Local Malliavin Calculus for Lévy Processes and Applications

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6th International Conference on Lévy Processes: Theory and Applications
Dresden, July 30, 2010
Outline

- Introduction
- Carlen–Pardoux approach to Malliavin derivative on the Poisson space
- Local Malliavin derivative for Lévy processes
- Some tools: Derivative of multiple integrals
- Application. Existence of densities: A Nourdin–Simon type result
Motivation

The first extensions of the Malliavin calculus to processes with jumps,

- Bismut (1983)
- Leandre (1985)
- Bichteler, Graveraux and Jacod (1987)

although be very powerful, they excluded the case of fixed jumps.
For the standard Poisson process

- **Malliavin derivative as a chaos anihilation operator:** Nualart and Vives (1990). That operator coincides with a difference operator with the addition of an extra jump. It does not satisfy the chain rule. It is useful for anticipative stochastic calculus.

- **Malliavin derivative as a true derivative operator:** Carlen and Pardoux (1990). It satisfies the chain rule and allows to study the absolute continuity of the solutions of stochastic differential equations driven by a Poisson process.
After the works of Carlen–Pardoux and Nualart–Vives it appeared the necessity to extend these methods for Lévy processes with a more general Lévy measure.

Working with Lévy processes, there are two different lines:

▶ Chaos annihilation operator.
▶ A true derivative operator that satisfies the chain rule.

In the Wiener case the annihilation and the derivative operator coincide.


Our aim is to contribute on the second line, and so to have a chain rule in order to use the usual proof for obtain an absolute continuity criterium. Then to apply to some stochastic differential equations driven by Lévy processes.
Carlen-Pardoux approach to Malliavin derivative on the Poisson space

\{N_t, t \in [0, T]\} \text{ Poisson process} of intensity 1 defined on the canonical space

\[\Omega = \{\omega : [0, T] \rightarrow \mathbb{N}, \omega(0) = 0, \text{increasing, cadlag, finitely many jumps}\}\]

\[\text{Diagram:}\]

- Vertical axis labeled \(n\)
- Horizontal axis labeled with \(t_1, t_2, t_3, \ldots, t_n, T\)
- Jumps at \(t_1, t_2, t_3, T\) with values 1, 2, respectively.
The shift function

\[ T_\varepsilon : \Omega \rightarrow \Omega, \quad P \circ T_\varepsilon^{-1} \ll P \quad \text{and} \quad N \text{ is still a } P \circ T_\varepsilon^{-1} - \text{Poisson process} \]
Malliavin derivative

The shift $\mathcal{T}_\varepsilon$ is defined using a centered function $g \in L^2_0(0, T)$.

$$
\mathbb{D}_g^0 = \{ F \in L^2(\Omega) : \text{exists } L^2(\Omega) - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\mathcal{T}_\varepsilon F - F) \}.
$$

For $F \in \mathbb{D}_g^0$,

$$
D_g F := L^2(\Omega) - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\mathcal{T}_\varepsilon F - F).
$$

There is a stochastic process

$$
\{ D_t F, 0 \leq t \leq T \}
$$

such that

$$
\int_0^T D_t F g(t) \, dt = D_g F, \ \forall g \in L^2_0(0, T).
$$
Our starting point: Leon–Tudor result

Write $\tilde{N}_t = N_t - t$ the compensated Poisson process. Denote by $T_1 < \cdots < T_n, \ldots$ the jump times of $N$. Let $h \in C^1([0, T])$. Then the random variable

$$F = \int_0^T h(s) \, d\tilde{N}_s = \sum_{T_n \leq T} h(T_n) - \int_0^T h(s) \, ds$$

is in the domain of the Malliavin derivative (Carlen-Pardoux sense) and

$$D_t F = \int_0^T h'(s)\left(\frac{s}{T} - 1_{(t,T]}(s)\right) \, dN_s.$$
Extension to Lévy process

