Subordination in the Sense of S. Bochner – An Approach through Pseudo Differential Operators

By Niels Jacob of Munich and René L. Schilling of Erlangen

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1. Introduction

In his monograph Harmonic Analysis and The Theory of Probability [3] Bochner showed that for the investigation of Lévy processes the Fourier transform is one of the most important tools. Let \( \{X_t\}_{t \geq 0} \) be a Lévy process taking values in \( \mathbb{R}^n \) and starting at \( 0 \in \mathbb{R}^n \). Then we have

\[
E^0(e^{-i(\xi,X_t)}) = e^{-t\psi(\xi)}
\]

(1.1)

with a continuous negative definite function \( \psi : \mathbb{R}^n \to \mathbb{C} \). It is useful to rewrite (1.1) in terms of the underlying semigroup \( \{\mu_t\}_{t \geq 0} \) of sub-probability measures

\[
\int_{\mathbb{R}^n} e^{-i(x,\xi)} \mu_t(dx) = e^{-t\psi(\xi)}
\]

(1.2)

thus establishing a one-one correspondence between \( \psi \) and \( \{\mu_t\}_{t \geq 0} \). By

\[
T_t u(x) = \int_{\mathbb{R}^n} u(x-y) \mu_t(dy)
\]

(1.3)

the convolution semigroup \( \{\mu_t\}_{t \geq 0} \) always defines a (strongly continuous) contraction semigroup of linear operators \( \{T_t\}_{t \geq 0} \) whose generator \( A \) has — in suitable function spaces — the representation

\[
Au(x) = - (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x,\xi)} \psi(\xi) \hat{\mu}(\xi) \, d\xi.
\]

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Conversely, starting with the generator $A$ it is possible to construct $\{T_t\}_{t \geq 0}$ as well as $\{X_t\}_{t \geq 0}$. These considerations show that the whole information on $\{X_t\}_{t \geq 0}$, $\{T_t\}_{t \geq 0}$, or $A$ is already contained in the function $\psi : \mathbb{R}^n \to \mathbb{C}$. Now, the prominent rôle played by the Fourier transform becomes clear.

In his book Bochner also handles the question how to get new Lévy processes from a given one. An important possibility to do this is a procedure called subordination. This concept is most simply explained either on the level of the convolution semigroup $\{\mu_t\}_{t \geq 0}$ or on the level of the process $\{X_t\}_{t \geq 0}$ itself. Suppose that $\{\rho_t\}_{t \geq 0}$ is another convolution semigroup of sub-probability measures on $[0, \infty)$. Then the vague integral

$$\mu^f_t := \int_0^\infty \mu_s \rho_t(ds)$$

defines an new convolution semigroup of sub-probabilities on $\mathbb{R}^n$ and, hence, both a new operator semigroup $\{T^f_t\}_{t \geq 0}$ and a new Lévy process $\{X^f_t\}_{t \geq 0}$. The superscript $f$ refers to the subordinator $\{\rho_t\}_{t \geq 0}$ which is completely characterized by one (and only one) function,

$$\int_0^\infty e^{-rs} \rho_t(ds) = e^{-rf(r)},$$

where $f : (0, \infty) \to \mathbb{R}$ is a Bernstein function. The semigroup $\{T^f_t\}_{t \geq 0}$ can be calculated as a Bochner integral

$$T^f_t u = \int_0^\infty T_s u \rho_t(ds)$$

its generator being given by

$$A^f u = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \xi)} f(\psi(\xi)) \hat{u}(\xi) \, d\xi.$$

The corresponding process $\{X^f_t\}_{t \geq 0}$ allows the following interpretation: the subordinator $\{\rho_t\}_{t \geq 0}$ defines a Lévy process $\{Y_t\}_{t \geq 0}$ taking values in $[0, \infty)$, starting almost surely in $0 \in \mathbb{R}$, and having almost surely increasing sample paths. Without loss of generality, we may assume that the processes $\{X_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$ are stochastically independent. Rewriting (1.5)

$$\mathbf{P}^0\{\omega : X^f_t(\omega) \in B\} = \mathbf{P}^0\{\omega : X_{Y_t(\omega)}(\omega) \in B\}, \text{ for all Borel sets } B \subset \mathbb{R}^n,$$

we may identify the processes $X^f_t(\omega) = X_{Y_t(\omega)}(\omega)$.

The subordination of Lévy processes and the related potential theory is well studied. Next to the book by Bochner [3] and the references given there we refer to [2, 7, 8]. It is, for example, possible to introduce a symbolic calculus for $A$ and $A^f$ and related operators, e.g., the resolvent $R_\lambda$ of $A$, see [2, 15] and other authors.

The concept of subordination is, however, not only restricted to semigroups arising from convolution semigroups, or — on the level of Markov processes — to (translation invariant) Lévy processes. Suppose that $L(x, D)$ is a symmetric second order
elliptic differential operator generating a Feller (or a sub-Markovian) semigroup, that is, a strongly continuous semigroup of contraction operators $T_t : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$, $t \geq 0$, such that $0 \leq u(x) \leq 1$ implies $0 \leq T_t u(x) \leq 1$. (In case of a sub-Markovian semigroup one replaces $C_0^\infty(\mathbb{R}^n)$ by $L^2(\mathbb{R}^n)$ with the inequalities holding almost everywhere only.) Again, we can define the semigroup $\{T_t^f\}_{t \geq 0}$ by (1.7) and the subordinated process by

$$P^\omega\left\{\omega : X_t^f(\omega) \in B\right\} = P^\omega\left\{\omega : X_{Y_t}(\omega) \in B\right\},$$

for all Borel sets $B \subset \mathbb{R}^n$.

In this setting, it is, however, much more difficult to find the generator of $\{T_t^f\}_{t \geq 0}$. Moreover, the corresponding process is, clearly, no Lévy process and its properties are difficult to study. Since the symbol $L(x, \xi)$ of $L(x, D)$ and the Bernstein function $f$ are the only known objects it should be possible to trace all interesting properties of $\{T_t^f\}_{t \geq 0}$, $A^f$, and $\{X_t^f\}_{t \geq 0}$ back to $L(x, \xi)$ and $f$.

In this paper we want to examine Bochner’s theory of subordination by using methods from the theory of pseudo differential operators. Roughly speaking, our idea is to work on the level of the generator and to use a type of symbolic calculus in order to get approximations for $A^f$ and $\{T_t^f\}_{t \geq 0}$ which will also lead to an approximation on the level of the process.

In some sense, the paper is introductory for we only consider one rather simple type of generators, a strongly elliptic symmetric differential operator

$$A \equiv L(x, D) = - \sum_{k, l=1}^n a_{kl}(x) \frac{\partial^2}{\partial x_k \partial x_l} + c(x),$$

imposing rather restrictive assumptions on the coefficients $a_{kl}$ and $c$. Although it would be sometimes possible to use some existing symbolic calculus, we strive to avoid using any of these calculi. This is for two reasons: firstly, we will work on $\mathbb{R}^n$ and not on a compact manifold without boundary, thus, estimates in Sobolev spaces $H^s(\mathbb{R}^n)$, $s \geq 0$, can only be obtained under restrictions on the coefficients, see, f.ex., [13], and functional calculi as were developed by Seeley [27] and Strichartz [28] cannot be adapted easily. Secondly, we will later extend our results to much more general operators, pseudo differential operators with non-classical symbols as considered in [19, 17]. Therefore, in order to estimate the resolvent $\{R_\lambda\}_{\lambda \geq 0}$ of $A$, we also avoid using a calculus for parameter dependent symbols as was developed by Grubb [11, 12] and others.

Let us briefly outline the plan of the paper. In Section 2 we recall some results on negative definite functions, Bernstein functions and Bochner’s theory of subordination. In particular, we give a formula for the generator of a subordinate semigroup in terms of the corresponding Bernstein function, the generator of the original semigroup, and its resolvent. The third section is devoted to introducing the operator $A = - L(x, D)$ (as generator of a semigroup) and a suitable class of function spaces to handle $A^f$. We determine the domain of $A^f$ in terms of these function spaces. The pseudo differential operator $p(x, D)$ with symbol $f(L(x, \xi))$ is introduced in Section 4 in order
to approximate $A^{f}$. Since the symbol $f(L(x,\xi))$ does, in general, not belong to a classical symbol class, $p(x, D)$ is already a non-classical pseudo differential operator. Using results from [20, 19] we prove that $p(x, D)$ itself extends both to a generator of a Feller semigroup and to a generator of a sub-Markovian semigroup. In order to control $A^{f} - p(x, D)$ we need estimates for the resolvent $R_{\lambda}$ of $A$. Again, we construct an approximation $q_{\lambda}(x, D)$ of $R_{\lambda}$. Here, the operator $q_{\lambda}(x, D)$ is a pseudo differential operator with symbol $(L(x, \xi) + \lambda)^{-1}$. Our assumptions on $a_{kl}$ and $c$ guarantee that $q_{\lambda}(x, D)$ is continuous as an operator from $H^{2}(\mathbb{R}^{n})$ onto $L^{2}(\mathbb{R}^{n})$, and, for suitable values of $s \geq 0$, even from $H^{s+2}(\mathbb{R}^{n})$ into $H^{s}(\mathbb{R}^{n})$. Furthermore, we prove that the operator $K_{\lambda}(x, D) := L_{\lambda}(x, D) \circ q_{\lambda}(x, D) - \text{id}$ satisfies certain estimates similar to the resolvent estimates, cf. Theorem 5.9 and Corollary 5.10 below. This leads to estimates for $q_{\lambda}(x, D) - R_{\lambda}$; in particular, this operator is of order $-3$.

In Section 6 we return to studying the operators $A^{f}$ and $p(x, D)$. Using results from the previous sections, we show for a large class of Bernstein functions the following estimates

\begin{equation}
\|f(L(x, D))u - p(x, D)u\|_{0} \leq c\|u\|_{0}
\end{equation}

which, in turn, gives estimates for $T_{i}^{f} - T_{i}^{(2)}$ and $T_{i}^{f} - T_{i}^{(\infty)}$. Here, $\left\{ \overline{T}_{i}^{f} \right\}_{i \geq 0}$ denotes the Fellerian semigroup obtained by subordinating the sub-Markovian and $\left\{ \overline{T}_{i}^{(2)} \right\}_{i \geq 0}$ the Fellerian semigroup generated by $L(x, D)$, while $\left\{ T_{i}^{(2)} \right\}_{i \geq 0}$ and $\left\{ T_{i}^{(\infty)} \right\}_{i \geq 0}$ are the sub-Markovian, resp., Fellerian semigroups generated by $-p(x, D)$. In fact, we have for all $u \in L^{2}(\mathbb{R}^{n})$

\begin{equation}
\left\| T_{i}^{f}u - T_{i}^{(2)}u \right\|_{0} \leq ct\|u\|_{0}
\end{equation}

and for $u \in H^{k, a^{2}}(\mathbb{R}^{n})$, $k > n/2$, $k \in \mathbb{N}$,

\begin{equation}
\left\| \overline{T}_{i}^{f}u - T_{i}^{(\infty)}u \right\|_{\infty} \leq c't\|u\|_{k, a^{2}}.
\end{equation}

Of course, these inequalities can be interpreted as first (and rather crude) estimates for the transition probabilities for the corresponding Markov and Feller processes.

As already mentioned, the operator $L(x, D)$ should be looked upon as a model operator. In forthcoming papers we will treat more general generators of Fellerian and sub-Markovian semigroups. There, the probabilistic consequences of our results and the relations to the work of BOULEAU [4] and HIRSCH [16] will be examined more carefully. Moreover, the case of an elliptic differential operator on a manifold without boundary will be handled separately using the classical symbolic calculus.

2. Negative definite functions, Bernstein functions, and subordination in the sense of Bochner

In this section we discuss those parts of Bochner's theory of subordination which will be used later on. Our main references are the books [3] by BOCHNER and [2] by BERG and FORST.
**Definition 2.1.** A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called *negative definite* if for all $m \in \mathbb{N}$ and $\xi^1, \ldots, \xi^m \in \mathbb{R}^n$ the matrix $(\psi(\xi^i) + \psi(\xi^j) - \psi(\xi^i - \xi^j))_{i,j=1}^m$ is positive Hermitian.

