Summary. Starting with a class of regular symmetric Dirichlet forms we determine explicitly the form of the infinitesimal generator. The generator turns out to have representations as integro-differential and as pseudo differential operator, and it is possible to find both the (semimartingale) characteristics and the symbol of the operator. We use then the symbol of the operator to obtain estimates for the mean sojourn time in balls of the associated stochastic process. These estimates, combined with a perturbation argument, enable us to apply the technique of Bass and co-authors to prove the Hölder regularity of the corresponding (perturbed) resolvent and semigroup and to use this to show that the (perturbed and unperturbed) semigroup is a Feller semigroup.

MSC 2000: Primary 31C45; 60J35. Secondary 60J75; 60J25; 60G40; 60G52; 47G20; 47G30.

Key Words: Dirichlet forms; Beurling-Deny formula; generator; pseudo differential operator; integro-differential operator; Feller process; stable-like process; jump measure.

1 Introduction

In his seminal 1971 paper Dirichlet spaces and strong Markov processes [7] M. Fukushima established a one-to-one correspondence between regular (symmetric) Dirichlet forms and symmetric Hunt processes. Ever since Dirichlet forms have been a central tool to study and construct stochastic processes and many authors have developed the theory—see e.g. [37, 8, 4, 12]—and extended it in various directions, e.g. quasi-regular processes and non-symmetric forms [1, 24], time-dependent forms [27, 28] or generalized Dirichlet forms [39]. Many concrete (jump-type) examples were given by Jacob who used pseudo differential
operators to construct stochastic processes and Dirichlet forms, [18] and [19, 21]; the most general results in this direction are to our knowledge due to Hoh [13, 14, 15, 16].

It is not by accident that pseudo differential operators enter the scene. A well-known result by Courrège, cf. [20, 19] for a survey, implies that infinitesimal generators of ‘regular’ Feller semigroups are pseudo differential operators—by ‘regular’ we mean that the test functions $C^\infty_0$ are contained in the domain of the generator. Since many Feller processes are Hunt processes, it is possible to study these processes via Dirichlet forms, see the references above; such Dirichlet forms are, of course, generated by pseudo differential operators. The connection between pseudo differential operators and Dirichlet forms runs, however, deeper. One indication is the Beurling-Deny formula which gives the generator of the form in ‘implicit’ form, meaning that one should—we take the analogy to the diffusion case—perform some kind of ‘integration by parts’. Some sufficient conditions for the generator of a Dirichlet form to be a pseudo differential operator were given in [34].

A drawback of the otherwise very powerful approach via Dirichlet forms is the problem of non-uniqueness in the sense of Theorem 4.2.7 in [12]. That is, any two stochastic processes (associated with the same Dirichlet form) are equivalent if there exists a common properly exceptional set $N$ such that the transition functions coincide outside of $N$. This leaves a question whether one can find a nice representative for the equivalence class of all associated processes which starts at each point in a natural way.

Of course, some conditions, e.g. the (strong) Feller property, are known in order to construct stochastic processes from every starting point in a unique way. One possibility, originally proposed by Malliavin [25] and for Dirichlet forms by Fukushima and Kaneko [11], see also [22, 10, 9], is to use redefinitions and $(r,p)$-capacities. If there exist enough $(r,p)$-quasi continuous functions and if $\text{cap}_{r,p}(A) = 0$ implies for large values of $p$ and $r$ that $A = \emptyset$, then we can choose a Feller representative. In [5, 6] this was worked out using characterisations of $\mathcal{D}(A)$ in terms of concrete Bessel-potential spaces for which Sobolev-type embeddings are available.

A more direct approach is possible due to the work of Bass and Levin. In [3] they obtained a Harnack inequality for a non-local pure jump type integral operator on $\mathbb{R}^n$ and showed that the corresponding harmonic functions are Hölder continuous. Note that they assumed that there exists a strong Markov process associated with the operator as infinitesimal operator and that the process solves the martingale problem for each starting point in $\mathbb{R}^n$. In [38], Song and Vondraček extended the work of Bass and co-authors ([3, 2]) to a larger class of Markov processes.

In the present paper we start with a symmetric regular Dirichlet form. From the general theory of Dirichlet forms we know that in this situation there is a stochastic ‘Dirichlet’ process associated with the form—but it is unique only up to a set of capacity
zero, say, $N$. To overcome this we show that the method of Bass et al. is applicable at all points outside $N$. In particular, we show that the Harnack inequality and the Hölder continuity of the harmonic functions associated with the Dirichlet process hold outside the exceptional set. This allows us to show that the resolvent and the semigroup of the Dirichlet process can be modified to a Feller resolvent and semigroup. Since the Dirichlet process admits a modification as a Feller process, Courrège’s theorem implies that process and form were generated by a pseudo differential operator in the first place. (The regularity of the Dirichlet form should guarantee that the domain of the generator is sufficiently rich.)