Let $X$ be a Lévy process with Lévy-Itô representation:

$$X_t = \gamma t + \sigma W_t + \int \int_{(0,t] \times \{|x| > 1\}} x dN(s, x) +$$

$$\lim_{\epsilon \to 0} \int \int_{(0,t] \times \{|x| \leq 1\}} x d\tilde{N}(s, x),$$

where $W$ is a standard Brownian motion, $N$ is the jump measure of the process and $d\tilde{N}(t, x) = dN(t, x) - dt\nu(dx)$. Moreover $W$ and $N$ are independent. The limit is a.s., uniform in $t$ in any bounded interval.
Itô (1956) proved that $X$ determines a centered independently scattered random measure $M$ on $[0, \infty) \times \mathbb{R}$. For $E \in \mathcal{B}([0, \infty) \times \mathbb{R})$ such that $\mu(E) < \infty$,

$$M(E) = \sigma \int_{E(0)} dW_t + \lim_{n \to \infty} \int \int \left\{ (t, x) \in E' : \frac{1}{n} \leq |x| \leq n \right\} x d\tilde{N}(t, x).$$

For $A, B \in \mathcal{B}([0, \infty) \times \mathbb{R})$, with $\mu(A) < \infty$ and $\mu(B) < \infty$, we have

$$\mathbb{E}[M(A)M(B)] = \mu(A \cap B),$$

where

$$\mu(dt, dx) = \lambda(dt)\delta_0(dx) + \lambda(dt)x^2\nu(dx),$$

and $\lambda$ is the Lebesgue measure in $\mathbb{R}$. 
For a function \( h \in L^2([0, T] \times \mathbb{R}, \mu) \), we can construct the integral (in the \( L^2(\Omega) \) sense)

\[
M(h) := \int_{[0, T] \times \mathbb{R}} h(t, x) \, dM(t, x),
\]

which is the multiple integral of order 1 defined by Itô. This integral is centered, and for \( g, h \in L^2([0, T] \times \mathbb{R}, \mu) \),

\[
E[M(h)M(g)] = \int_{[0, T] \times \mathbb{R}} h g \, d\mu.
\]

We can write this integral as

\[
M(h) = \sigma \int_0^T h(t, 0) \, dW_t + \int_{[0, T] \times \mathbb{R}_0} h(t, x) x \, d\tilde{N}(t, x).
\]
Smooth random variables

The set of smooth random variables $S$ is the family of all functionals of the form

$$f(M(h_1), \ldots, M(h_n)),$$

where

- $f \in C^\infty_b(\mathbb{R}^n)$ ($f$ and all its partial derivatives are bounded),
- $h_1, \ldots, h_n \in L^2([0, T] \times \mathbb{R}, \mu)$ and for all $x \neq 0$,
- $h_i(\cdot, x) \in C^1([0, T])$, $i \in \{1, \ldots, n\}$, and
- $\partial h_i \in L^2([0, T] \times \mathbb{R}, \mu)$ where $\partial$ means the partial derivative with respect to time.

$S$ is dense in $L^2(\Omega)$.

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We will also consider the family $\mathcal{K}$ of all bounded functions $k : [0, T] \times \mathbb{R}_0 \to \mathbb{R}$ such that they and their partial derivative with respect time are in $L^2([0, T] \times \mathbb{R}_0, \lambda \times \nu) \cap L^1([0, T] \times \mathbb{R}_0, \lambda \times \nu)$,
The local Malliavin derivative

Given $F = f(M(h_1), \ldots, M(h_n)) \in S$, $k \in \mathcal{K}$, and $\Lambda \in \mathcal{B}(\mathbb{R})$, we define

$$D_{t}^{\Lambda,k} F = \sum_{i=1}^{n} (\partial_i f)(M(h_1), \ldots, M(h_n)) D_{t}^{\Lambda,k} M(h_i), \quad t \in [0, T],$$

where

$$D_{t}^{\Lambda,k} M(h) = 1_{\Lambda}(0) \sigma h(t, 0)$$

$$+ \int_{0}^{T} \int_{\Lambda \cap \mathbb{R}_0} k(s, y) \partial_s h(s, y)y\left(\frac{s}{T} - 1_{[t, T]}(s)\right) dN(s, y).$$

We call $D_{t}^{\Lambda,k} F$ the **local Malliavin derivative** of $F$. 
The function $k(t, x)$ is an essential ingredient of the derivative, and it may change from one application to another. To short the notation, in general we will omit that $k$ in the expression $D_t^{\Lambda,k} F$.