For continuous negative definite functions we have the Lévy–Khinchine representation

$$
(2.1) \quad \psi(\xi) = c_0 + i(\ell, \xi) + q(\xi) + \int_{\mathbb{R}^n} \left(1 - e^{-i(x, \xi)} - \frac{i(x, \xi)}{1 + |x|^2}\right) \frac{1 + |x|^2}{|x|^2} \nu(d\xi),
$$

with a positive constant $c_0 \geq 0$, a vector $\ell \in \mathbb{R}^n$, a positive semi-definite quadratic form $q(\cdot)$ on $\mathbb{R}^n$, and a finite Borel measure $\nu$ not charging the origin.

In this paper we will always consider real-valued continuous negative definite functions. Such a function, $\psi : \mathbb{R}^n \to \mathbb{R}$ say, satisfies

$$
(2.2) \quad 0 \leq \psi(\xi) \leq c_\psi (1 + |\xi|^2) \quad \text{and} \quad \psi(\xi) = \psi(-\xi)
$$

and, for this reason, we denote a real-valued continuous negative definite function by $a^2(\cdot)$. In this case, the Lévy–Khinchine formula (2.1) becomes

$$
(2.3) \quad a^2(\xi) = c_0 + q(\xi) + \int_{\mathbb{R}^n} (1 - \cos(x, \xi)) \frac{1 + |x|^2}{|x|^2} \nu(dx).
$$

The next class of functions we need is the class of Bernstein functions.

**Definition 2.2.** An arbitrarily often differentiable function $f : (0, \infty) \to \mathbb{R}$ is called *Bernstein function* if

$$
(2.4) \quad f \geq 0 \quad \text{and} \quad (-1)^{j-1} f^{(j)} \geq 0
$$

hold for all $j \in \mathbb{N}$. By $BF$ we denote the set of all Bernstein functions and we set

$$
BF_0 := \{ f \in BF : f(0) = 0 \}.
$$

The following analogue of the Lévy–Khinchine formula holds for Bernstein functions

$$
(2.5) \quad f(s) = c_1 + c_2 s + \int_0^\infty (1 - e^{-sr}) \mu(dr),
$$

with constants $c_1, c_2 \geq 0$ and a Borel measure $\mu$ on the open interval $(0, \infty)$ satisfying

$$
\int_0^\infty \frac{s^{\frac{r}{s+1}}}{s+1} \mu(dx) < \infty.
$$

**Remark 2.3.** Using (2.5) it is easy to see that $f$ has a holomorphic extension onto the right half-plane $\{ z \in \mathbb{C} : \text{Re} z \geq 0 \}$ which is continuous up to the boundary. When appropriate, we will use this extension without change of notation. The function $\psi : \mathbb{R} \to \mathbb{R}$

$$
\psi(s) := f(is), \quad s \in \mathbb{R},
$$

is continuous and — comparing (2.5) with (2.1) — also negative definite. This shows the close relationship between continuous negative definite and Bernstein functions.
A crucial point in our considerations is the fact that the composition of a continuous negative definite function \( a^2 : \mathbb{R}^n \to \mathbb{R} \) with a Bernstein function \( f : [0, \infty) \to \mathbb{R} \),

\[
f \circ a^2 : \mathbb{R}^n \to \mathbb{R}
\]
is again a continuous negative definite function.

Let \( a^2 : \mathbb{R}^n \to \mathbb{R} \) be a continuous negative definite function and \( f : [0, \infty) \to \mathbb{R} \) be a Bernstein function. With each of these functions we can associate convolution semigroups of sub-probability measures. In the first case, we have a convolution semigroup \( \{ \mu_t \}_{t \geq 0} \) of symmetric measures on \( \mathbb{R}^n \) related with \( a^2 \) via

\[
\mu_t(x) = \int_{\mathbb{R}^n} e^{-i(x, \xi)} \mu_t(\xi) d\xi = e^{-ta^2(\xi)},
\]

while \( f \) gives rise to a convolution semigroup \( \{ \rho_t \}_{t \geq 0} \) of sub-probability measures on the open interval \((0, \infty)\),

\[
\rho_t(r) = \int_0^\infty e^{-rs} \rho_t(ds) = e^{-tf(r)}.
\]

Combining (2.3), (2.5), (2.6), and (2.7) we find for the function \( f \circ a^2 \)

\[
f \circ a^2(x) = c_1 + c_2a^2(x) + \int_0^\infty \left( 1 - e^{-ra^2(\xi)} \right) \mu(dr) = c_1 + c_0c_2 + c_2q(\xi) + \int_{\mathbb{R}^n} \left( 1 - \cos(x, \xi) \right) \frac{1 + |x|^2}{|x|^2} \nu_f(dx),
\]

where \( \nu_f \) is given by the vague integral

\[
\nu_f(dx) := \frac{|x|^2}{1 + |x|^2} \int_0^\infty \mu_t(dx) \mu(dt) + c_2 \nu(dx).
\]

The convolution semigroup referring to \( f \circ a^2 \) can be explicitly calculated.

**Proposition 2.4.** ([2], p. 70.) Let \( a^2 : \mathbb{R}^n \to \mathbb{R} \) and \( f : [0, \infty) \to \mathbb{R} \) be as above. The convolution semigroup \( \{ \mu_t \}_{t \geq 0} \) associated with \( f \circ a^2 \) is given by the vague integral

\[
\mu_t(dx) = \int_0^\infty \mu_s(dx) \rho_t(ds)
\]

where \( \{ \mu_s \}_{s \geq 0} \) and \( \{ \rho_t \}_{t \geq 0} \) are the semigroups attached to \( a^2 \) and \( f \), respectively.

**Definition 2.5.** The semigroup \( \{ \mu_t \}_{t \geq 0} \) of sub-probability measures on \( \mathbb{R}^n \) is called *subordinate* to \( \{ \mu_t \}_{t \geq 0} \) with respect to \( f \) (or \( \{ \rho_t \}_{t \geq 0} \)).

Later on, we will extend this notion of a subordinate semigroup. We need some further results on a subclass of Bernstein functions.
Definition 2.6. A Bernstein function $f$ is said to be a complete Bernstein function if the representing measure $\mu$ of (2.5) is absolutely continuous with respect to Lebesgue measure on $[0, \infty)$, the density being given by

$$m(s) = \int_0^\infty r e^{-sr} \rho(dr), \quad s \geq 0,$$

where $\rho$ is a measure on the open interval $(0, \infty)$ satisfying $\int_0^\infty \frac{1}{r+1} \rho(dr) < \infty$. The set of all complete Bernstein functions is denoted by $CB\mathcal{F}$ and we set

$$CB\mathcal{F}_0 := \{ f \in CB\mathcal{F} : f(0) = 0 \}.$$

Remark 2.7. Some authors call the functions in $CB\mathcal{F}$ also operator monotone for they are the only functions preserving positivity of the quadratic form associated with a Hilbert space operator, see Löwner [22] and Heinz [14].

For complete Bernstein functions $f \in CB\mathcal{F}$ we have the following characterization:

Lemma 2.8. A function $f$ belongs to $CB\mathcal{F}$ if and only if

$$s \mapsto \frac{f(s)}{s}, \quad s > 0,$$

is a Stieltjes transform, that is, if

$$\frac{f(s)}{s} = c_1 \frac{1}{s} + c_2 + \int_0^\infty \frac{1}{s + r} \rho(dr)$$

holds with $c_1, c_2$ as in (2.5) and $\rho$ as in (2.11).

A proof of this lemma and further characterizations of complete Bernstein functions $f \in CB\mathcal{F}$ are given in [25], Theorem 5.6, see also [26] and the related paper [1].

Example 2.9. The following functions are in $CB\mathcal{F}_0$:

$$s^\alpha = \frac{\sin(\alpha \pi)}{\pi} \int_0^\infty \frac{s}{s + r} r^{\alpha - 1} dr, \quad \alpha \in (0, 1);$$

$$\frac{s}{s + \lambda} = \int_0^\infty \frac{s}{s + r} \delta_\lambda(dr), \quad \lambda > 0;$$

$$\log(1 + s) = \int_1^\infty \frac{s}{r(s + r)} dr;$$

$$\sqrt{s} \arctan \frac{\lambda}{\sqrt{s}} = \int_0^{\lambda^2} \frac{s}{s + r} \cdot \frac{1}{2 \sqrt{r}} dr, \quad \lambda > 0.$$

Let us prove some auxiliary results for complete Bernstein functions.
Lemma 2.10. For all \( f \in \mathcal{B}F \) we have

\[
\frac{f'(s)}{f(s)} \leq \frac{1}{s}, \quad s > 0.
\]

If \( f \in \mathcal{CB}F \),

\[
\left| \frac{f^{(k+1)}(s)}{f^{(k)}(s)} \right| \leq \frac{k + 1}{s}, \quad s > 0,
\]

holds for all \( k \in \mathbb{N} \) and, in particular,

\[
\left| f^{(k+1)}(s) \right| \leq (k + 1)! \frac{1}{s^{k+1}} f(s), \quad s > 0,
\]

is valid for all \( k \in \mathbb{N}_0 \).

Proof. By the inequality \( 1 + s \leq e^{rs} \) or, equivalently, \( s e^{-rs} \leq 1 - e^{-rs} \), we obtain by differentiating (2.5) under the integral sign

\[
s f'(s) = c_2 s + \int_0^\infty s e^{-rs} \mu(dr) \leq c_1 + c_2 s + \int_0^\infty (1 - e^{-rs}) \mu(dr) = f(s)
\]

which proves (2.17).

If \( f \in \mathcal{CB}F \), we obtain by \((k + 1)-times differentiating (2.12)\)

\[
f^{(k+1)}(s) = \int_0^\infty \frac{(-1)^{k+1}(k + 1)! r}{(s + r)^{k+2}} \rho(dr)
\]

which yields

\[
\left| f^{(k+1)}(s) \right| = \int_0^\infty \frac{(k + 1)! r}{(s + r)^{k+2}} \rho(dr) \leq \int_0^\infty \frac{k! r}{(s + r)^{k+1}} \frac{k + 1}{s} \rho(dr)
\]

and the lemma follows. \( \Box \)

Remark 2.11. The function \( s \mapsto 1 - e^{-\beta s}, \beta > 0 \), is a Bernstein function that is not completely Bernstein. A direct computation shows that (2.18) does not hold for this function.

Bernstein functions are known to be subadditive. In fact, we even have the following stronger property:

Lemma 2.12. If \( f \in \mathcal{B}F \), then

\[
\frac{1}{c} f(s) \leq f(cs) \leq cf(s), \quad s > 0,
\]

holds for all \( c \geq 1 \).
Proof. Since \( f \geq 0 \) and \( f' \geq 0 \), the first inequality in (2.20) is obvious. Using \( f'' \leq 0 \) and (2.17) we find
\[
f'(r) \leq f'(s) \leq \frac{f(s)}{s}, \quad 0 < s \leq r.
\]
Integrating this, we get
\[
\int_s^{cs} f'(r) \, dr \leq \int_s^{cs} \frac{f(s)}{s} \, dr = (c-1)f(s),
\]
which proves the second inequality in (2.20).

