace exponent
  \[
  \phi(\lambda) = \int_0^1 \lambda^\alpha p(d\alpha)
  \]

  where \( p \) is a measure supported on \((0, 1)\).
  This is an example of a distribution with no finite moments.

- Integrals with respect to a Poisson measure.
- Integrals with respect to a Lévy measure.
**α-stable distributions**

**Figure:** \(\alpha\)-stable PDFs (left) and CDFs (right) for \(\alpha = i/10\) for \(i = 1, 2, \ldots, 9\). The PDF for \(\alpha = 0.9\) corresponds to the right-most peak, and \(\alpha = .1\) corresponds to the left-most peak (which isn’t visible). The CDF with the steepest slope around \(x = 0.8\) corresponds to \(\alpha = 0.9\).
Mixtures of $\alpha$-stable distributions

Figure: (Left figure) This is the plot of the PDF corresponding to the choice $p_\omega(du) = (1 - \omega)\delta(\alpha - 0.4) + \omega\delta(\alpha - 0.8)$ for various choices of $0 \leq \omega \leq 1$. By increasing $\omega$, the PDF for $\alpha = 0.4$ (on the left) is morphing into the PDF for $\alpha = 0.8$ (on the right). (Right figure) These are PDFs corresponding to the mixtures $p_m(du) = \frac{1}{m}\sum_{i=1}^{m} \delta(\alpha - \frac{i}{m+1})$ for $m = 1, 2, \ldots, 5$ together with the uniform continuous mixture on $(0, 1)$, which has the highest peak.
Integrals with respect to a Poisson measure

Let $N$ be a Poisson measure with mean measure $\mu$. Consider the stochastic integral

$$I(g) = \int_{\mathbb{R}^n} g(x) N(dx)$$

for a function $g \geq 0$ on $\mathbb{R}^n$, for which $\int_{\mathbb{R}^n} \min(1, g(x)) \mu(dx) < \infty$. Then

$$\mathbb{E}e^{-\lambda I(g)} = \exp\left(-\int_{\mathbb{R}^n} (1 - e^{-\lambda g(s)}) \mu(ds)\right), \quad \lambda > 0.$$

In order to compute the PDF and CDF of $I(g)$ using our method, this must first be rewritten in LK form through a change of variables, so that

$$\mathbb{E}e^{-\lambda I(g)} = \exp\left(-\int_0^\infty (1 - e^{-\lambda u_1}) \Pi_g(du_1)\right).$$
PDF and CDF of an Ornstein-Uhlenbeck Poisson integral

Figure: Plots of the PDF and CDF for the Ornstein-Uhlenbeck Poisson stochastic integral $I(e^{-s/\eta})$ with various values of the parameter $\eta$, which plays the role of a shape parameter. The PDF for $\eta = 0.5$ is the highest on the left.
Integrals with respect to a Lévy measure

Replace the Poisson measure \( N \) by a Lévy measure \( L \). Then

\[
I_L(g) = \int_0^\infty g(s) L(ds) \equiv \int_0^\infty \int_0^\infty u g(z) N(du, dz),
\]

where the control measure of \( N \) is given now by \( \Pi(du) \mu(dz) \).

Observe that the kernel \( g \) must now satisfy

\[
\int_0^\infty \int_0^\infty \min(1, u g(z)) \Pi(du) \mu(dz) < \infty.
\]

After a change of variables (assuming for simplicity that \( \Pi \) has a density:

\[
\mathbb{E} e^{-\lambda I_L(g)} = \exp \left( - \int_0^\infty (1 - e^{-\lambda v}) \Pi'(v) dv \right).
\]
Example: Lévy measure is Gamma

\[ I_L(g) = \int_0^\infty e^{-z/\eta} L(dz) = \int_0^\infty \int_0^\infty u e^{-z/\eta} N(du, dz) \]

where \( g(z) = e^{-z/\eta} \) with \( \eta > 0 \), \( \mu(dz) \) is Lebesgue, and \( \Pi(du) = \kappa u^{-1} e^{-u/\theta} du \) is the Lévy measure corresponding to the Gamma distribution with shape \( \kappa > 0 \) and scale \( \theta > 0 \). Then

\[ \Pi'_g(u) = \frac{\eta \kappa}{u} \Gamma \left( 0, \frac{u}{\theta} \right), \]

\[ \phi(\lambda) = \eta \kappa \int_0^\lambda \frac{\log(1 + t\theta)}{t} dt, \]

\[ \phi^{(n)}(\lambda) = (-1)^{n+1} \eta \kappa \frac{(n-1)!}{\lambda^n} \left[ \log(1 + \lambda \theta) - \sum_{m=1}^{n-1} \frac{(\lambda \theta)^m}{m (1 + \lambda \theta)^m} \right]. \]

The distribution depends only on \( \rho = \eta \kappa \) and \( \theta \).
PDF and CDF of an Ornstein-Uhlenbeck Lévy integral

Figure: Plots of the PDF and CDF for the Lévy stochastic integral $I_L(e^{-s/\eta})$ where $L$ is Gamma with shape parameter $\kappa > 0$ and scale $\theta = 1$. The distribution depends only on $\rho = \eta \kappa$. 
We presented an efficient technique for computing numerically the PDF and CDF of non-negative infinitely divisible random variables.