It is therefore natural to consider the generator and to search for an explicit formula for its symbol; obviously an ‘integration by parts’ in the Beurling-Deny representation of the form should do. This is done in §§1,2. Having calculated the generator we need to get the method developed by Bass et al. to work; this requires further properties of the symbol and, in particular, estimates for the mean sojourn time of the process for small balls. We use a new method using the symbol of the generator to derive such estimates, cf. §3. This section is based on results from the paper [33] which was written for Feller processes. As a matter of fact, this paper uses Feller processes only to guarantee that the infinitesimal generator is a pseudo differential operator; all other calculations only need this particular form of the generator and strong Markovianity. Since we are in a situation where the generator is a pseudo differential operator and the process is a Hunt process, we can use the results without changes. The proof of the Harnack inequality and the Hölder estimates is modeled on the papers [3, 2] and [38]; since we need some refinements and alterations of some steps we decided to give full details of all critical steps. This should also enhance the readability of our paper. The principal innovation is that we use a perturbation result—the abstract version is contained in §4 and §7—which allows us to pose only assumptions on the small-jump part and to choose the large-jump part as convenient as necessary. Since we are more interested in the method and do not strive for greatest possible generality in the present paper, we only work out bounded perturbations, more general schemes are outlined in §7.

Interestingly, our paper is also a contribution to the question if it is possible to construct a Feller process if a certain type of symbol is given. This problem is discussed in Jacob [19, 20, 21], and the most general solutions up to now need (at least) some differentiability of the symbol. Since we start with Dirichlet forms, we do not require differentiability of the symbol at all.

**Dedication.** We dedicate this paper to Professor M. Fukushima on the occasion of his 70th birthday. We are grateful for his constant encouragement and his continuing interest in our work.
**Notation.** We are switching between pseudo differential operators and their symbols and integro-differential operators and their Lévy/semimartingale characteristics which makes the paper quite technical. To make things easier for the readers we list here some frequently used assumptions and where to find them in the paper:

(A), (B), (C),—cf. p. 6 (Theorem 2.2);
(A'), (B'), (C'),—cf. p. 11 (before Corollary 2.4),
(D'),—cf. p. 17 (Lemma 3.5);
(10), (10'), (11), (11'), (12),—cf. p. 12 (Example 2.7).

As usual, we write $a \vee b$ and $a \wedge b$ for the maximum and minimum of $a, b \in \mathbb{R}$. Unless otherwise indicated, $L^p(\mathbb{R}^n)$ means the usual $L^p$ space with respect to $n$-dimensional Lebesgue measure $dx$; $C^k_0$ (resp. $C^k_0$) denote the $k$ times continuously differentiable functions which are bounded with all their derivatives (resp. with compact support) and by $C_\infty$ we mean the continuous functions vanishing at infinity. We also use the Landau symbols $g(x) = O(f(x)), x \to a$ or $h(x) = o(f(x)), x \to a$ to indicate that $|g(x)/f(x)|$ stays bounded as $x \to a$ or that $\lim_{x \to a} h(x)/f(x) = 0$. In the second part of the paper we use $\tau_U$ for the first entrance time into a set $U$ and $\sigma_V$ for the sojourn or first exit time from a set $V$:

$$\tau_U := \inf\{t \geq 0 : X_t \in U\}, \quad \sigma_V := \inf\{t \geq 0 : X_t \notin V\}, \quad \text{and} \quad \sigma^x := \sigma_{B_r(x)}.$$

All other notation should be standard or self-explanatory.

## 2 Generator and symbol of a jump-type Dirichlet form

Throughout this paper we will always consider quadratic forms of the following type:

$$\mathcal{E}(u, v) := \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) n(x, y) \, dx \, dy$$

(1)

$$\mathcal{D}(\mathcal{E}) := \{ u \in L^2(\mathbb{R}^n) : \mathcal{E}(u, u) < \infty \}$$

where $n(x, y)$ is a positive measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Denote by $C^0_0(\mathbb{R}^n)$ the set of all uniformly Lipschitz continuous functions on $\mathbb{R}^n$ with compact support. In [41], see also [40], the second-named author considered ‘stable-like’ forms $\mathcal{E}(\cdot, \cdot)$ where $n(x, y) = |x - y|^{-\alpha(x,y)-n}$ and gave conditions on the exponent $\alpha$ which turn $\mathcal{E}(\cdot, \cdot)$ into a Dirichlet form. Since

$$n(x, y) = |x - y|^{-\alpha(x,y)-n} \iff \alpha(x, y) = -\frac{\log n(x, y)}{\log |x - y|} - n,$$
these results extend to all quadratic forms of type (1). One of the conditions in [41] is that \( \{ \alpha(x, y) = \pm \infty \} \subset \mathbb{R}^{2n} \) should be a Lebesgue null set. It is not hard to see that this condition can be weakened to the requirement that \( \{(x, y) : |x - y| < 1, \alpha(x, y) = -\infty \} \subset \mathbb{R}^{2n} \) and \( \{(x, y) : |x - y| > 1, \alpha(x, y) = +\infty \} \subset \mathbb{R}^{2n} \) should be Lebesgue null sets. Taking this into account we can rephrase Theorems 2.1 and 2.2 of [41] in the following way:

Theorem 2.1. Let \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) be as above.

(i) If the set \(\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : n(x, y) = +\infty \}\) is a Lebesgue null set, then \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is a Dirichlet form on \(L^2(\mathbb{R}^n)\) in the wide sense (i.e., \(\mathcal{E}\) satisfies all requirements for a Dirichlet form except that \(\mathcal{D}(\mathcal{E})\) might be not dense in \(L^2(\mathbb{R}^n)\)).