Lemma 2.13. Let \( f \in CBF \) and denote by \( \mu \) and \( \rho \) the representing measures from (2.5) and (2.11), respectively. Then we have for \( \sigma > -1 \)
\[
\int_0^1 s^\sigma \mu(ds) < \infty
\]
if and only if
\[
\int_1^\infty \frac{1}{r^\sigma} \rho(dr) < \infty.
\]

Proof. Using (2.5) and (2.11) we find
\[
\int_0^1 s^\sigma \mu(ds) = \int_0^1 s^\sigma \int_0^\infty r e^{-sr} \rho(dr) \, ds
\]
\[
= \int_0^\infty \int_0^1 s^\sigma r e^{-sr} \, ds \, \rho(dr)
\]
\[
= \int_0^\infty \int_r^\infty \frac{r^\sigma}{r} e^{-r} \, d\tau \, \rho(dr)
\]
\[
= \int_0^1 \int_0^r r^\sigma e^{-r} \, d\tau \, r^{-\sigma} \rho(dr) + \int_1^\infty \int_0^r r^\sigma e^{-r} \, d\tau \, r^{-\sigma} \rho(dr).
\]
Thus, we find
\[
\int_0^1 r^\sigma e^{-r} \, d\tau \int_1^\infty r^{-\sigma} \rho(dr) \leq \int_0^1 s^\sigma \mu(ds)
\]
\[
\leq c_\sigma \int_0^1 \tau \rho(dr) + \int_0^\infty r^\sigma e^{-r} \, d\tau \int_1^\infty r^{-\sigma} \rho(dr)
\]
and the lemma follows as \( \int_0^1 \tau \rho(dr) < \infty \) and \( \int_1^\infty \tau^\sigma e^{-\tau} \, d\tau < \infty \).

In example (2.13), we have \( \rho(dr) = \frac{\sin \alpha \pi}{\pi} r^{-1-\alpha} \, dr \) and \( \mu(ds) = \frac{\alpha}{\Gamma(1-\alpha)} s^{-\alpha-1} \, ds \). Hence, Lemma 2.13 yields for \( 0 \leq \sigma \leq 1 \) that
\[
\int_1^\infty \frac{\rho(dr)}{r^\sigma} \sim c + c' \int_0^\infty r^{(\sigma-\alpha)-1} \, dr < \infty
\]
is finite if and only if $\sigma > \alpha$.

Next, we will turn our attention to contraction semigroups on a Hilbert space. Let $\{T_t\}_{t \geq 0}$ be a semigroup of strongly continuous symmetric contractions on the Hilbert space $H$ and denote by $A$ its generator and by $\{R_\lambda\}_{\lambda \geq 0}$ the corresponding resolvent. Clearly, $A$ is a self adjoint nonpositive operator with dense domain $D(A)$, and $R_\lambda$ is given by $R_\lambda = (\lambda - A)^{-1}$. As above, pick $f \in BF_0$ with corresponding convolution semigroup $\{\rho_t\}_{t \geq 0}$, see (2.7). Since for all $u \in H$

$$\left\| \int_0^\infty T_s u \rho_t(ds) \right\| \leq \int_0^\infty \|T_s\| \rho_t(ds) \|u\| \leq \|u\|$$

the following definition makes sense.

**Definition 2.14.** The semigroup $\{T_t^f\}_{t \geq 0}$ defined on $H$ by the Bochner integral

$$T_t^f u := \int_0^\infty T_s u \rho_t(ds)$$

is called subordinate to $\{T_t\}_{t \geq 0}$ with respect to $f$ (or $\{\rho_t\}_{t \geq 0}$).

It is well-known, see [7] or [24], that $\{T_t^f\}_{t \geq 0}$ is again a strongly continuous contraction semigroup of symmetric operators with self adjoint generator $A^f$. In the sequel, we will often need the following (Bochner) integral representation of $A^f$:

**Theorem 2.15.** Let $f \in CBF$. For all $u \in D(A)$ we have $u \in D(A^f)$ and

$$A^f u = c_1 u + c_2 Au + \int_0^\infty R_\lambda Au \rho(d\lambda)$$

where $c_1, c_2$ are the constants from (2.5) and $\rho$ is the measure given by (2.11).

A proof of Theorem 2.15 is given in [25] Satz 5.10, Korollar 5.11, see also [26] and [1]. Later, we will use (2.25) only in the case $c_1 = c_2 = 0$.

3. **The operators $L(x, D)$ and $f(L(x, D))$**

Let $L(x, D)$ be the differential operator

$$L(x, D) = - \sum_{k, l=1}^n a_{kl}(x) \frac{\partial^2}{\partial x_k \partial x_l} + c(x)$$

where $a_{kl} : \mathbb{R}^n \to \mathbb{R}, 1 \leq k, l \leq n$, are continuously differentiable functions such that $a_{kl}(x) = a_{lk}(x)$ and

$$\kappa_1 |\xi|^2 \leq \sum_{k, l=1}^n a_{kl}(x) \xi_k \xi_l \leq \kappa_2 |\xi|^2$$

for all $\xi \in \mathbb{R}^n$. The operators $L(x, D)$ and $f(L(x, D))$ are then defined as above.
hold for all \( x \in \mathbb{R}^n \) and \( \xi \in \mathbb{R}^n \) with suitable constants \( 0 < \kappa_1 \leq \kappa_2 \). Without loss of generality we may assume that \( \kappa_1 \leq 1 \). Moreover, we assume that

\[
\sum_{k=1}^{n} \frac{\partial a_{kl}}{\partial x_k} = 0
\]

(3.3)

holds for \( 1 \leq l \leq n \) and that \( c : \mathbb{R}^n \rightarrow \mathbb{R} \) is a continuous and bounded function satisfying

\[
c(x) \geq \nu_0 > 0.
\]

(3.4)

For \( u, v \in C_0^\infty(\mathbb{R}^n) \) we find

\[
\int_{\mathbb{R}^n} v(x) L(x, D) u(x) \, dx = \sum_{k,l=1}^{n} \int_{\mathbb{R}^n} a_{kl}(x) \frac{\partial u(x)}{\partial x_k} \cdot \frac{\partial v(x)}{\partial x_l} \, dx + \int_{\mathbb{R}^n} c(x) u(x) v(x) \, dx
\]

implying that \( L(x, D) \) extends to a self adjoint operator on \( L^2(\mathbb{R}^n) \) with domain \( H^2(\mathbb{R}^n) \). Moreover, \( L(x, D) \) is positive, i.e.,

\[
( L(x, D) u, u )_0 \geq \kappa_1 \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx + \nu_0 \int_{\mathbb{R}^n} |u(x)|^2 \, dx \geq 0.
\]

(3.5)

From (3.5) we deduce that for any \( \lambda > -\nu_0 \) the operator

\[
L_\lambda(x, D) := L(x, D) + \lambda \text{id}
\]

(3.6)

has a bounded inverse \( R_\lambda \), its resolvent at \( \lambda \). The resolvent maps \( L^2(\mathbb{R}^n) \) continuously and bijectively onto \( H^2(\mathbb{R}^n) \) and (3.5) implies the resolvent estimate

\[
||R_\lambda|| \leq \frac{1}{\nu_0 + \lambda}
\]

(3.7)

where \( ||R_\lambda|| \) is the operator norm of \( R_\lambda \) (considered as a bounded operator on \( L^2(\mathbb{R}^n) \)). Thus, we find that \( -L(x, D) \) generates a strongly continuous contraction semigroup \( \{ T_t^f \} \) of operators on \( L^2(\mathbb{R}^n) \) which satisfy

\[
||T_t|| \leq e^{-\nu_0 t}.
\]

(3.8)

Since \( -L(x, D) \) is a Dirichlet operator in the sense of [5], the semigroup \( \{ T_t \} \) is sub–Markovian (in the sense of [9]) and the quadratic form

\[
B(u, v) := \int_{\mathbb{R}^n} a_{kl}(x) \frac{\partial u(x)}{\partial x_k} \cdot \frac{\partial v(x)}{\partial x_l} \, dx + \int_{\mathbb{R}^n} c(x) u(x) v(x) \, dx
\]

(3.9)

defined on \( H^1(\mathbb{R}^n) \) is a Dirichlet form.

Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be a complete Bernstein function in the sense of Definition 2.6 and assume that in its representation (2.5) \( c_1 = c_2 = 0 \) hold. By \( \{ T_t^f \} \) we denote the semigroup obtained from \( \{ T_t \} \) by subordination with respect to \( f \in CBF_0 \).
Clearly, \( \{T_t^f\}_{t\geq 0} \) is again a symmetric sub-Markovian semigroup on \( L^2(\mathbb{R}^n) \). In order to determine the domain of the generator \( A^f \) of \( \{T_t^f\}_{t\geq 0} \), we need some function spaces. The function \( a^2 : \mathbb{R}^n \to \mathbb{R} \),

\[
a^2(\xi) := f(|\xi|^2), \quad \xi \in \mathbb{R}^n,
\]

is a continuous negative definite function and for \( s \geq 0 \) the norm

\[
\|u\|_{s,a^2} := \left( \int_{\mathbb{R}^n} (1 + a^2(\xi))^{2s} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}
\]

turns the space

\[
H^{s,a^2}(\mathbb{R}^n) := \{ u \in L^2(\mathbb{R}^n) : \|u\|_{s,a^2} < \infty \}
\]

into a Hilbert space with inner product \( (\cdot, \cdot)_{s,a^2} \) and \( C_0^{\infty}(\mathbb{R}^n) \) as a dense subspace. Introducing the negative norms in the sense of Lax,

\[
\|u\|_{-s,a^2} := \sup_{0 \neq v \in H^{s,a^2}(\mathbb{R}^n)} \frac{|(u,v)_0|}{\|v\|_{s,a^2}}
\]

it follows by standard arguments that \( \left[H^{s,a^2}(\mathbb{R}^n)\right]^* \) can be identified with the space

\[
H^{-s,a^2}(\mathbb{R}^n) := \{ u \in S'(\mathbb{R}^n) : \|u\|_{-s,a^2} < \infty \}.
\]

In particular, for \( u \in L^2(\mathbb{R}^n) \) we find

\[
\|u\|_{-s,a^2}^2 = \int_{\mathbb{R}^n} (1 + a^2(\xi))^{-2s} |\hat{u}(\xi)|^2 \, d\xi
\]

with \( L^2(\mathbb{R}^n) \) being a dense subspace of \( H^{-s,a^2}(\mathbb{R}^n) \), \( s \geq 0 \).

In order to get some embedding theorems, we assume

\[
f(\lambda) \geq c \lambda^{r/2}
\]

to hold for some \( r > 0 \) and all \( \lambda \geq \rho > 0 \). This implies

\[
1 + a^2(\xi) = 1 + f(|\xi|^2) \geq c'(1 + |\xi|^2)^{r/2}, \quad \xi \in \mathbb{R}^n,
\]

which leads to the continuous embeddings

\[
H^{s,a^2}(\mathbb{R}^n) \hookrightarrow H^{sr}(\mathbb{R}^n)
\]

and — by Sobolev’s embedding theorem — for \( sr > \frac{n}{2} \)

\[
H^{s,a^2}(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n)
\]

where \( C_0(\mathbb{R}^n) \) denotes the space of all continuous functions vanishing at infinity. Further results on the anisotropic Sobolev spaces \( H^{s,a^2}(\mathbb{R}^n) \), \( s \in \mathbb{R} \), can be found in [18] and [19].
The following result is taken from [25] Section 5.3, see also [26].

**Theorem 3.1.** By \(- f(L(x, D))\) we denote the generator of the subordinate semigroup \(\{T^f_t\}_{t \geq 0}\). Then

\[
D(- f(L(x, D))) = H^1, a^2(\mathbb{R}^n)
\]

is valid.

**Remark 3.2.** The notation \(f(L(x, D))\) is formal. However, it is possible to show (compare Theorem 2.15) that for \(u \in D(L(x, D)) = H^2(\mathbb{R}^n)\) we have

\[
f(L(x, D))u(x) = \int_0^\infty L(x, D)R_\lambda u(x) \rho(d\lambda) = \int_0^\infty R_\lambda L(x, D)u(x) \rho(d\lambda)
\]

where we used the representation (2.12) of \(f \in CBF\). For a proof of (3.21) we refer to [1, 25, 26].

It should be noted that \(- L(x, D)\) also extends to a generator of a Feller semigroup on \(C_\infty(\mathbb{R}^n)\).