If $\Lambda = \{0\}$, then $D^\Lambda$ is the Malliavin’s derivative with respect to the Brownian part of the Lévy process $X$. 
If \( \Lambda = \{x\} \) for some \( x \neq 0 \) with \( \nu(\{x\}) \neq 0 \), we obtain

\[
D_t^{\{x\}} M(h) = x \int_0^T k(s, x) \partial_s h(s, x) \left( \frac{s}{T} - 1_{[t,T]}(s) \right) dN_s^x,
\]

where \( N_s^x \) is the Poisson process on \([0, T]\) that counts the number of jumps of height \( x \).

Moreover, if the Lévy process is the standard Poisson process, \( (x = 1) \), and we take \( k(t, x) \) independent of the time parameter, we obtain

\[
D_t^{\{1\}} M(h) = k(1) \int_0^T \partial_s h(s, 1) \left( \frac{s}{T} - 1_{[t,T]}(s) \right) dN_s^1.
\]

So it coincides with Carlen-Pardoux derivative.
Integration by parts formula

**Theorem**

Let $F \in S$, and $g$ be a measurable and bounded function on $[0, T]$. Then

\[
\mathbb{E} \left[ \int_0^T (D_t^\Lambda F) g(t) dt \right] = \mathbb{E} \left[ F 1_{\Lambda}(0) \int_0^T \sigma g(s) dW_s \right] \\
+ \mathbb{E} \left[ F \int_0^T \int_{\Lambda \cap \mathbb{R}_0} \left( g(s) - \frac{1}{T} \int_0^T g(t) dt \right) k(s, y) dN(s, y) \right] \\
- \mathbb{E} \left[ F \left( \int_0^T \int_{\Lambda \cap \mathbb{R}_0} \partial_s k(s, y) \left[ \int_0^T g(t) \left( \frac{s}{T} - 1_{[0,s]}(t) \right) dt \right] dN(s, y) \right) \right].
\]
Theorem

The operator $D^{\wedge,k}$ is an unbounded densely defined and closable operator from $L^2(\Omega)$ into $L^2([0, T] \times \Omega)$.

In particular, we have that the operator $D^\wedge$ has a closed extension, which is also written by $D^\wedge$. The domain of this operator is denoted by Dom $D^\wedge$. 
1. **Chain rule.** Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives, and let $F = (F_1, \ldots, F_n)$ a random vector such that $F_j \in \text{Dom } D^\Lambda$, for $j = 1, \ldots, n$. Then $\Phi(F) \in \text{Dom } D^\Lambda$ and

$$D^\Lambda(\Phi(F)) = \sum_{j=1}^{n} \partial_j \Phi(F) D^\Lambda F_j.$$

2. **Derivative of a product.** Let $F, G \in \text{Dom } D^\Lambda$ such that $G D^\Lambda F, F D^\Lambda G \in L^2([0, T] \times \Omega)$. Then $F G \in \text{Dom } D^\Lambda$ and

$$D^\Lambda(F G) = G D^\Lambda F + F D^\Lambda G.$$
Multiple integrals

Let $\Theta \in B(\mathbb{R})$ be a bounded set such that $0 \notin \overline{\Theta}$ (away from 0); then $\nu(\Theta) < \infty$. Write

$$N^\Theta(B) = \# \{ t : (t, \Delta X_t) \in B \text{ and } \Delta X_t \in \Theta \}, \quad B \in B((0, \infty) \times \mathbb{R}_0).$$

It is Poisson random measure with intensity $\lambda \otimes \nu_\Theta$, where $\nu_\Theta(C) = \nu(\Theta \cap C)$, for $C \in B(\mathbb{R})$. Write

$$N^\Theta_t = N^\Theta([0, t] \times \Theta) < \infty, \text{ a.s., } \quad t \in [0, T]$$

Then we can order the jumps in the interval $[0, T]$, say $T_1 < \cdots < T_{N^\Theta_T}$, and

$$N^\Theta = \sum_{j=1}^{N^\Theta_T} \delta(T_j, \Delta X_{T_j}).$$
Let $\phi: [0, T] \times \mathbb{R}_0 \to \mathbb{R}$ be a measurable function

$$J_1^\Theta(\phi) := \int_{[0, T] \times R_0} \phi(t, x) dN^\Theta(t, x)$$

has a compound Poisson distribution.