(ii) The domain \(\mathcal{D}(\mathcal{E})\) contains the set \(C^{0,1}_0(\mathbb{R}^n)\) if, and only if, for all compact sets \(K\) and all relatively compact open sets \(G \supset K\) the following condition is satisfied:

\[
\iint_{K \times K} |x - y|^2 (n(x, y) + n(y, x)) \, dx \, dy + \iint_{K \times G^c} (n(x, y) + n(y, x)) \, dx \, dy < \infty.
\]

For \(\mathcal{F} := \overline{C^{0,1}_0(\mathbb{R}^n)}^{\mathcal{E}(\bullet) + \|\bullet\|_{L^2}}\), i.e. the closure of \(C^{0,1}_0(\mathbb{R}^n)\) under \(\mathcal{E}(\bullet, \bullet) + \langle \bullet, \bullet \rangle_{L^2}\), the form \((\mathcal{E}, \mathcal{F})\) becomes a regular (symmetric) Dirichlet form.

Throughout this paper we will assume that \((\mathcal{E}, \mathcal{F})\) satisfies the conditions of Theorem 2.1

According to the general theory of Dirichlet forms, see the monographs of Fukushima [8], Fukushima, Oshima and Takeda [12] or Ma and Röckner [24], we can associate with every regular Dirichlet form a symmetric Hunt process \(M = (X_t, \mathbb{P}_x)\). Note that the family of probability measures \(\mathbb{P}_x\) is uniquely determined only up to a capacity-zero set \(N\) of starting points \(x\).

All regular Dirichlet forms can be written in terms of their Beurling-Deny decomposition, see [12, Theorem 3.2.1, Lemma 4.5.4]. In our situation it is easy to see that

\[
\mathcal{E}(u, v) = \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) \, J(dx, dy)
\]

holds for all \(u, v \in C^{0,1}_0(\mathbb{R}^n)\) or, more generally, for all quasi-continuous (modifications of) \(u, v\) from \(\mathcal{F}\). The symmetric jump measure \(J(dx, dy)\) is given by

\[
J(dx, dy) = j(x, y) \, dx \, dy \quad \text{with} \quad j(x, y) = n(x, y) + n(y, x).
\]

Intuitively, see [12, Theorem 4.5.2] for a precise statement, \(j(x, y)\) is the rate at which the paths of the associated Hunt process jump from the current position \(X_{t-} = x\) to the
point $X_t = y \neq x$. It is sometimes helpful not to look at the rate for the new position but at the rate of the jump size $X_t - X_{t-} = x - y =: h$. Doing so, we get

$$E(u, v) = \frac{1}{2} \int \int_{h \neq 0} (u(x + h) - u(x))(v(x + h) - v(x)) j(x, x + h) \, dx \, dh,$$

and the conditions of Theorem 2.1 become for some (hence, all) $\epsilon > 0$

(A) \[ \int_{|h| \leq \epsilon} |h|^2 j(\bullet, \bullet + h) \, dh \in L^1_{\text{loc}}(\mathbb{R}^n), \]

(B) \[ \int_{|h| \geq \epsilon} j(\bullet, \bullet + h) \, dh \in L^1_{\text{loc}}(\mathbb{R}^n). \]

Another way to describe Dirichlet forms is through their $L^2$-generator $(A, D(A))$. The connection between form and generator is given by

$$E(u, v) = -\langle u, Av \rangle_{L^2}, \quad u \in \mathcal{F}, \ v \in D(A).$$

For (jump-type) Dirichlet forms it is, in general, difficult to find a closed expression for $A$ if only the form is known. In the present situation this is, however, possible if we make some more assumptions on the jump density $j(x, y)$.

We call a rotationally invariant, measurable function $\chi : \mathbb{R}^n \rightarrow [0,1]$ a centering function if $\chi(h)$ decays for $|h| \rightarrow \infty$ at least as fast as $|h|^{-1}$ and if $\lim_{h \rightarrow 0} |\chi(h)h - h|/|h|^2 = 0$. Typical examples are $\chi(h) = (1 + |h|^2)^{-1}$ and $\chi(h) = 1_{B_1(0)}(h)$.

**Theorem 2.2.** Let $(E, D(E))$ be given by (1) and assume that conditions (A) and (B) hold. Moreover we assume that the jump density $j(x, y)$ satisfies

(C) \[ \int_{|h| \leq \epsilon} |h||j(x, x + h) - j(x, x - h)| \, dh < \infty. \]

Then $E(u, \phi) = -\langle u, A\phi \rangle_{L^2}$ for all $u, \phi \in C_0^2(\mathbb{R}^n)$ with the operator $A$ given by

$$A\phi(x) = \int_{h \neq 0} \left( \phi(x + h) - \phi(x) - \chi(h)h \cdot \nabla \phi(x) \right) j(x, x + h) \, dh$$

$$+ \frac{1}{2} \int_{h \neq 0} \chi(h)h( j(x, x + h) - j(x, x - h) ) \, dh \cdot \nabla \phi(x)$$

$$= \text{p.v.} \int \left( \phi(x + h) - \phi(x) \right) j(x, x + h) \, dh.$$

Here $\chi(h)$ denotes a compactly supported centering function and p.v. $\int \ldots \, dh$ means the Cauchy principal value integral.
Proof. Let $u, \phi \in C_0^2(\mathbb{R}^n)$ and fix some centering function $\chi(h)$. Since $u(x) - u(y), \phi(x) - \phi(y)$ are of order $O(1)$ as $|x - y| \to \infty$ and $O(|x - y|)$ as $|x - y| \to 0$, we see that