Although we possess a handsome representation formula for \(f(L(x, D))\), it is desirable to have a more concrete formula for this operator. Consider, therefore, \(L(x, D)\) as pseudo differential operator. Its symbol is then determined by

\[
sigma(L(x, D))(x, \xi) = L(x, \xi) = e^{i(x, \xi)}L(x, D)e^{-i(x, \xi)} = \sum_{k, l=1}^n a_{kl}(x)\xi_k \xi_l + c(x),
\]

thus

\[
L(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \xi)}L(x, \xi) \hat{u}(\xi) d\xi.
\]

We are looking for a representation of \(f(L(x, D))\) as a pseudo differential operator, i.e., we want to find a function \(\tilde{p} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) such that

\[
f(L(x, D))u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \xi)} \tilde{p}(x, \xi) \hat{u}(\xi) d\xi
\]

holds. Clearly, one would expect

\[
\tilde{p}(x, \xi) = e^{i(x, \xi)}f(L(x, D))e^{-i(x, \xi)}
\]

but this expression is not well-defined. Instead of (3.25) we will use the symbol

\[
p(x, \xi) := f(L(x, \xi))
\]

and the associated operator

\[
p(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \xi)} p(x, \xi) \hat{u}(\xi) d\xi.
\]
In some sense, \( p(x, D) \) turns out to be a good approximation of \( f(L(x, D)) \). We can, however, prove a bit more, namely, that \( -p(x, D) \) itself extends to a generator of a sub-Markovian and even Fellerian semigroup. This will be discussed in the next section.

4. The operator \( p(x, D) \)

Let \( L(x, D) \) and \( f \in CBF_0 \) be as in Section 3. We define the symbol

\[
p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}
\]

by

\[
p(x, \xi) := f(L(x, \xi)) = f \left( \sum_{k, l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x) \right).
\]

Since for fixed \( x \in \mathbb{R}^n \) the function \( \xi \mapsto L(x, \xi) \) is continuous and negative definite, so is \( p(x, \xi) \), cf. Section 2. For some fixed \( x_0 \in \mathbb{R}^n \) we put

\[
p(x, \xi) = p(x_0, \xi) + (p(x, \xi) - p(x_0, \xi)) \equiv p_1(\xi) + p_2(x, \xi).
\]

**Lemma 4.1.** For all \( \xi \in \mathbb{R}^n \) and \( x \in \mathbb{R}^n \) the estimate

\[
|p_2(x, \xi)| \leq \phi_0(x)(f(|\xi|^2) + 1)
\]

holds where \( \phi_0 \) is given by

\[
\phi_0(x) = \max \{1, \nu_0\} \cdot \max \left\{ \left| \frac{c(x) - c(x_0)}{\nu_0} \right|, \frac{n}{\kappa_1} \max_{1 \leq k, l \leq n} |a_{kl}(x) - a_{kl}(x_0)| \right\}.
\]

**Proof.** Using the representation (2.12) for \( f \) we find by (3.2) and (3.4)

\[
|p_2(x, \xi)|
\]

\[
= \left| f \left( \sum_{k, l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x) \right) - f \left( \sum_{k, l=1}^{n} a_{kl}(x_0)\xi_k \xi_l + c(x_0) \right) \right|
\]

\[
\leq \int_0^\infty \frac{\sum_{k, l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x)}{\lambda + \sum_{k, l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x)} - \frac{\sum_{k, l=1}^{n} a_{kl}(x_0)\xi_k \xi_l + c(x_0)}{\lambda + \sum_{k, l=1}^{n} a_{kl}(x_0)\xi_k \xi_l + c(x_0)} \left| \rho(d\lambda) \right|
\]

\[
= \int_0^\infty \frac{\lambda \left( \sum_{k, l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x) \right) - \lambda \left( \sum_{k, l=1}^{n} a_{kl}(x_0)\xi_k \xi_l + c(x_0) \right)}{\left( \lambda + \sum_{k, l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x) \right) \left( \lambda + \sum_{k, l=1}^{n} a_{kl}(x_0)\xi_k \xi_l + c(x_0) \right)} \left| \rho(d\lambda) \right|
\]
\[
\begin{align*}
&\leq \int_0^\infty \left| \lambda \sum_{k,l=1}^n a_{kl}(x)\xi_k \xi_l - \lambda \sum_{k,l=1}^n a_{kl}(x_0)\xi_k \xi_l \right| \frac{\rho(d\lambda)}{(\lambda + \sum_{k,l=1}^n a_{kl}(x)\xi_k \xi_l) (\lambda + \sum_{k,l=1}^n a_{kl}(x_0)\xi_k \xi_l + \nu_0)} \\
&\quad + \int_0^\infty \frac{\lambda |c(x) - c(x_0)|}{(\lambda + \sum_{k,l=1}^n a_{kl}(x)\xi_k \xi_l) (\lambda + \sum_{k,l=1}^n a_{kl}(x_0)\xi_k \xi_l + \nu_0)} \rho(d\lambda)
\end{align*}
\]

\[
\leq \max_{1 \leq k, l \leq n} |a_{kl}(x) - a_{kl}(x_0)| \int_0^\infty \frac{\lambda}{\lambda + \sum_{k,l=1}^n a_{kl}(x_0)\xi_k \xi_l + \nu_0} \times
\frac{\sum_{k,l=1}^n |\xi_k \xi_l|}{\rho(d\lambda)}
\]

\[
\leq \frac{n}{\kappa_1} \max_{1 \leq k, l \leq n} |a_{kl}(x) - a_{kl}(x_0)| \int_0^\infty \frac{\kappa_1|\xi|^2}{\lambda + \kappa_1|\xi|^2 + \nu_0} \rho(d\lambda)
\]

\[
+ |c(x) - c(x_0)| \int_0^\infty \frac{1}{\lambda + \kappa_1|\xi|^2 + \nu_0} \rho(d\lambda)
\]

\[
\leq \tilde{\phi}_0(x) \int_0^\infty \frac{\kappa_1|\xi|^2 + \nu_0}{\lambda + \kappa_1|\xi|^2 + \nu_0} \rho(d\lambda)
\]

\[
= \tilde{\phi}_0(x) f(\kappa_1|\xi|^2 + \nu_0),
\]

where

\[
(4.5) \quad \tilde{\phi}_0(x) = \max \left\{ \frac{1}{\nu_0} |c(x) - c(x_0)|; \frac{n}{\kappa_1} \max_{1 \leq k, l \leq n} |a_{kl}(x) - a_{kl}(x_0)| \right\}.
\]

Since Bernstein functions are subadditive we find by Lemma 2.12, using \( \kappa_1 \leq 1 \),

\[
(4.6) \quad f(\kappa_1|\xi|^2 + \nu_0) \leq f(|\xi|^2) + \nu_0 \leq \max\{1, \nu_0\}(1 + f(|\xi|^2))
\]

which gives (4.3).

In the sequel, we will always assume that the function \( \phi_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). Thus, we have

\[
(4.7) \quad |p_2(x, \xi)| \leq \phi_0(x) (f(|\xi|^2) + 1)
\]

\(\square\)
with \( \phi_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \).

We need estimates like (4.7) for certain partial derivatives of \( p_2(x, \xi) \).

**Lemma 4.2.** Let \( L(x, D) \) and \( f \) be as above and suppose that \( \partial_{x}^\alpha a_{kl} \) and \( \partial_{x}^\alpha c \) belong to \( L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) for \( \alpha \in \mathbb{N}_0^n \), \( 0 < |\alpha| \leq q \). Then there exist functions \( \phi_\alpha \) from \( L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) such that
\[
|\partial_{x}^\alpha p_2(x, \xi)| \leq \phi_\alpha(x)(f(|\xi|^2) + 1), \quad 0 < |\alpha| \leq q,
\]
holds.

**Proof.** Since \( \alpha \neq 0 \), we find
\[
\partial_{x}^\alpha p_2(x, \xi) = \partial_{x}^\alpha \left( f \left( \sum_{k, l=1}^{n} a_{kl}(x_0)\xi_k \xi_l + c(x) \right) \right) = \partial_{x}^\alpha f \left( \sum_{k, l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x) \right).
\]

Using the formula for the \( \alpha \)-th derivative of a composite function, \( \partial^\alpha(f \circ g) \), where \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \),
\[
\partial^\alpha(f \circ g) = \sum_{j=1}^{|\alpha|} f^{(j)}(g(\cdot)) \sum_{\beta, \gamma, \ldots, \omega} \frac{\alpha!}{\delta_\beta! \delta_\gamma! \ldots \delta_\omega!} \left( \frac{\partial^\beta g}{\beta!} \right)^{\delta_\beta} \left( \frac{\partial^\gamma g}{\gamma!} \right)^{\delta_\gamma} \ldots \left( \frac{\partial^\omega g}{\omega!} \right)^{\delta_\omega}
\]
with the second sum running over all pairwise different multiindices \( 0 \neq \beta, \gamma, \ldots, \omega \) in \( \mathbb{N}_0^n \) and all \( \delta_\beta, \delta_\gamma, \ldots, \delta_\omega \in \mathbb{N} \) such that \( \delta_\beta \beta + \delta_\gamma \gamma + \ldots + \delta_\omega \omega = \alpha \) and \( \delta_\beta + \delta_\gamma + \ldots + \delta_\omega = j \) hold (compare [10] 0.430 for the one-dimensional case), the estimate (2.19), the ellipticity estimate (3.2), and (3.4), we find for \( 0 < |\alpha| \leq q \) the desired inequalities once we know that \( \partial_{x}^\alpha a_{kl}, \partial_{x}^\alpha c \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). \( \square \)

The next result will us enable to apply most of the estimates done in [19] to the operator \( p(x, D) \). To do this, we need the following notations:
\[
\tilde{p}_2(\eta, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \eta)} p_2(x, \xi) dx
\]
and for \( N \in \mathbb{N} \) pick \( \tilde{\gamma}_N \) such that
\[
(1 + |\eta|^2)^{N/2} \leq \tilde{\gamma}_N \sum_{|\alpha| \leq N} |\eta^\alpha|.
\]

**Lemma 4.3.** Let \( q \in \mathbb{N}_0 \) and \( \partial_{x}^\alpha a_{kl}, \partial_{x}^\alpha c \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) for \( 0 < |\alpha| \leq q \) and suppose that (4.7), (4.8) hold. Then we have for all \( N \in \mathbb{N}_0, N \leq q \),
\[
\tilde{p}_2(\eta, \xi) \leq \tilde{\gamma}_N \sum_{|\alpha| \leq N} \|\phi_\alpha\|_{L^1}(1 + |\eta|^2)^{-N/2}(f(|\xi|^2) + 1).
\]
Proof. For $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq N$, we have

$$
\left| \eta^\alpha \int_{\mathbb{R}^n} e^{-i(x,\xi)} p_2(x, \xi) \, dx \right| = \left| \int_{\mathbb{R}^n} \left( \partial_x^\alpha e^{-i(x,\xi)} \right) p_2(x, \xi) \, dx \right|
$$

$$
= \left| \int_{\mathbb{R}^n} e^{-i(x,\xi)} \partial_x^\alpha p_2(x, \xi) \, dx \right|
$$

$$
\leq \int_{\mathbb{R}^n} |\phi_\alpha(x)| \left( f(|\xi|^2) + 1 \right) \, dx
$$

$$
= \|\phi_\alpha\|_{L^1} \left( f(|\xi|^2) + 1 \right).
$$

This implies

$$
(1 + |\eta|^2)^{N/2} |\tilde{p}_2(\eta, \xi)| \leq \tilde{c}_N \sum_{|\alpha| \leq N} \|\phi_\alpha\|_{L^1} \left( f(|\xi|^2) + 1 \right)
$$

and the assertion follows. \qed

Let $p(x, \xi)$ be as above and suppose that $\sum_{|\alpha| \leq q} \|\phi_\alpha\|_{L^1}$ is small compared with $k_1$ for sufficiently large $q \in \mathbb{N}$. Then it is shown in [19] that $-p(x, D)$ extends to a generator of a Feller semigroup. For we will need further estimates of $p(x, D)$ and $p_\lambda(x, D) := p(x, D) + \lambda \text{id}$ we recall the results of [19] in greater detail. The crucial point to transfer the results in [19] is the validity of Lemma 4.3.