Also multiple integrals can be considered: Denote by $S_n(\Theta)$ the simplex

$$\{ (t_n, x_1; \ldots; t_n, x_n) \in ([0, T] \times \Theta)^n : t_1 < \cdots < t_n \}.$$

For $\phi: S_n(\Theta) \to \mathbb{R}$, define

$$J_n^\Theta(\phi) = \int \cdots \int_{S_n(\Theta)} \phi(t_1, x_1; \ldots; t_n, x_n) dN^\Theta(t_1, x_1) \cdots dN^\Theta(t_n, x_n)$$

$$= \sum_{1 \leq i_1 < \cdots < i_n \leq N_T^\Theta} \phi(T_{i_1}, \Delta X_{T_{i_1}}; \ldots; T_{i_n}, \Delta X_{T_{i_n}}).$$

with the convention that the sum is zero if $N_T^\Theta < n$. 
Theorem
Let $\Theta$ away from zero, $\Theta \subset \Lambda$, and $\phi \in L^2(S_n(\Theta), (\lambda \otimes \nu)^n)$ such that for every $(x_1, \ldots, x_n) \in \Theta^n$, $\phi(\cdot, x_1; \ldots; \cdot, x_n) \in C^\infty(\overline{S_n})$, where $\overline{S_n} = \{0 \leq t_1 \leq \cdots \leq t_n \leq T\}$. Then $J_n^\Theta(\phi) \in \text{Dom } D^\Lambda$ and

$$D_t^\Lambda J_n^\Theta(\phi) = \sum_{j=1}^n J_n^\Theta \left( k(s_j, x_j) \partial_{s_j} \phi(s_1, x_1; \ldots; s_n, x_n) \left( \frac{s_j}{T} - 1_{(t, T]}(s_j) \right) \right).$$
Absolute continuity criterium

The chain rule allows to prove a criterion for the absolutely continuity of a random variable $F$ in a similar way that the Bouleau-Hirsch criterium in the Brownian case.

Theorem

Let $\Lambda \in \mathcal{B}(\mathbb{R})$, $k \in \mathcal{K}$ and $F \in \text{Dom} \ D^\Lambda$ such that

$$\int_0^T (D^\Lambda_t F)^2 dt > 0$$

a.s. on a measurable set $A \in \mathcal{F}$. Then, $P \circ F^{-1}$ is absolutely continuous on $A$ (i.e., $\lambda(B) = 0$ implies that $P(\{F \in B\} \cap A) = 0$).

It is worth to remark that the criterion is true for every set $\Lambda$ and weight function $k \in \mathcal{K}$. This is very interesting for applications because we can choose an appropriate $\Lambda$ and $k$ depending of $F$. 

Applications: An equation with continuous part

\[ Z_t = x_0 + \int_0^t b(Z_s) ds + \int_0^t \sigma(Z_s) dW_s + \int_0^t \int_{\mathbb{R}_0} l(y) y dN(s, y), \quad t \in [0, T] \]

where \( l \in \mathcal{K} \) and \( b, \sigma \) differentiable with bounded derivative.

Choose \( \Lambda = \{0\} \) and \( k \in \mathcal{K} \) arbitrary. Then

\[ D^{\{0\}, k}_t Z_T = \sigma(Z_t) \exp \left( \int_0^t \sigma'(Z_s) dW_s + \int_0^t \left\{ b'(Z_s) - \frac{1}{2} (\sigma'(Z_s))^2 \right\} ds \right) \]

**Theorem**

The random variable \( Z_T \) is absolutely continuous with respect to the Lebesgue measure on the set \( \{ S < T \} \), where

\[ S = \inf \left\{ t \in [0, T] : \int_0^t 1_{\{\sigma(Z_s) \neq 0\}} ds > 0 \right\}. \]
A pure discontinuous equation with a monotone drift

Consider the following equation with an additive jump noise

\[ Z_t = x + \int_0^t f(Z_s)ds + \int_0^t \int_{\mathbb{R}_0} h(y)dN(s, y), \quad t \in [0, T]. \]

where \( x \in \mathbb{R} \), \( f \) differentiable with bounded derivative and \( h \in L^2(\mathbb{R}_0, \nu) \cap L^1(\mathbb{R}_0, \nu) \). It is well-known that equation has a unique square-integrable solution. We prove that

\[
D_{t, R_0}^{R_0, k} Z_T = \int_0^T \int_{\mathbb{R}_0} \exp \left( \int_s^T f'(Z_u)du \right) (f(Z_{s-}) - f(Z_s)) k(s, y) \left( \frac{s}{T} - 1_{[0,s]}(t) \right) dN(s, y)
\]
Idea of the proof