$$E(u, \phi) = \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))((\phi(x) - \phi(y)) j(x, y) \, dx \, dy = \frac{1}{2} \iint_{h \neq 0} (u(x + h) - u(x))((\phi(x + h) - \phi(x)) j(x, x + h) \, dx \, dh$$

is well-defined. By the Taylor formula we see that the expressions

$$\phi(x + h) - \phi(x) - \chi(h)h \cdot \nabla \phi(x + h) \quad \text{and} \quad \phi(x + h) - \phi(x) - \chi(h)h \cdot \nabla \phi(x)$$

behave like $O(1)$ for large $|h|$ and like $O(|h|^2)$ for $|h| \to 0$. Because of the conditions (A), (B), this enables us to rewrite $E(u, \phi)$ in the following form

$$E(u, \phi) = \frac{1}{4} \iint_{h \neq 0} (u(x + h) - u(x))(\phi(x + h) - \phi(x) - \chi(h)h \nabla \phi(x + h)) \, j(x, x + h) \, dx \, dh + \frac{1}{4} \iint_{h \neq 0} (u(x + h) - u(x))(\phi(x + h) - \phi(x) - \chi(h)h \nabla \phi(x)) \, j(x, x + h) \, dx \, dh + \frac{1}{4} \iint_{h \neq 0} (u(x + h) - u(x))\chi(h)h(\nabla \phi(x + h) + \nabla \phi(x)) \, j(x, x + h) \, dx \, dh$$

$$= \frac{1}{4} \iint_{h \neq 0} u(x + h)(\phi(x + h) - \phi(x) - \chi(h)h \nabla \phi(x + h)) \, j(x, x + h) \, dx \, dh - \frac{1}{4} \iint_{h \neq 0} u(x)(\phi(x + h) - \phi(x) - \chi(h)h \nabla \phi(x + h)) \, j(x, x + h) \, dx \, dh + \frac{1}{4} \iint_{h \neq 0} u(x + h)(\phi(x + h) - \phi(x) - \chi(h)h \nabla \phi(x)) \, j(x, x + h) \, dx \, dh - \frac{1}{4} \iint_{h \neq 0} u(x)(\phi(x + h) - \phi(x) - \chi(h)h \nabla \phi(x)) \, j(x, x + h) \, dx \, dh + \frac{1}{4} \iint_{h \neq 0} (u(x + h) - u(x))\chi(h)h(\nabla \phi(x + h) + \nabla \phi(x)) \, j(x, x + h) \, dx \, dh.$$

Now we change variables in the first and third double integrals according to $x \rightsquigarrow x - h$ and then $h \rightsquigarrow -h$ and we use the symmetry of the jump density $j(x, y) = j(y, x)$; after that we expand the second and third double integrals by inserting an ‘artificial zero’,
\[ \nabla \phi(x) - \nabla \phi(x), \text{ and arrive at} \]
\[ \mathcal{E}(u, \phi) = \frac{1}{4} \int \int_{h \neq 0} u(x) \left( \phi(x) - \phi(x + h) + \chi(h) h \nabla \phi(x) \right) j(x, x + h) \, dx \, dh \]
\[ - \frac{1}{4} \int \int_{h \neq 0} u(x) \left( \phi(x + h) - \phi(x) - \chi(h) h \nabla \phi(x) \right) j(x, x + h) \, dx \, dh \]
\[ - \frac{1}{4} \int \int_{h \neq 0} u(x) \chi(h) h \left( \nabla \phi(x) - \nabla \phi(x + h) \right) j(x, x + h) \, dx \, dh \]
\[ + \frac{1}{4} \int \int_{h \neq 0} u(x) \left( \phi(x + h) - \phi(x) - \chi(h) h \nabla \phi(x) \right) j(x, x + h) \, dx \, dh \]
\[ - \frac{1}{4} \int \int_{h \neq 0} u(x) \chi(h) h \left( \nabla \phi(x) - \nabla \phi(x + h) \right) j(x, x + h) \, dx \, dh \]
\[ + \frac{1}{4} \int \int_{h \neq 0} (u(x + h) - u(x)) \chi(h) h \left( \nabla \phi(x + h) + \nabla \phi(x) \right) j(x, x + h) \, dx \, dh. \]

Note that all integrals converge as \( \nabla \phi(x) - \nabla \phi(x + h) \) is \( O(1) \) resp. \( O(|h|) \) as \( |h| \to \infty \) resp. \( |h| \to 0 \). Thus,
\[ \mathcal{E}(u, \phi) = - \int \int_{h \neq 0} u(x) \left( \phi(x + h) - \phi(x) - \chi(h) h \nabla \phi(x) \right) j(x, x + h) \, dx \, dh \]
\[ - \frac{1}{2} \int \int_{h \neq 0} u(x) \chi(h) h \left( \nabla \phi(x) - \nabla \phi(x + h) \right) j(x, x + h) \, dx \, dh \]
\[ + \frac{1}{4} \int \int_{h \neq 0} (u(x + h) - u(x)) \chi(h) h \left( \nabla \phi(x + h) + \nabla \phi(x) \right) j(x, x + h) \, dx \, dh \]
\[ =: - I_1 - \frac{1}{2} I_2 + \frac{1}{4} I_3. \]