It applies to Poisson and Lévy integrals.

Our method produces the best results if the PDF and/or CDF are smooth functions.

The method still produces useful results in the non-smooth case, however it may fail to converge near points where the function lacks smoothness.

This method should not be used for distributions which contain atoms, such as the Poisson or compound distribution. For such distributions we developed another method based on the Kolmogorov-Feller forward equation.
Danke!
The Fourier transform alternative

One needs to compute accurately two integrals with oscillating integrands:

$$\text{Chf}(\theta) = - \int_0^\infty (1 - e^{i\theta u})\Pi(du)$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\theta x} \text{Chf}(\theta) d\theta,$$

whereas we have only to compute integrals with exponentially decreasing integrands:

$$\phi^{(n)}(\lambda) = \begin{cases} 
\int_0^\infty (1 - e^{-\lambda u})\Pi(du), & n = 0 \\
(-1)^{n+1} \int_0^\infty u^n e^{-\lambda u}\Pi(du), & n \geq 1
\end{cases}$$
The case of $X \in \mathbb{R}$

If $\int_{\mathbb{R}} |u| \Pi(du) < \infty$, we can split the sum of the positive jumps and the sum of the negative jumps, which are both infinitely divisible:

$$
Chf(\theta) = \int_{-\infty}^{0} (e^{i\theta u} - 1) \Pi(du) + \int_{0}^{\infty} (e^{i\theta u} - 1) \Pi(du) + e^{i\theta a},
$$

analyze each separately, and convolve the two PDFs.

We are also working on eliminating the condition $\int_{\mathbb{R}} |u| \Pi(du) < \infty$, to allow the case where the sum of the small jumps is infinite.
The general Lévy-Khinchine formula

\[ \mathbb{E} e^{i\theta X} = \exp \left\{ i(a, u) - \frac{1}{2}(u, Au) \right. \\
+ \left. \int_{\mathbb{R}^d - \{0\}} \left[ e^{i(u, y)} - 1 - i(u, y) \kappa_B(y) \right] \Pi(dy) \right\} \]

where \( a \in \mathbb{R} \), \( A \) is a \( d \times d \) positive definite symmetric matrix, \( \Pi \) is the Lévy-Khinchine measure on \( \mathbb{R}^d - \{0\} \) and \( B \) is the open ball of radius 1 centered at 0.
Chi-squared distribution with one degree of freedom

\[ f_{\chi^2}(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}, \quad F_{\chi^2}(x) = \text{erf}\left(\sqrt{\frac{x}{2}}\right) \]

\[ \psi_{\chi^2}(\lambda) = \tilde{f}_{\chi^2}(\lambda) = (1 + 2\lambda)^{-1/2} \]

\[ \phi_{\chi^2}(\lambda) = -\log \psi_{\chi^2}(\lambda) = \frac{1}{2} \log(1 + 2\lambda) = \int_{0}^{\infty} (1 - e^{-\lambda u}) \left[ \frac{e^{-u/2}}{2u} \right] \, du. \]

\[ \phi^{(n)}_{\chi^2}(\lambda) = (-1)^{n+1} \int_{0}^{\infty} u^n e^{-\lambda n} \left[ \frac{e^{-u/2}}{2u} \right] \, du \]

\[ = \frac{(-1)^{n+1}}{2} \int_{0}^{\infty} u^{n-1} e^{-u(\lambda + 1/2)} \, du = \frac{(-1)^{n+1}(n - 1)!}{2} \left(\frac{1}{2} + \lambda\right)^{-n}, \quad n \geq 1. \]
Inverse Gaussian (IG) distribution

\[ f_{IG}(x) = \frac{1}{\sqrt{4\pi x^3}} e^{-1/(4x)} , \quad F_{IG}(x) = 1 - \text{erf} \left( \frac{1}{\sqrt{2x}} \right) \]

\[ \psi_{IG}(\lambda) = \tilde{f}_{IG}(\lambda) = e^{-\sqrt{\lambda}} \]