We assume that $q > n + \frac{3n}{2} + 10$, that $\partial_x^\alpha a_{kl}, \partial_x^\alpha c \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $0 < |\alpha| \leq q$, and that $\phi_0 \in L^1(\mathbb{R}^n)$ where $\phi_0$ is given by (4.4). The number $r \leq 2$ has already been introduced in (3.16). Since $rq > \frac{n}{2}$ we get by Sobolev’s embedding theorem the continuous inclusions

$$
H^{q+1, a^2}(\mathbb{R}^n) \hookrightarrow H^{q, a^2}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n)
$$

with $a^2(\xi) = f(|\xi|^2)$. By $\gamma_b$ we denote the constant

$$
\gamma_b := \tilde{c}_q \sum_{|\alpha| \leq q} \|\phi_\alpha\|_{L^1} \int_{\mathbb{R}^n} \left( 1 + |\tau|^2 \right)^{\frac{1+a}{2}} \, d\tau
$$

where $\phi_0$ is given by (4.4) whereas the $\phi_\alpha$, $0 < |\alpha| \leq q$, are taken from Lemma 4.2. Note that $\gamma_b$ is finite since $q > n + 1$. The above mentioned smallness condition imposed on $\sum_{|\alpha| \leq q} \|\phi_\alpha\|_{L^1}$ can now be made explicit

$$
\gamma_b \leq (1 - \epsilon)k_1 \text{ for some } \epsilon \in \left( 1 - \frac{1}{\sqrt{24}}, 1 \right).
$$

It should be noted that all considerations remain valid if we define

$$
\tilde{c} := \sup \left\{ \mu > 0 : \mu|\xi|^2 \leq \sum_{k, l=1}^n a_{kl}(x)\xi_k \xi_l \right\}
$$
and require
\begin{equation}
\gamma_b \leq (1 - \epsilon)\bar{\kappa} \text{ for some } \epsilon \in \left( 1 - \frac{1}{\sqrt{24}}, 1 \right)
\end{equation}
instead of (4.15).

On $C_0^\infty(\mathbb{R}^n)$ we consider the operator
\begin{equation}
p(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \xi)} p(x, \xi) \tilde{u}(\xi) \, d\xi
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \xi)} f \left( \sum_{k, i=1}^{n} a_{ki}(x) x_k \xi_i + c(x) \right) \tilde{u}(\xi) \, d\xi
\end{equation}
and the associated bilinear form $B(\cdot, \cdot)$ given by
\begin{equation}
B(u, v) := (p(x, D)u, v)_0.
\end{equation}
As usual, we set $B_\lambda(u, v) := B(u, v) + \lambda(u, v)_0$ for $\lambda \in \mathbb{R}$. The following estimates are taken from [19]; the proofs rely mainly on Lemma 4.3.

**Theorem 4.4.** Suppose that $q > n + \frac{3n}{r} + 10$ and that (4.15) holds. Then we have for all $u, v \in C_0^\infty(\mathbb{R}^n)$ and $0 \leq t \leq \frac{n}{2r} + 1$
\begin{align}
|B(u, v)| &\leq C_1 \|u\|_{t, a_2} \|v\|_{t, a_2}; \\
B(u, u) &\geq \epsilon \kappa_1 \|u\|_{t, a_2}^2 - C_2 \|u\|_0^2; \\
\|p(x, D)u\|_{t, a_2} &\leq C_3 \|u\|_{t+1, a_2}; \\
\|u\|_{t+1, a_2} &\leq C_4 (\|p(x, D)u\|_{t, a_2} + \|u\|_0).
\end{align}

**Remark 4.5.** (i) We can enlarge the range of $t$ in (4.22) and (4.23) if we enlarge $q$.

(ii) The conditions $1 - \frac{1}{\sqrt{24}} < \epsilon < 1$ and $q > n + \frac{3n}{r} + 10$ are by no means sharp, see HOH [17]. Since we want to give a precise citation, we took the weaker, but exact, conditions from [19].

Theorem 4.4 allows us to extend $B$ as a continuous bilinear form onto $H^{t+1, a_2}(\mathbb{R}^n)$ and to extend $p(x, D)$ as a continuous linear operator from $H^{t+1, a_2}(\mathbb{R}^n)$ into $H^{t, a_2}(\mathbb{R}^n)$ provided that $t \leq \frac{n}{2r} + 1$. The estimates (4.20) – (4.23) remain valid for these extensions and, in what follows, we will not distinguish between the original operators, $B$ and $p(x, D)$, and their extensions.

In the proof of Theorem 4.4 we use commutator estimates of the type
\begin{equation}
\left\| \left[ \left( 1 + a^2(D) \right)^s, p_{s, D} \right] u \right\|_{t, a_2} \leq c \|u\|_{s+t+\frac{1}{2}, a_2},
\end{equation}
cf. [19] Theorem 2.1. Moreover, using the Friedrichs mollifier, Theorem 4.4 implies that, if
\begin{equation}
B_\lambda(u, v) = (f, \phi)_0, \quad \phi \in C_0^\infty(\mathbb{R}^n),
\end{equation}
holds for some \( f \in H^{\frac{n}{2}+1, a^2}(\mathbb{R}^n) \), the left-hand side \( u \) is already of class \( H^{\frac{n}{2}+2, a^2}(\mathbb{R}^n) \).

By an approximation argument, see [18] Theorem 9.3 or [21] Proposition 4.1, we deduce that \( -p(x, D) \) satisfies the positive maximum principle on \( H^{\frac{n}{2}+2, a^2}(\mathbb{R}^n) \).

**Theorem 4.6.** Under the above assumptions, \( -p(x, D) \), defined on \( H^{\frac{n}{2}+2, a^2}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), extends to (i) a generator of a Feller semigroup \( \{T_t^{(\infty)}\}_{t \geq 0} \) and (ii) to a generator of a sub-Markovian semigroup \( \{T_t^{(2)}\}_{t \geq 0} \).

The domain of the generator of \( \{T_t^{(2)}\}_{t \geq 0} \) is given by \( H^{1, a^2}(\mathbb{R}^n) \).

**Remark 4.7.** (i) The proof that \( -p(x, D) \) extends to a generator of a contractive sub-Markovian semigroup is given in [20].

(ii) The smallness condition (4.15) is due to the method, in particular to the fact that we want to have the estimates (4.20)–(4.23). Using a martingale problem approach HOH [17] succeeded in constructing the Feller semigroup without a smallness condition like (4.15). He, however, could not derive (4.20)–(4.23) for the corresponding bilinear form and operator, respectively.

(iii) Note that (4.20) – (4.23) also imply that the sub-Markovian semigroup \( \{S_t\}_{t \geq 0} \), obtained by extending \( \{T_t^{(\infty)}\}_{t \geq 0} \) onto \( L^2(\mathbb{R}^n) \), coincides with \( \{T_t^{(2)}\}_{t \geq 0} \). In particular, this yields \( T_t^{(\infty)} u(x) = T_t^{(2)} u(x) \) almost everywhere for all \( u \in H^{1, a^2}(\mathbb{R}^n) \) if \( H^{1, a^2}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n) \). The proof relies, essentially, on the fact that \( H^{\frac{n}{2}+2, a^2}(\mathbb{R}^n) \) is an operator core for both the generator of \( \{S_t\}_{t \geq 0} \) and \( p(x, \xi) \).

We have now two pairs of semigroups: On the one hand, the Feller semigroups \( \{T_t^{(\infty)}\}_{t \geq 0} \) and \( \{\tilde{T}_t^{f}\}_{t \geq 0} \) generated by \( -p(x, D) \) and \( -f(L(x, D)) \) (considered as operators on \( C_\infty(\mathbb{R}^n) \)), respectively. On the other hand, the sub-Markovian semigroups \( \{T_t^{(2)}\}_{t \geq 0} \) and \( \{T_t^{f}\}_{t \geq 0} \) generated by \( -p(x, D) \) and \( -f(L(x, D)) \) (considered as operators on \( L^2(\mathbb{R}^n) \)), respectively.

Our aim is to compare \( \{T_t^{(2)}\}_{t \geq 0} \) and \( \{T_t^{f}\}_{t \geq 0} \), \( \{T_t^{(\infty)}\}_{t \geq 0} \) and \( \{\tilde{T}_t^{f}\}_{t \geq 0} \), and \( p(x, D) \) and \( f(L(x, D)) \). For this purpose we need an approximation of the resolvent \( \{R_\lambda\}_{\lambda \geq 0} \) of \( L(x, D) \).

### 5. A first-order approximation of \( R_\lambda \)

Let \( L(x, D) \) be as in Section 3 and \( \lambda > -\nu_0 \). We want to construct an approximation of the resolvent \( R_\lambda \) of \( L(x, D) + \lambda \text{id} \). Since the symbol of \( L(x, D) + \lambda \text{id} \) is given by \( L(x, \xi) + \lambda \), a natural candidate of such an approximation is the pseudo differential operator

\[
q_\lambda(x, D) u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \xi)} q_\lambda(x, \xi) \hat{u}(\xi) \, d\xi
\]
whose symbol $q_\lambda(x, \xi)$ is given by

$$
q_\lambda(x, \xi) = \frac{1}{L(x, \xi) + \lambda} = \frac{1}{\sum_{k, l} a_{kl}(x)\xi_k\xi_l + c(x) + \lambda}.
$$

Our first aim is to prove that for suitable $s \geq 0$ the operator $q_\lambda(x, D)$ maps $H^s(\mathbb{R}^n)$ continuously into $H^{s+2}(\mathbb{R}^n)$. To this end we need some auxiliary considerations.

Fix a point $x_0 \in \mathbb{R}^n$ and decompose $q_\lambda(x, \xi)$

$$
q_\lambda(x, \xi) := q_\lambda(x_0, \xi) + r_\lambda(x, \xi)
$$

where $r_\lambda(x, \xi)$ is given by

$$
r_\lambda(x, \xi) = \frac{1}{\sum_{k, l} a_{kl}(x)\xi_k\xi_l + c(x) + \lambda} - \frac{1}{\sum_{k, l} a_{kl}(x_0)\xi_k\xi_l + c(x_0) + \lambda}.
$$

By $\widetilde{\psi}_0$ we denote the function

$$
\widetilde{\psi}_0(x) := \max \left\{ n \max_{1 \leq k, l \leq n} |a_{kl}(x) - a_{kl}(x_0)|, |c(x) - c(x_0)| \right\}
$$

and by $\psi_0$

$$
\psi_0(x) := \frac{1}{\kappa_1} \widetilde{\psi}_0(x).
$$

By the assumptions made in Section 4, we have $\psi_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

**Lemma 5.1.** Let $r_\lambda(x, \xi)$ and $\psi_0(x)$ be as above. Then we have

$$
|r_\lambda(x, \xi)| \leq \psi_0(x) \frac{|\xi|^2 + 1}{(|\xi|^2 + \nu_0 + \lambda)^2}.
$$

**Proof.** Without loss of generality we may assume that $L(x, \xi) \leq L(x_0, \xi)$ and $\kappa_1 \leq 1$. By the definition of $r_\lambda(x, \xi)$ we find

$$
r_\lambda(x, \xi) = \int_{L(x, \xi)}^{L(x_0, \xi)} \frac{1}{(\tau + \lambda)^2} d\tau,
$$

hence,

$$
|r_\lambda(x, \xi)| \leq |L(x, \xi) - L(x_0, \xi)| \sup_{L(x, \xi) \leq \tau \leq L(x_0, \xi)} \frac{1}{(\tau + \lambda)^2}

\leq |L(x, \xi) - L(x_0, \xi)| \frac{1}{(\kappa_1|\xi|^2 + \nu_0 + \lambda)^2}

\leq \frac{1}{\kappa_1^2} |L(x, \xi) - L(x_0, \xi)| \frac{1}{(|\xi|^2 + \nu_0 + \lambda)^2}.
$$
On the other hand,

\[ |L(x, \xi) - L(x_0, \xi)| = \left| \sum_{k,l=1}^{n} (a_{kl}(x) - a_{kl}(x_0))\xi_k \xi_l + (c(x) - c(x_0)) \right| \]

\[ \leq n \max_{1 \leq k, l \leq n} |a_{kl}(x) - a_{kl}(x_0)| |\xi|^2 + |c(x) - c(x_0)| \]

yields

\[ |r_\lambda(x, \xi)| \leq \psi_0(x) \frac{|\xi|^2 + 1}{(|\xi|^2 + \nu_0 + \lambda)^2} \]

proving (5.7). The case \( L(x, \xi) \geq L(x_0, \xi) \) can be handled in a similar fashion. \( \square \)

In order to get estimates for \( \partial_x^\alpha r_\lambda(x, \xi) \), \( 0 < |\alpha| \leq q \), we observe that for \( \alpha \neq 0 \)

\[ \partial_x^\alpha r_\lambda(x, \xi) = \partial_x^\alpha q_\lambda(x, \xi) \]

holds.