Again, changing variables according to \( x \leftrightarrow x - h \) and \( h \leftrightarrow -h \), and using the symmetry of \( j(x, y) \) leads to
\[ I_2 = - \int \int_{h \neq 0} u(x + h) \chi(h) h \left( \nabla \phi(x + h) - \nabla \phi(x) \right) j(x, x + h) \, dx \, dh. \]

Averaging this and the original representation of \( I_2 \) yields
\[ I_2 = \frac{1}{2} \int \int_{h \neq 0} (u(x) + u(x + h)) \chi(h) h \left( \nabla \phi(x) - \nabla \phi(x + h) \right) j(x, x + h) \, dx \, dh. \]
Therefore, we have
\[
\frac{1}{4} I_3 - \frac{1}{2} I_2 = \frac{1}{4} \iint_{h \neq 0} \left[(u(x+h) - u(x))\chi(h)h(\nabla \phi(x+h) + \nabla \phi(x)) \right.
\]
\[
- (u(x) + u(x+h))\chi(h)(\nabla \phi(x) - \nabla \phi(x+h)) \right] j(x, x+h) \, dx \, dh
\]
\[
= \frac{1}{2} \text{p.v.} \iint \chi(h)((u(x+h)h \cdot \nabla \phi(x+h) - u(x)h \cdot \nabla \phi(x)) j(x, x+h) \, dx \, dh
\]
with the principal value integral \(\text{p.v.} \iint \ldots dx \, dh := \lim_{\epsilon \to 0} \iint_{|h| > \epsilon} \ldots dx \, dh\). Since for all \(\epsilon > 0\)
\[
\iint_{|h| > \epsilon} u(x+h)\chi(h)h \cdot \nabla \phi(x+h) j(x, x+h) \, dh \, dx = \iint_{|h| > \epsilon} u(x)\chi(h)h \cdot \nabla \phi(x) j(x, x-h) \, dh \, dx,
\]
we see that
\[
\frac{1}{4} I_3 - \frac{1}{2} I_2 = \frac{1}{2} \text{p.v.} \iint u(x)\chi(h)h \cdot \nabla \phi(x) (j(x, x-h) - j(x, x+h)) \, dh \, dx;
\]
because of (C) the above principal value integral is absolutely convergent.

Piecing things together we obtain for all \(u, \phi \in C_0^2(\mathbb{R}^n)\) and a fixed centering function \(\chi(h)\) the following formula
\[
\mathcal{E}(u, \phi) = - \iint_{h \neq 0} u(x)(\phi(x+h) - \phi(x) - \chi(h)h \cdot \nabla \phi(x)) j(x, x+h) \, dh \, dx
\]
\[
- \frac{1}{2} \iint_{h \neq 0} u(x)\chi(h)h \cdot \nabla \phi(x) (j(x, x+h) - j(x, x-h)) \, dh \, dx.
\]
Note also that \(A\) does not depend on the centering function \(\chi\).

**Corollary 2.3.** Let \((\mathcal{E}, \mathcal{F})\) and \(A\) be as in Theorem 2.2. Then \(A\) can be extended to the bounded and twice differentiable functions \(C_0^2(\mathbb{R}^n)\). For \(\phi \in C_0^2(\mathbb{R}^n)\) we have
\[
|A\phi(x)| \leq C \left[ \int_{h \neq 0} \frac{|h|^2}{1 + |h|^2} j(x, x+h)dh + \int |h|\chi(h)|j(x, x+h) - j(x, x-h)|dh \right] \sum_{|\alpha| \leq 2} \|\partial^\alpha \phi\|_\infty
\]
for all \(x \in \mathbb{R}^n\). Moreover, \(A|_{C_0^\infty(\mathbb{R}^n)}\) is a pseudo differential operator
\[
A\phi(x) = -p(x, D)\phi(x) = (2\pi)^{-n/2} \int p(x, \xi)\hat{\phi}(\xi)e^{ix\xi} \, d\xi
\]
\( \hat{\phi}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \phi(x) \, dx \) denotes the Fourier transform. With negative definite symbol \( p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C} \) which is given by the Lévy-Khinchine-type formula

\[
(6) \quad p(x, \xi) = \int_{h \neq 0} (1 - \cos h \xi) j(x, x + h) \, dh - \frac{1}{2} \int_{h \neq 0} i\sin \xi h (j(x, x + h) - j(x, x - h)) \, dh
\]

(7) \quad = \text{p.v.} \int (1 - e^{i\xi h}) j(x, x + h) \, dh.