\[ \phi_{IG}(\lambda) = -\log \psi_{IG}(\lambda) = \sqrt{\lambda} = \int_{0}^{\infty} (1 - e^{-\lambda u}) \left[ \frac{u^{-3/2}}{2\sqrt{\pi}} \right] du \]

\[ \phi_{IG}^{(n)}(\lambda) = \frac{(-1)^{n+1}}{2\sqrt{\pi}} \int_{0}^{\infty} u^{n-3/2} e^{-\lambda u} du \]

\[ = \frac{(-1)^{n+1}}{2\sqrt{\pi}} \Gamma(n - 1/2)\lambda^{1/2-n} = \frac{(-1)^{n+1}(2n-3)!}{2^{2(n-1)}(n-2)!} \lambda^{1/2-n} \]
Mixtures of stable distributions

\[ \phi(\lambda) = \int_0^1 \lambda^\beta p(d\beta) = \int_0^\infty (1 - e^{-\lambda u}) \left[ \int_0^1 \frac{u^{-\beta-1}}{\Gamma(-\beta)} p(d\beta) \right] du, \]

where \( p \) is a measure supported on \((0, 1)\).

\[ \phi^{(n)}(\lambda) = -\frac{1}{\lambda^n} \sum_{m=1}^{n} c_m(\lambda) S_n^{(m)} \]

where \( c_m(\lambda) = \int_0^1 \beta^m \lambda^\beta p(d\beta) \) and \( \{S_n^{(m)}\}, \ n \geq 0, \ 0 \leq m \leq n \) are the Stirling numbers of the first kind. These are such that

\[ \prod_{m=0}^{n-1} (\beta - m) = \sum_{m=1}^{n} S_n^{(m)} \beta^m, \]

and can computed with a triangular array similarly to Pascal’s triangle using the recursion formula

\[ S_0^{(0)} = 1, \quad S_n^{(0)} = 0, \quad n \geq 1 \]

\[ S_n^{(m)} = S_{n-1}^{(m-1)} - (n - 1) S_{n-1}^{(m)}, \quad n, m \geq 1. \]
A single right-skewed stable distribution

\[
\phi^{(n)}(\lambda) = (-1)^{n+1} \frac{\lambda^{\alpha-n} \Gamma(n-\alpha)}{\Gamma(-\alpha)} \\
= -\lambda^{\alpha-n} \prod_{m=0}^{n-1} (\alpha - m) \\
= -\lambda^{\alpha-n} \sum_{m=1}^{n} \alpha^m S^{(m)}_n.
\]
Sums of right-skewed stable distributions

\[ p(d\beta) = \sum_{j=1}^{r} d_j \delta(\beta - \alpha_j) d\beta, \quad d_j \geq 0, \quad \sum_{j=1}^{r} d_j = 1 \]

\[ \phi^{(n)}(\lambda) = (-1)^{n+1} \sum_{j=1}^{r} d_j \frac{\lambda^{\alpha_j - n} \Gamma(n - \alpha)}{\Gamma(-\alpha_j)} \]

\[ = - \sum_{j=1}^{r} d_j \lambda^{\alpha_j - n} \prod_{m=0}^{n-1} (\alpha_j - m) \]

\[ = - \sum_{j=1}^{r} d_j \lambda^{\alpha_j - n} \left( \sum_{m=1}^{n} \alpha_j^m S_n^{(m)} \right). \]
“Uniform mixture” of $\alpha$-stable distributions

$$p(d\beta) = d\beta, \ \beta \in (0, 1).$$

This is an example of a distribution with no finite moments. The Laplace exponent is given by

$$\phi(\lambda) = \int_0^1 \lambda^\beta d\beta = \int_0^1 e^{\beta \log \lambda} d\beta = \frac{(\lambda - 1)}{\log \lambda} \quad \text{if } \lambda \neq 1,$$

and is 1 if $\lambda = 1$.

The coefficients $c_m(\lambda), \ m = 1, 2, \ldots, n$ can be computed in a few different ways.
Ornstein-Uhlenbeck Poisson integral

\[ I(g) = \int_{\mathbb{R}} g(x) N(dx) \]  

(1)

The integrand is \( g(s) = e^{-s/\eta} \), \( \eta > 0 \) is a parameter and the control measure \( \mu \) is Lebesgue. Then

\[ \phi(\lambda) = \int_0^1 \left( 1 - e^{-\lambda u} \right) \left( \frac{\eta}{u} \right) du = \eta \int_0^\lambda \frac{1 - e^{-x}}{x} dx. \]

\[ \phi^{(n)}(\lambda) = \frac{(-1)^{n+1} \eta}{\lambda^n} \left( (n - 1)! - e^{-\lambda} \sum_{m=0}^{n-1} \frac{(n - 1)!}{m!} \lambda^m \right), \quad n \geq 1. \]