**Lemma 5.2.** For \( 0 < |\alpha| \leq q \) we assume \( \partial_x^{\alpha_k} a_{kl}, \partial_x^{\alpha_c} c \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). Then there exist functions \( \psi_\alpha \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) such that

\[ |\partial_x^\alpha r_\lambda(x, \xi)| \leq \psi_\alpha(x) \frac{|\xi|^2 + 1}{(|\xi|^2 + \nu_0 + \lambda)^2} \]

holds.

**Proof.** In view of (5.8) it is enough to estimate

\[ \partial_x^\alpha q_\lambda(x, \xi) = \partial_x^\alpha \left( \frac{1}{\sum_{k,l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x) + \lambda} \right), \quad 0 < |\alpha| \leq q. \]

Suppose that \( |\alpha| = 1 \), i.e., that \( \partial_x^\alpha = \frac{\partial}{\partial x_j} \) for some \( 1 \leq j \leq n \). This gives

\[ \left| \frac{\partial}{\partial x_j} r_\lambda(x, \xi) \right| = \frac{|\partial_x^{\alpha_k} L(x, \xi)|}{(L(x, \xi) + \lambda)^2} = \frac{\sum_{k,l=1}^{n} \frac{\partial a_{kl}(x)}{\partial x_j} \xi_k \xi_l + \frac{\partial c(x)}{\partial x_j} \xi_l + \frac{\partial c(x)}{\partial x_j} \xi_l}{\left( \sum_{k,l=1}^{n} a_{kl}(x)\xi_k \xi_l + c(x) + \lambda \right)^2} \]

\[ \leq \frac{n \max_{1 \leq k, l \leq n} \left| \frac{\partial a_{kl}(x)}{\partial x_j} \right| \left| \xi_l^2 + \frac{\partial c(x)}{\partial x_j} \right|}{(\kappa_1 |\xi|^2 + \nu_0 + \lambda)^2} \]

\[ \leq \frac{1}{\kappa_1^2} \max \left\{ \sum_{1 \leq k, l \leq n} \left| \frac{\partial a_{kl}(x)}{\partial x_j} \right| ; \left| \frac{\partial c(x)}{\partial x_j} \right| \right\} \frac{|\xi|^2 + 1}{(|\xi|^2 + \nu_0 + \lambda)^2}, \]
where we again used $\kappa_1 \leq 1$. To prove the general estimate we can either proceed by induction with respect to $|\alpha|$ or use — as in the proof of Lemma 4.2 — formula (4.9) for the partial derivatives of a composite function.

The next Lemma follows from Lemmas 5.1 and 5.2 by very much the same arguments as did Lemma 4.3 from 4.1 and 4.2.

**Lemma 5.3.** Let $q \in \mathbb{N}_0$ and $\partial_x^\alpha a_{k1}, \partial_x^\alpha c \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $0 < |\alpha| \leq q$ and suppose that $\psi_0 \in L^1(\mathbb{R}^n)$, cf. (5.5) and (5.6). Then we have

\[
|\hat{r}_\lambda(\eta, \xi)| \leq c_q \sum_{|\alpha| \leq q} \|\psi_\alpha\|_{L^1} (1 + |\eta|^2)^{-q/2} \frac{|\xi|^2 + 1}{(|\xi|^2 + \nu_0 + \lambda)^2}.
\]

The continuity of $q_\lambda(x, D)$ is now easy to prove.

**Theorem 5.4.** Suppose that for $q > n + 2$ the assumptions of Lemma 5.3 are fulfilled. Then the operator $q_\lambda(x, D)$, $\lambda \geq 0$, maps $L^2(\mathbb{R}^n)$ continuously into $H^2(\mathbb{R}^n)$ and we have the estimate

\[
\|q_\lambda(x, D)u\|_2 \leq c \|u\|_0
\]

where the constant $c$ depends only on $\kappa_1, \nu_0$ and $n$.

**Proof.** We decompose $q_\lambda(x, D)$ similarly to (5.3),

\[
q_\lambda(x, D) = q_\lambda(x_0, D) + r_\lambda(x, D).
\]

Clearly,

\[
\|q_\lambda(x_0, D)u\|_2^2 = \int_{\mathbb{R}^n} \left( \sum_{k, l=1}^n a_{k1}(x_0) \xi_k \xi_l + c(x) + \lambda \right)^2 |\hat{u}(\xi)|^2 d\xi
\]

\[
\leq \frac{1}{\kappa_1^2} \int_{\mathbb{R}^n} \frac{(|\xi|^2 + 1)^2}{(|\xi|^2 + \nu_0 + \lambda)^2} |\hat{u}(\xi)|^2 d\xi \leq c \|u\|_0
\]

which proves (5.11) for the operator $q_\lambda(x_0, D)$. We now turn to $r_\lambda(x, D)$. For $v \in L^2(\mathbb{R}^n) \subset H^{-2}(\mathbb{R}^n)$ we find

\[
|\langle r_\lambda(x, D)u, v \rangle|
\]

\[
= (2\pi)^{-n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{r}_\lambda(\xi - \eta, \xi) \hat{u}(\eta) \overline{\hat{v}(\xi)} d\eta d\xi \right|
\]

\[
\leq c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{|\alpha| \leq q} \|\psi_\alpha\|_{L^1} (1 + |\xi - \eta|^2)^{-q/2} \frac{1 + |\eta|^2}{(|\eta|^2 + \nu_0 + \lambda)^2} |\hat{u}(\eta)||\hat{v}(\xi)| d\eta d\xi \leq
\]

\[
\leq c \|u\|_0 \|v\|_0.
\]
implying \( L^2(\mathbb{R}^n) \) is dense in \( H^{-2}(\mathbb{R}^n) \). Therefore, we get for \( \varphi \in L^2(\mathbb{R}^n) \)
\[
\left| \langle T(\xi, D) \varphi, \varphi \rangle \right| \leq c \left\| \varphi \right\|_0 \left\| \varphi \right\|_{-2},
\]
where we used Peetre’s inequality \((1 + |\xi|^2)(1 + |\eta|^2)^{-1} \leq 2(1 + |\xi - \eta|^2)\). Thus, we get for \( \varphi \in L^2(\mathbb{R}^n) \)
\[
\left| \langle \tau_\lambda(x, D) \varphi, \varphi \rangle \right| \leq c \left\| \varphi \right\|_0 \left\| \varphi \right\|_{-2},
\]
implying \( L^2(\mathbb{R}^n) \) is dense in \( H^{-2}(\mathbb{R}^n) \) — that \( \| \tau_\lambda(x, D) \varphi \|_2 \leq c \| \varphi \|_0 \) and the theorem follows.

**Corollary 5.5.** Suppose that for \( q > n + 2 + s \) the assumptions of Lemma 5.3 are fulfilled. Then the operator \( q_\lambda(x, D) \) satisfies

\[
\| q_\lambda(x, D) u \|_{s+2} \leq c_s \| u \|_s.
\]

**Proof.** For \( q_\lambda(x_0, D) \) the above estimate is trivial. In order to estimate \( r_\lambda(x, D) \) we proceed as in Theorem 5.4. If \( \varphi \in L^2(\mathbb{R}^n) \subset H^{-2-s}(\mathbb{R}^n) \), we find

\[
\left| \langle \tau_\lambda(x, D) \varphi, \varphi \rangle \right| \leq c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-\frac{s+2}{2}} |\hat{\varphi}(\eta)|(1 + |\xi|^2)^{-1} |\hat{\varphi}(\xi)| \, d\eta \, d\xi
\]

\[
= c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-\frac{s+2}{2}} \frac{(1 + |\xi|^2)^{s/2}}{(1 + |\eta|^2)^{s/2}} \times
\]

\[
\times (1 + |\eta|^2)^{s/2} |\hat{\varphi}(\eta)|(1 + |\xi|^2)^{-\frac{s-2}{2}} |\hat{\varphi}(\xi)| \, d\eta \, d\xi
\]

\[
\leq c_s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-\frac{s+2}{2}} (1 + |\eta|^2)^{s/2} |\hat{\varphi}(\eta)|(1 + |\xi|^2)^{-\frac{s-2}{2}} |\hat{\varphi}(\xi)| \, d\eta \, d\xi
\]

\[
\leq c_s' \| u \|_s \| \varphi \|_{-s-2},
\]

which gives (5.13) for \( \tau_\lambda(x, D) \) and the corollary is proved.

Our next aim is to prove that the operator

\[
K_\lambda(x, D) := L_\lambda(x, D) \circ q_\lambda(x, D) - \text{id}
\]
is of order $-1$, that is, for all $u \in L^2(\mathbb{R}^n)$ we have

\begin{equation}
\|K_\lambda(x,D)u\|_1 \leq c\|u\|_0. 
\end{equation}

To do this, note that $K_\lambda(x,D)$ is a pseudo differential operator whose symbol $K_\lambda(x,\xi)$ is given by

\[
K_\lambda(x,\xi) = \frac{2i}{L_\lambda(x,\xi)^2} \left( \sum_{k,l=1}^{n} a_{\mu\nu}(x) \xi_{\mu} \frac{\partial}{\partial x_{\mu}} (a_{kl}(x) \xi_{k} \xi_{l} + c(x)) \right) 
+ \frac{1}{L_\lambda(x,\xi)^3} \left( \sum_{k,l=1}^{n} a_{\mu\nu}(x) \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} (a_{kl}(x) \xi_{k} \xi_{l} + c(x)) \right) 
- \frac{2}{L_\lambda(x,\xi)^3} \left( \sum_{\mu,\nu=1}^{n} a_{\mu\nu}(x) \left( \sum_{k,l=1}^{n} \frac{\partial}{\partial x_{\mu}} (a_{kl}(x) \xi_{k} \xi_{l} + c(x)) \right) \times \sum_{s,t=1}^{n} \frac{\partial}{\partial x_{\nu}} (a_{st}(x) \xi_{s} \xi_{t} + c(x)) \right) 
= K_1^1(x,\xi) + K_2^2(x,\xi) + K_3^3(x,\xi).
\]

In order to estimate the $K_\lambda^1(x,D)u$'s, we need some analogues of Lemma 5.2 and Lemma 5.3. For later use, we derive estimates of type

\[
\|K_\lambda(x,D)u\|_{\sigma} \leq \frac{c}{(\nu_0 + \lambda)^{\tau}}
\]

for certain values of $\sigma$ and $\tau$.

By our assumptions we find for $K_1^1(x,\xi)$

\begin{equation}
K_1^1(x,\xi) = \frac{1}{L_\lambda(x,\xi)^2} \left( \sum_{k,l,m=1}^{n} \psi_{klm}(x) \xi_{k} \xi_{l} \xi_{m} + \sum_{k=1}^{n} \psi_k(x) \xi_{k} \right)
\end{equation}

where $\partial^\alpha_x \psi_{klm}$, $\partial^\alpha_x \psi_k \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $|\alpha| \leq q - 1$. Clearly, (5.16) implies for some $\phi_0^1 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

\begin{equation}
|K_1^1(x,\xi)| \leq \phi_0^1(x) \frac{1}{(|\xi|^2 + \nu_0 + \lambda)^{1/2}}.
\end{equation}

Using the formula for derivatives of composite functions we find

\begin{equation}
|\partial^\alpha_x K_1^1(x,\xi)| \leq \phi_0^1(x) \frac{1}{(|\xi|^2 + \nu_0 + \lambda)^{1/2}}
\end{equation}

with $\phi_0^1 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $|\alpha| \leq q - 1$. If $0 \leq \sigma \leq \frac{1}{2}$, we may deduce

\begin{equation}
|\partial^\alpha_x K_1^1(x,\xi)| \leq \frac{1}{(\nu_0 + \lambda)^{\sigma}} \phi_0^1(x) \frac{1}{(|\xi|^2 + 1)^{1/2 - \frac{2\sigma}{2}}}
\end{equation}
If \( q > n + 2 \), this proves the following proposition.