Let $\Theta_m = \{ x \in \mathbb{R} : 1/m < |x| < m \}$, and consider

$$Z_t^{(m)} = x + \int_0^t f(Z_s^{(m)}) ds + \int_{[0,t] \times \Theta_m} h(y) dN(s, y), \quad t \in [0, T].$$

That equation can be solved pathwise using the flow $\{ \Phi_t(s, x) : 0 \leq s \leq t \leq T, \ x \in \mathbb{R} \}$. associated with the equation

$$z_t = x + \int_0^t f(z_s) ds, \quad t \in [0, T].$$
Figura: A trajectory of $Z_t^{(m)}$. 
The derivative of $Z_T^{(m)}$ can be computed, and passing to the limit,

$$D_{t}^{\mathbb{R}_0,k}Z_T = \int_0^T \int_{\mathbb{R}_0} \exp \left( \int_s^T f'(Z_u)du \right) (f(Z_{s-}) - f(Z_s)) k(s,y) \left( \frac{s}{T} - 1_{[0,s]}(t) \right) dN(s,y).$$
Case with finite Lévy measure.

\[
Z_t = x + \int_0^t f(Z_s) \, ds + \int_0^t \int_{\mathbb{R}_0} h(y) \, dN(s, y), \quad t \in [0, T].
\]

**Theorem**

Assume that \( \nu(\mathbb{R}_0) < \infty \) and \( h(y) \neq 0 \) for \( y \in \mathbb{R}_0 \) and that \( f \) is a monotone function. Then \( Z_T \) is absolutely continuous on the set \( \{ N_{\mathbb{R}_0}^T \geq 1 \} \).

Take

\[
k(t, x) = (-h(x) \wedge 1) \vee (-1)
\]

(independent of \( t \))
Case with infinite Lévy measure: A Nourdin–Simon type result

Now we deal with the case that $f$ is only monotone on an neighborhood of the initial condition $x$. A similar problem has been analyzed by Nourdin and Simon using an stratification method.

$$Z_t = x + \int_0^t f(Z_s)ds + \int_0^t \int_{\mathbb{R}_0} h(y)dN(s, y), \quad t \in [0, T].$$

Here $f$ is differentiable with bounded derivative and $h \in L^2(\mathbb{R}_0, \nu) \cap L^1(\mathbb{R}_0, \nu)$.

**Theorem**

Assume that $\nu(\mathbb{R}_0) = \infty$, $h(y) \neq 0$ for $y \neq 0$, and that $f$ is a monotone function on a neighborhood of the point $x$. Then, the random variable $Z_T$ is absolutely continuous.
A pure discontinuous equation with no zero Wronskian

Assume $\nu(\mathbb{R}) < \infty$. Consider

$$Z_t = x + \int_0^t f(Z_s) ds + \int_0^t \int_{\mathbb{R}_0} h(y) g(Z_{s-}) dN(s, y), \quad t \in [0, T].$$

where $h \neq 0$ is a bounded function, $f \in C^2_b$ and $g \in C^1_b$. Assume also

$$|h(y)W(g, f)(x)| > \frac{1}{2} ||f''||_\infty ||h||^2_\infty ||g||^2_\infty, \quad x \in \mathbb{R} \text{ and } y \in \mathbb{R}_0,$$

where $W(g, f) = g'f - f'g$ is the Wronskian of $g$ and $f$.

The existence of a density for $Z_T$ was analyzed by Carlen and Pardoux in the case that the involved Lévy process is a Poisson process.
\[ Z_t = x + \int_0^t f(Z_s)ds + \int_0^t \int_{R_0} h(y)g(Z_s-)dN(s,y), \quad t \in [0, T]. \]

**Theorem**

Let \( Z_t \) be the solution of equation and \( k \in K \). Then \( Z_T \) is in the domain of \( D_{R_0,k} \) and it has a density on the set \( \{ N_{R_0} > 0 \} \).