**Proof.** Once we have shown that \( A \) can be extended to all \( C_0^2 \)-functions, we may substitute \( \phi(x) \) in (2) for \( e_\xi(x) = e^{i\xi x} \). With some routine calculations—see e.g. Jacob [20, 21]—we then see that \( p(x, \xi) = e^{-i\xi \cdot A \xi(x)} \) and that \( p(x, \xi) \) is given by

\[
p(x, \xi) = \int_{h \neq 0} (1 - e^{ih\xi} + i\xi h \chi(h)) j(x, x + h) \, dh - \frac{1}{2} \int_{h \neq 0} i\xi h \chi(h) (j(x, x + h) - j(x, x - h)) \, dh.
\]

Observe now that the functions \( 1 - \cos h \xi \) and \( \chi(h)h\xi - \sin h \xi \) are bounded for large \( |h| \) and behave like \( O(|h|^2) \) as \( h \to 0 \). Therefore we find

\[
\text{Re} \, p(x, \xi) = \int_{h \neq 0} (1 - \cos h \xi) j(x, x + h) \, dh
\]

\[
\text{Im} \, p(x, \xi) = \int_{h \neq 0} (\xi h \chi(h) - \sin h \xi) j(x, x + h) \, dh - \frac{1}{2} \int_{h \neq 0} \xi h \chi(h) (j(x, x + h) - j(x, x - h)) \, dh.
\]

Since \( h \mapsto \xi h \chi(h) - \sin h \xi \) is an odd function, the change of variables \( h \mapsto -h \) gives

\[
\int_{h \neq 0} (\xi h \chi(h) - \sin h \xi) j(x, x + h) \, dh = - \int_{h \neq 0} (\xi h \chi(h) - \sin h \xi) j(x, x - h) \, dh
\]

and averaging over these two terms yields

\[
\text{Im} \, p(x, \xi) = \frac{1}{2} \int_{h \neq 0} (\xi h \chi(h) - \sin h \xi) (j(x, x + h) - j(x, x - h)) \, dh
\]

\[
- \frac{1}{2} \int_{h \neq 0} \xi h \chi(h) (j(x, x + h) - j(x, x - h)) \, dh
\]

which shows (6).

To see that \( A \) extends to \( C_0^2(\mathbb{R}^n) \) it is clearly enough to prove (4). Using Taylor’s formula we get for \( \phi \in C_0^2(\mathbb{R}^n) \)

\[
| \phi(x + h) - \phi(x) - \chi(h)h\nabla \phi(x) | \leq c(\chi) \frac{|h|^2}{1 + |h|^2} \sum_{|\alpha| \leq 2} \| \partial^\alpha \phi \|_\infty.
\]

The estimate (5) follows now immediately from the representation (2) of the operator \( A \).
In order to identify \( A|_{C_0^2(\mathbb{R}^n)} \) as the (restriction of the \( L^2 \)-) generator of the Dirichlet form \((\mathcal{E}, \mathcal{F})\) we have to show that \( A \) maps \( C_0^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \). For this we have to replace the conditions (A)-(C) by the following uniform versions

\[(A') \quad \sup_x \int_{|h| \leq \epsilon} |h|^2 j(x, x+h) \, dh < \infty,\]

\[(B') \quad \sup_x \int_{|h| \geq \epsilon} j(x, x+h) \, dh < \infty,\]

\[(C') \quad \sup_x \int_{|h| \leq \epsilon} |h||j(x, x+h) - j(x, x-h)| \, dh < \infty.\]

**Corollary 2.4.** Let \((\mathcal{E}, \mathcal{F})\) and \( A \) be as in Theorem 2.2. If \((A')-(C')\) hold, then the operator \( A \) has bounded coefficients in the sense that there exist constants \( C, K > 0 \) such that

\[(8) \quad \|A\phi\|_\infty \leq C \sum_{|\alpha| \leq 2} \|\partial^\alpha \phi\|_\infty, \quad \phi \in C_0^2(\mathbb{R}^n),\]

and for and its symbol \(-p(x, \xi)\)

\[(9) \quad \sup_x |p(x, \xi)| \leq K (1 + |\xi|^2), \quad \xi \in \mathbb{R}^n.\]

**Proof.** The assumptions guarantee that all integrals appearing in the estimate (4) of \( A\phi(x) \) converge uniformly for all \( x \). This proves (8). The second inequality follows immediately from (9) since the symbol \(-p(x, \xi) = e^{-ix\xi}Ae^{ix\xi}(x), e_{\xi}(x) := e^{ix\xi}, \) see the proof of Corollary 2.3.

**Proposition 2.5.** Let \((\mathcal{E}, \mathcal{F})\) and \( A \) be as in Theorem 2.2. If \((A')-(C')\) hold, then \( A \) maps \( C_0^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \). In particular, \( A \) coincides on \( C_0^2(\mathbb{R}^n) \) with the generator \((A, \mathcal{D}(A))\) of the Dirichlet form \((\mathcal{E}, \mathcal{F})\) and \( C_0^2(\mathbb{R}^n) \subset \mathcal{D}(A) \).

**Proof.** Pick some \( \phi \in C_0^2(\mathbb{R}^n) \) and choose \( r > 0 \) so large that \( \text{supp} \phi \subset B_r(0) \). Because of (8) we have

\[\|A\phi\|_{L^2} \leq \|1_{B_{2r}(0)}A\phi\|_{L^2} + \|1_{B_{2r}(0)}A\phi\|_{L^2} \leq C \sqrt{\lambda^n(B_{2r}(0))} \sum_{|\alpha| \leq 2} \|\partial^\alpha \phi\|_\infty + \|1_{B_{2r}(0)}A\phi\|_{L^2}.\]