**Proposition 5.6.** Under the above assumptions we have for \( 0 \leq \sigma \leq \frac{1}{2} \)

\[
\| K_1^1(x, D)u \|_{L^2} \leq \frac{c}{(\nu_0 + \lambda)^{\sigma}} \| u \|_0,
\]

\( i.e., \) the operator \( K_1^1(x, D) \) maps \( L^2(\mathbb{R}^n) \) continuously into \( H^{1-2\sigma}(\mathbb{R}^n) \).

By the same reasoning as above, we find for \( K_2^1(x, D) \)

\[
|\partial_x^\alpha K_2^1(x, \xi) | \leq \frac{1}{(\nu_0 + \lambda)^{\sigma}} \phi_\alpha^2(x) \frac{1}{((\xi)^2 + 1)^{1-\sigma}}, \quad |\alpha| \leq q - 2,
\]

with \( \phi_\alpha^2 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) and \( 0 \leq \sigma \leq 1 \). This yields for \( q > n + 3 \) the following result:

**Proposition 5.7.** Under the above assumptions we have for \( 0 \leq \sigma \leq 1 \)

\[
\| K_3^1(x, D)u \|_{L^2} \leq \frac{c}{(\nu_0 + \lambda)^{\sigma}} \| u \|_0,
\]

\( i.e., \) the operator \( K_3^1(x, D) \) maps \( L^2(\mathbb{R}^n) \) continuously into \( H^{2-2\sigma}(\mathbb{R}^n) \).

Finally we get for \( 0 \leq \sigma \leq 1 \)

\[
|\partial_x^\alpha K_3^1(x, \xi) | \leq \frac{1}{(\nu_0 + \lambda)^{\sigma}} \phi_\alpha^3(x) \frac{1}{((\xi)^2 + 1)^{1-\sigma}}, \quad |\alpha| \leq q - 1,
\]

with \( \phi_\alpha^3 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). This shows for \( q > n + 2 \) the following proposition:

**Proposition 5.8.** Under the above assumptions we have for \( 0 \leq \sigma \leq 1 \)

\[
\| K_4^1(x, D)u \|_{L^2} \leq \frac{c}{(\nu_0 + \lambda)^{\sigma}} \| u \|_0,
\]

\( i.e., \) the operator \( K_4^1(x, D) \) maps \( L^2(\mathbb{R}^n) \) continuously into \( H^{2-2\sigma}(\mathbb{R}^n) \).

Let us sum up our results in the following theorem.

**Theorem 5.9.** Let \( q > n + 3 \) and \( 0 \leq \sigma \leq \frac{1}{2} \). Then there exists a constant \( c \), not depending on \( \lambda \), such that

\[
\| K_{1}(x, D) \|_{L^2} \leq \frac{c_\sigma}{(\nu_0 + \lambda)^{\sigma}} \| u \|_0
\]

holds. In particular, taking \( \sigma = 0 \), the operator \( K_{1}(x, D) \) is of order \(-1\), i.e., it maps \( L^2(\mathbb{R}^n) \) continuously into \( H^{1}(\mathbb{R}^n) \).
Corollary 5.10. For sufficiently large $q \in \mathbb{N}$, $0 \leq \sigma < \frac{1}{2}$ and all $s \in \mathbb{R}$ we have

\begin{equation}
\| K_\lambda(x, D) u \|_{s+1+2\sigma} \leq \frac{c_{s, \sigma}}{(\nu_0 + \lambda)^{\sigma}} \| u \|_s. \tag{5.26}
\end{equation}

Proof. Combining (5.19), (5.21), and (5.23) we find

\begin{equation}
| \partial_\alpha^\sigma K(x, \xi) | \leq \frac{1}{(\nu_0 + \lambda)^{\sigma}} \phi_\alpha(x) \frac{1}{(|\xi|^2 + 1)^{(1-2\sigma)/2}}
\end{equation}

with $\phi_\alpha \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $|\alpha| \leq q - 2$. Mimicking the proof of Lemma 4.3 we find for $v \in L^2(\mathbb{R}^n)$

\begin{align*}
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{K}_\lambda(\xi - \eta, \eta) \tilde{u}(\eta) \tilde{v}(\xi) \, d\eta \, d\xi \right| &
\leq \frac{c}{(\nu_0 + \lambda)^{\sigma}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-q/2} (1 + |\eta|^2)^{\sigma - 1/2} |\tilde{u}(\eta)| |\tilde{v}(\xi)| \, d\eta \, d\xi \\
&= \frac{c}{(\nu_0 + \lambda)^{\sigma}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{-q/2} \left(1 + |\eta|^2\right)^{2\sigma - s - 1/2} (1 + |\xi|^2)^{2\sigma - s - 1/2} \\
&\quad \times (1 + |\eta|^2)^{s/2} |\tilde{u}(\eta)| (1 + |\xi|^2)^{2\sigma - s - 1} |\tilde{v}(\xi)| \, d\eta \, d\xi \\
&\leq \frac{c'}{(\nu_0 + \lambda)^{\sigma}} \| u \|_s \| v \|_{2\sigma - s - 1}
\end{align*}

which implies (5.26). \qed

In particular, for $\sigma = \frac{1}{2}$ and $s > \frac{n}{2}$ we get

\begin{equation}
\| K_\lambda(x, D) u \|_\infty \leq \frac{c'}{(\nu_0 + \lambda)^{1/2}} \| u \|_s. \tag{5.27}
\end{equation}

By our construction, the pseudo differential operator $K_\lambda(x, D)$ with symbol (5.16) satisfies (5.14). Thus, under the above assumptions we can restate our results in the following way:

Theorem 5.11. There is a family of pseudo differential operators $K_\lambda(x, D)$, $\lambda \geq 0$, of order $-1$ satisfying (5.14), i.e., $K_\lambda(x, D) \equiv L_\lambda(x, D) \circ q_\lambda(x, D) - \text{id}$.

Since $L_\lambda(x, D)$ has for all $\lambda > -\nu_0$ a bounded inverse $R_\lambda$, the resolvent of $L(x, D)$ at $\lambda$, we know

\begin{equation}
L_\lambda(x, D) R_\lambda = \text{id} \tag{5.28}
\end{equation}

while Theorem 5.11 implies

\begin{equation}
L_\lambda(x, D) q_\lambda(x, D) = \text{id} + K_\lambda(x, D). \tag{5.29}
\end{equation}
Combining (5.28) and (5.29) we find

\[ q_\lambda(x, D) - R_\lambda = R_\lambda K_\lambda(x, D) \]

which gives us various estimates for \( q_\lambda(x, D) - R_\lambda \). By the resolvent estimate, we have

\[ \|R_\lambda\| \leq \frac{1}{\nu_0 + \lambda}, \quad \lambda > -\nu_0, \]

thus, by Theorem 5.9, for \( 0 \leq \sigma \leq \frac{1}{2} \)

\[ \|(q_\lambda(x, D) - R_\lambda)u\|_0 = \|R_\lambda K_\lambda(x, D)u\|_0 \]
\[ \leq \frac{1}{\nu_0 + \lambda} \|K_\lambda(x, D)u\|_0 \leq \frac{1}{(\nu_0 + \lambda)^{\sigma + 1}} \|u\|_0. \]

This implies for \( 0 \leq \sigma \leq \frac{1}{2} \)

\[ \|(q_\lambda(x, D) - R_\lambda)u\|_0 = \|R_\lambda K_\lambda(x, D)\|_0 \leq \frac{1}{(\nu_0 + \lambda)^{\sigma + 1}}. \]

Moreover, \( q_\lambda(x, D) \) and \( R_\lambda \) map \( L^2(\mathbb{R}^n) \) continuously into \( H^2(\mathbb{R}^n) \), hence,

\[ \|(q_\lambda(x, D) - R_\lambda)u\|_2 \leq c(\nu_0, \lambda)\|u\|_0. \]

Under our smoothness assumptions on \( a_{kl} \) and \( c \) we know that

\[ R_\lambda : H^1(\mathbb{R}^n) \longrightarrow H^3(\mathbb{R}^n) \]

is also a bounded operator. Using (5.30) this gives

\[ \|(q_\lambda(x, D) - R_\lambda)u\|_3 = \|R_\lambda K_\lambda(x, D)u\|_3 \leq \frac{c}{\nu_0 + \lambda} \|K_\lambda(x, D)u\|_1 \leq \frac{c'}{\nu_0 + \lambda} \|u\|_0, \]

hence

\[ \|(q_\lambda(x, D) - R_\lambda)u\|_3 \leq \frac{c'}{\nu_0 + \lambda} \|u\|_0. \]

The last inequality has a nice interpretation: the operator \( q_\lambda(x, D) \) approximates \( R_\lambda \) with the difference \( q_\lambda(x, D) - R_\lambda \) being not only of order \(-2\) (as would have been expected) but even of order \(-3\).

We will use these estimates in the next section in order to compare the operators \( f(L(x, D)) \) and \( p(x, D) \) as well as the associated semigroups \( \{T^f_t\}_{t \geq 0} \) and \( \{T^{(2)}_t\}_{t \geq 0} \), and \( \{\tilde{T}^f_t\}_{t \geq 0} \) and \( \{T^{(\infty)}_t\}_{t \geq 0} \), respectively.

6. **A comparison of** \( f(L(x, D)) \) **and** \( p(x, D) \) **and of the related semigroups**

We suppose that \( L(x, D) \) and \( f \in CBF_0 \) are as in the above sections. In particular, \( f \) has the representation

\[ f(s) = \int_0^\infty \frac{s}{s + \lambda} \rho(d\lambda) \]
with the measure \( \rho \) satisfying \( \int_1^\infty \frac{1}{\lambda} \rho(d\lambda) < \infty \), see (2.12). Thus, for any continuous negative definite function \( \psi : \mathbb{R}^n \to \mathbb{R} \) we have

\[
(6.1) \quad f(\psi(\xi)) = \int_0^\infty \frac{\psi(\xi)}{\psi(\xi) + \lambda} \rho(d\lambda).
\]

By (2.25), we know that

\[
(6.2) \quad f(L(x,D))u = \int_0^\infty L(x,D)r_\lambda u \rho(d\lambda) = \int_0^\infty r_\lambda L(x,D)u \rho(d\lambda)
\]

holds for all \( u \in D(L(x,D)) = H^2(\mathbb{R}^n) \). For these \( u \) we find

\[
f(L(x,D))u
= \int_0^\infty L(x,D)r_\lambda u \rho(d\lambda)
= \int_0^\infty L(x,D)(\lambda + L(x,D))^{-1}u \rho(d\lambda)
= \int_0^\infty ((\lambda + L(x,D))(\lambda + L(x,D))^{-1}u - \lambda(\lambda + L(x,D))^{-1}u) \rho(d\lambda)
= \int_0^\infty (u - \lambda r_\lambda u) \rho(d\lambda)
= \int_0^\infty (u - \lambda q_\lambda(x,D)u) \rho(d\lambda) + \int_0^\infty \lambda(q_\lambda(x,D)u - r_\lambda u) \rho(d\lambda)
\]

\[
(6.3) \quad = \int_0^\infty (u - \lambda q_\lambda(x,D)u) \rho(d\lambda) + \int_0^\infty \lambda r_\lambda K_\lambda(x,D)u \rho(d\lambda).
\]

The first integral can be explicitly calculated,

\[
\int_0^\infty (u - \lambda q_\lambda(x,D)u) \rho(d\lambda)
= (2\pi)^{-n/2} \int_0^\infty \int_{\mathbb{R}^n} e^{i(x,\xi)} \left( \hat{u}(\xi) - \frac{\lambda}{L(x,\xi) + \lambda} \hat{u}(\xi) \right) d\xi \rho(d\lambda)
= (2\pi)^{-n/2} \int_0^\infty \int_{\mathbb{R}^n} e^{i(x,\xi)} \hat{u}(\xi) \frac{L(x,\xi)}{L(x,\xi) + \lambda} d\xi \rho(d\lambda)
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x,\xi)} \hat{u}(\xi) \int_0^\infty \frac{L(x,\xi)}{L(x,\xi) + \lambda} \rho(d\lambda) d\xi
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x,\xi)} f(L(x,\xi)) \hat{u}(\xi) d\xi
= p(x,D)u(x).
\]

In view of (6.3), we thus have for \( u \in H^2(\mathbb{R}^n) \)

\[
(6.4) \quad f(L(x,D))u = p(x,D)u + \int_0^\infty \lambda r_\lambda K_\lambda(x,D)u \rho(d\lambda).
\]
Since \( f(L(x,D)) \) and \( p(x,D) \) are bounded operators from \( H^{1,a^2} (\mathbb{R}^n) \) into \( L^2 (\mathbb{R}^n) \), the operator \( Q \),

\[
Q u(x) := \int_0^\infty \lambda R_\lambda K_\lambda (x,D) u(x) \rho (d\lambda),
\]
maps \( H^{1,a^2} (\mathbb{R}^n) \) continuously into \( L^2 (\mathbb{R}^n) \). Under some additional assumptions, the operator \( Q \) behaves even better.

**Theorem 6.1.** Let \( L(x,D) \) and \( f \) be as above and assume that for \( 0 \leq \sigma \leq \frac{1}{2} \)

\[
(6.6) \quad \int_1^\infty \frac{1}{\lambda^\sigma} \rho (d\lambda) < \infty.
\]

Then the operator \( Q : H^{2\sigma-1} (\mathbb{R}^n) \to L^2 (\mathbb{R}^n) \) is bounded. In particular, setting \( \sigma = \frac{1}{2} \), we have

\[
(6.7) \quad \| (f(L(x,D)) - p(x,D)) u \|_0 \leq c \| u \|_0.
\]

**Proof.** Choosing \( s = 2\sigma - 1 \) in Corollary 5.10 we find for \( u \in L^2 (\mathbb{R}^n) \)

\[
\left\| \int_0^\infty \lambda R_\lambda K_\lambda (x,D) u \rho (d\lambda) \right\|_0 \leq \int_0^\infty \lambda \| R_\lambda \| \| K_\lambda (x,D) u \|_0 \rho (d\lambda)
\]

\[
\leq \int_0^\infty \frac{\lambda}{\nu_0 + \lambda} \cdot \frac{\lambda}{(\nu_0 + \lambda)^\sigma} \| u \|_{2\sigma-1} \rho (d\lambda)
\]

\[
= c \| u \|_{2\sigma-1}
\]

and the assertion follows by a standard density argument. \( \Box \)

Estimate (6.7) shows that the difference of the operators \( f(L(x,D)) \) and \( p(x,D) \) is of zero order, i.e., the space \( L^2 (\mathbb{R}^n) \) is mapped continuously into itself, while each single operator is only continuous from \( H^{1,a^2} \) into \( L^2 (\mathbb{R}^n) \). Since the difference has better regularity properties we may speak of a lower order perturbation. In particular, we can improve the estimate

\[
(6.8) \quad \| (f(L(x,D)) - p(x,D)) u \|_0 \leq c \| u \|_{1,a^2}.
\]

For any \( \epsilon > 0 \) we find

\[
(6.9) \quad \| (f(L(x,D)) - p(x,D)) u \|_0 \leq \epsilon \| u \|_{1,a^2} + c(\epsilon) \| u \|_0.
\]

Let us now recall Poincaré’s inequality. For all \( u \in H^1 (\mathbb{R}^n) \) with support in a compact set \( K \subset \mathbb{R}^n \)

\[
(6.10) \quad \| u \|_0 \leq g (\text{vol} (K)) \| u \|_1
\]

holds true with \( g(R) \searrow 0 \) as \( R \searrow 0 \). Thus, for all \( u \in H^1 (\mathbb{R}^n) \) having compact support with volume \( \text{vol} (\text{supp} u) \leq R \), Theorem 6.1 implies

\[
(6.11) \quad \| (f(L(x,D)) - p(x,D)) u \|_0 \leq g(R) \| u \|_1.
\]
Let us now apply these estimates to compare the semigroups \( \{ T^f_t \}_{t \geq 0} \) and \( \{ T^{(2)}_t \}_{t \geq 0} \).

In the framework of semigroup theory (6.7) means that the semigroup \( \{ T^f_t \}_{t \geq 0} \) is a bounded perturbation of \( \{ T^{(2)}_t \}_{t \geq 0} \). Clearly, we can interchange the roles played by \( \{ T^f_t \}_{t \geq 0} \) and \( \{ T^{(2)}_t \}_{t \geq 0} \). Therefore, \( \{ T^f_t \}_{t \geq 0} \) can be obtained from \( \{ T^{(2)}_t \}_{t \geq 0} \) and the operator \( Q \) (cf. (6.5)) using the Trotter product formula, see [6] p. 93,

\[
T^f_t u = \lim_{n \to \infty} \left( T^{(2)}_{t/n} S_{t/n} \right)^n u,
\]
where \( \{ S_t \}_{t \geq 0} \) denotes the contraction semigroup \( \{ e^{-tQ} \}_{t \geq 0} \) generated by the bounded operator \( -Q \).

Furthermore, we can control the norm of \( T^f_t u - T^{(2)}_t u \). Following [23] p. 79 we find for \( u \in L^2(\mathbb{R}^n) \)

\[
\begin{align*}
\left\| T^f_t u - T^{(2)}_t u \right\|_0 & \leq \int_0^t \left\| T^f_{t-s} \right\| \left\| f(L(x,D)) - p(x,D) \right\| \left\| T^{(2)}_s \right\| \| u \|_0 ds \\
& \leq t \left\| f(L(x,D)) - p(x,D) \right\| \| u \|_0 \\
& \leq ct \| u \|_0.
\end{align*}
\]

This proves the following theorem:

**Theorem 6.2.** Under the assumptions of Theorem 6.1 we have

\[
\left\| T^f_t - T^{(2)}_t \right\| \leq ct.
\]

From Theorem 6.2 (or (6.13)) we obtain estimates for the transition probabilities

\[
T^f_t 1_B(x) = p^f_t(x,B) \quad \text{and} \quad T^{(2)}_t 1_B(x) = p^{(2)}_t(x,B)
\]

of the corresponding Markov processes

\[
\left( \int_{\mathbb{R}^n} \left| p^f_t(x,B) - p^{(2)}_t(x,B) \right|^2 dx \right)^{1/2} \leq ct \left( \text{vol}(B) \right)^{1/2}
\]

for every bounded Borel set \( B \subset \mathbb{R}^n \).

So far, we have only used (5.25). Using (5.26) and (5.27) we get different estimates. Suppose that \( q \) is so large that (5.26) holds for \( s = k \in 2\mathbb{N}_0 \). For \( u \in H^k(\mathbb{R}^n) \), \( k \in 2\mathbb{N}_0 \), we find because of \( \| L(x,D) u \|_{k-2} \sim \| u \|_k \) - this is due to our smoothness assumptions on the coefficients \( a_{kl} \) and \( c \) - and \( [(L(x,D))^k, R_\lambda] = 0 \) that

\[
\| R_\lambda u \|_k \leq \frac{c}{\nu_0 + \lambda} \| u \|_k
\]
holds with a suitable constant \( c \). By (6.16), (5.26) with \( \sigma = \frac{1}{2} \), and choosing \( k > \frac{n}{2} \) (in order to have Sobolev's embedding theorem at hand) we get

\[
\left\| \int_0^\infty \lambda R_\lambda K_\lambda(x,D) u \rho(d\lambda) \right\|_k \leq c \left\| \int_0^\infty \lambda R_\lambda K_\lambda(x,D) u \rho(d\lambda) \right\|_k
\]
This is already the proof of the next theorem.

**Theorem 6.3.** Suppose that \( q \) is sufficiently large and \( 2N_0 \geq k > \frac{n}{2} \), i.e., such that \( H^k(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n) \) holds. Then we have for all \( u \in H^k(\mathbb{R}^n) \)

\[
\|f(L(x, D))u - p(x, D)u\|_\infty \leq c \|u\|_k.
\]

By Theorem 4.4 and the commutator estimate (4.24) we know that for any admissible \( k \in \mathbb{N}_0 \) the norms \( \|u\|_{k, a^2} \) and \( \|(p(x, D) + \text{id})^k u\|_0 \) are equivalent. Since \( p(x, D) \) generates \( \{T_i^{(2)}\}_{i \geq 0} \), we have \( \left[(p(x, D) + \text{id}), T_i^{(2)}\right] = 0 \) which yields

\[
\left\|T_i^{(2)}u\right\|_{k, a^2} \leq c \|u\|_{k, a^2}.
\]

If \( k > \frac{n}{2r} + 2 \), we have \( H^{k, a^2}(\mathbb{R}^n) \subset H^{\frac{n}{2r}+2, a^2}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n) \), thus, cf. Remark 4.7(iii), for any \( u \in H^{k, a^2}(\mathbb{R}^n) \) and \( t \geq 0 \)

\[
T_i^{(2)}u(x) = T_i^{(\infty)}u(x) \text{ almost everywhere.}
\]

**Theorem 6.4.** Suppose that \( q \) is large enough and \( k > \frac{n}{2r} + 2 \). Then we have

\[
\left\|\tilde{T}_i^f u - T_i^{(\infty)}u\right\|_\infty \leq c t \|u\|_{k, a^2}
\]

for all \( u \in H^{k, a^2}(\mathbb{R}^n) \).

**Proof.** Pick a \( u \in H^{k, a^2}(\mathbb{R}^n) \). Then we find from (6.18) that

\[
T_i^{(2)} \in H^{k, a^2}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n).
\]

Now, Theorem 6.3 shows that \( (f(L(x, D)) - p(x, D))T_i^{(2)}u \in H^{k, a^2}(\mathbb{R}^n) \) and the same argument that gave (6.19) also proves \( \tilde{T}_i^f u(x) = T_i^f u(x) \) almost everywhere for any \( v \in H^{\frac{n}{2r}+2, a^2}(\mathbb{R}^n) \). Thus, for sufficiently large \( k \),

\[
T_i^f (f(L(x, D)) - p(x, D))T_i^{(2)}u = \tilde{T}_i^f (f(L(x, D)) - p(x, D))T_i^{(2)}u
\]

almost everywhere. Using the formula

\[
\tilde{T}_i^f u - T_i^{(\infty)}u = T_i^f u - T_i^{(2)}u = \int_0^t T_{i-s}^f (f(L(x, D)) - p(x, D))T_i^{(2)}u ds
\]
the estimate
\[ \left\| \tilde{T}_t u - T_t^{(\infty)} u \right\|_\infty \]
\[ = \left\| \int_0^t \tilde{T}_{t-s} (f(L(x, D)) - p(x, D)) T_s^{(2)} u \, ds \right\|_\infty \]
\[ \leq \int_0^t \left\| \tilde{T}_{t-s} \right\|_\infty \to \infty \left\| f(L(x, D)) - p(x, D) \right\|_{k, a^2} \to \infty \left\| T_s^{(2)} \right\|_{k, a^2} \to k, a^2 \left\| u \right\|_{k, a^2} \, ds \]
\[ \leq c t \left\| u \right\|_{k, a^2} \]
finally follows. □

References

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Lehrstuhl für Angewandte Mathematik
Technische Universität München
Dachauerstraße 9a
D-80333 München
Germany

Mathematisches Institut
Universität Erlangen
Bismarckstraße 1a
D-91054 Erlangen
Germany