To see the finiteness of the second member we observe that (2) reduces for \(|x| \geq 2r\) to

\[1_{B_{2r}(0)}(x)A\phi(x) = \int_{h \neq 0} \phi(x+h)j(x, x+h) \, dh,\]
and since for $|h| \leq r$ and $|x| \geq 2r$ we have $|x + h| \geq |x| - |h| \geq r$ we conclude that

$$\|1_{B_{2r}(0)}A\phi\|_{L^2}^2 = \left(\int_{|x| \geq 2r} \left(\int_{|h| > r} \phi(x + h)j(x, x + h) \, dh\right)\right)^2 \, dx$$

$$\leq \|\phi\|_{L^2}^2 \int_{|x| \geq 2r} \left(\int_{|h| > r} \left[1_{B_r(0)}(x + h)j(x, x + h) \right] \, dh\right)^2 \, dx$$

$$\leq \|\phi\|_{L^2}^2 \sup_{x \in \mathbb{R}^n} \left[\int_{|h| > r} j(x, x + h) \, dh\right]^2 \left(\int_{|h| > r} j(y, y + h) \, dy\right)$$

For the last estimate we used the Cauchy-Schwarz inequality for the inner integral. We now interchange the order of integration and change variables according to $x \sim y - h$. Since $j(x, z) = j(z, x)$ we arrive at

$$\|1_{B_{2r}(0)}A\phi\|_{L^2}^2 \leq \|\phi\|_{L^2}^2 \sup_{x \in \mathbb{R}^n} \left[\int_{|h| > r} j(x, x + h) \, dh\right] \left(\int_{|h| > r} j(y, y + h) \, dy\right)$$

which is finite if we assume (A'–C'). Since $A(C^2_0(\mathbb{R}^n)) \subset L^2(\mathbb{R}^n)$ we conclude from Theorem 2.2 that $A$ coincides on $C^2_0(\mathbb{R}^n)$ with the generator of the Dirichlet form and that $C^2_0(\mathbb{R}^n) \subset D(A)$. 

**Remark 2.6.** Using the much finer estimates which were developed in [41] it is possible to prove $A(C^2_0(\mathbb{R}^n)) \subset L^2(\mathbb{R}^n)$ under the following weaker assumptions: the functions

$$\int_{|h| \leq \epsilon} |h|^2 j(x, x + h) \, dh, \quad \int_{|h| \geq \epsilon} j(x, x + h) \, dh, \quad \int_{|h| \leq \epsilon} |h| \left| j(x, x + h) - j(x, x - h) \right| \, dh$$

are in $L^2_{\text{loc}}(\mathbb{R}^n)$ and $\left\{x \mapsto \int_{|h| > \epsilon} 1_{B_{2r+1}(0)}(x + h)j(x, x + h) \, dh\right\} \in L^2(\mathbb{R}^n)$.

**Example 2.7.** Here is a simple condition—partly inspired by the form of the jump density for stable processes—that guarantees (A') and (B'): if there exist exponents $-\infty < \alpha \leq \beta < 2$ and constants $c, C > 0$ such that

$$\frac{c}{|x - y|^\alpha} \leq j(x, y) \leq \frac{C}{|x - y|^\beta} \quad \text{for all} \quad |x - y| \leq 1$$

then (A') holds; (B') is true, if we have for $0 < \gamma \leq \infty$ ($\gamma = \infty$ means that $j(x, y)$ vanishes if $|x - y| > 1$) and $K > 0$

$$0 \leq j(x, y) \leq \frac{K}{|x - y|^\gamma} \quad \text{for all} \quad |x - y| > 1.$$
The straightforward proof is left to the reader.

If we write $j(x, y)$ in the form $|x - y|^{-\alpha(x,y) - \gamma} + |x - y|^{-\alpha(y,x) - \gamma}$ (as, e.g., in [41]), then (10), (11) are essentially equivalent to the conditions

(10') $-\infty < \alpha \leq \alpha(x, y) \leq \beta < 2$ for all $|x - y| \leq 1$

(11') $0 < \gamma \leq \alpha(x, y) \leq \infty$ for all $|x - y| > 1$.

For (C) resp. (C') we have to make the additional assumption that $\alpha(x, y)$ is Lipschitz continuous for all $x, y$ from any compact set $K \subset \mathbb{R}^n$ (resp. globally Lipschitz), i.e., that

(12)

$$|\alpha(x, y) - \alpha(x, z)| + |\alpha(y, x) - \alpha(z, x)| \leq C_K |y - z|, \quad x, y, z \in K.$$ (and that, for (C'), the constants $C_K$ are uniformly bounded). If this is the case, we get for $x \in K$ and $|h| < 1$

$$|j(x, x + h) - j(x, x - h)| = |h|^{-n} |h|^{-\alpha(x,x+\theta)} + |h|^{-\alpha(x,x-\theta)} - |h|^{-\alpha(x,x-h)} - |h|^{-\alpha(x,x+h)}|$$

$$\leq |h|^{-n} (|h|^{-\alpha(x,x+\theta)} - |h|^{-\alpha(x,x-h)} + |h|^{-\alpha(x,x,h)} - |h|^{-\alpha(x,x+h)}).$$

The elementary formula $|t^{-\alpha} - t^{-\beta}| = \int_a^b |t^{-u} \log t| \, du$ shows for $|h| < 1$ that

$$|h|^{-\alpha(x,x+h)} - |h|^{-\alpha(x,x-h)}| = \int_{\alpha(x,x-h)}^{\alpha(x,x+h)} |h|^{-u} \log |h| \, du$$

$$\leq |\alpha(x, x + h) - \alpha(x, x - h)| \log \frac{1}{|h|} |h|^{-\max(\alpha(x,x+h),\alpha(x,x-h))}$$

$$\leq c_K \log \frac{1}{|h|} |h|^{1-\beta}.$$ Thus,

$$\int_{|h|<1} |h| |j(x, x + h) - j(x, x - h)| \, dh \leq c_K \int_{|h|<1} |h|^{2-\beta-n} \log \frac{1}{|h|} \, dh < \infty$$

uniformly for $x \in K$ (resp. uniformly in $x \in \mathbb{R}^n$).

Later on, we will use that $\beta < 2$ also guarantees that

$$\int_{|h|<1} |h| \log \frac{1}{|h|} |j(x, x + h) - j(x, x - h)| \, dh < \infty.$$ 

Exact knowledge of the generator, in particular, the fact that $A$ is a pseudo differential operator with symbol $-p(x, \xi)$, makes simple proofs of (global) properties of the process possible. For Feller processes this technique is by now well-known, see the survey in [19] and [20, 21]. As a matter of fact, most proofs only used the existence of the symbol and the (strong) Markov property of the underlying process. This explains why the proof of conservativeness from [31, 32] is easily adapted to the present situation. It should be noted that this is the ‘symbolic’ version of Oshima’s conservativeness criterion [26], see also [17].
Proposition 2.8. Let \((\mathcal{E}, \mathcal{F})\) be as in Theorem 2.2 and assume that the conditions \((A')\)–\((C')\) hold. Then the Dirichlet form \((\mathcal{E}, \mathcal{F})\) is conservative in the sense that for all \(t > 0\)

\[ T_t 1 = 1 \quad \text{Lebesgue a.e.,} \]

where \((T_t)_{t \geq 0}\) is the \(L^2\)-semigroup associated with the form \((\mathcal{E}, \mathcal{F})\).

Proof. As we have seen in Corollaries 2.3 and 2.4, the generator \(A\) of the Dirichlet form is a pseudo differential operator with symbol \(-p(x, \xi)\) which has bounded coefficients, i.e., \(|p(x, \xi)| \leq c(1 + |\xi|^2)\).

Since \((X_t)_{t \geq 0}\) is a Hunt process, we know that for (Lebesgue) almost all starting points \(x\) and all \(\phi \in D(A)\)

\[ T_t \phi(x) - \phi(x) = \mathbb{E}^x(\phi(X_t)) - \phi(x) = \int_0^t \mathbb{E}^x(A\phi(X_s)) \, ds = \int_0^t T_s A\phi(x) \, ds. \]

Pick \(\phi \in C_0^\infty(\mathbb{R}^n) \subset D(A)\) satisfying \(0 \leq \phi \leq 1\), \(\phi(0) = 1\), and define \(\phi_k(x) := \phi(x/k)\). Noting that \(\phi_k(x) \xrightarrow{k \to \infty} 1\) and that for the Fourier transform \(\hat{\phi}_k(\xi) = k^n \hat{\phi}(k\xi)\), we see

\[
|A\phi_k(x)| = |p(x, D)\phi_k(x)| = \left| (2\pi)^{-n/2} \int p(x, \xi) \hat{\phi}_k(\xi) e^{ix\xi} \, d\xi \right| \\
\leq (2\pi)^{-n/2} \int |p(x, \xi)| |\hat{\phi}(\xi)| e^{ix\xi/k} \, d\xi \\
\leq c (2\pi)^{-n/2} \int (1 + |\xi|^2) |\hat{\phi}(\xi)| \, d\xi \\
\leq c (2\pi)^{-n/2} \int (1 + |\xi|^2) |\hat{\phi}(\xi)| \, d\xi. 
\]

The last integral converges absolutely since \(\hat{\phi}\) is a rapidly decreasing Schwartz function. Since this estimate is uniform in \(k \in \mathbb{N}\), we can use dominated convergence in (13), (14), and conclude that

\[ A\phi_k \xrightarrow{k \to \infty} 0, \quad \text{and} \quad \sup_k \|A\phi_k\|_\infty < \infty. \]

Therefore, another application of the dominated convergence theorem shows that for almost all \(x\)

\[ |1 - T_t 1| = \lim_{k \to \infty} |\phi_k - T_t \phi_k| \leq \lim_{k \to \infty} \int_0^t |T_s A\phi_k| \, ds = \int_0^t \lim_{k \to \infty} |T_s A\phi_k| \, ds = 0, \]

whence \(T_t 1 = 1\) almost everywhere. \(\blacksquare\)
3 Sojourn times for small balls

In [33] one of us studied the growth behaviour of a class of Feller processes \((X_t)_{t\geq0}\) which are generated by pseudo differential operators. To do so, estimates for the running maxima and the sojourn times

\[ \sigma_{B_r}(x) := \inf\{t \geq 0 : |X_t - x| \geq r\}, \quad r > 0, \; x \in \mathbb{R}^n \]

were obtained. Although [33] was written for Feller processes, the necessary input for the estimates to work was that \((X_t)_{t\geq0}\) is a strong Markov process whose infinitesimal generator is a pseudo differential operator \(-p(x,D)-i.e.\ a\ operator\ of\ the\ form\ (5)-\) with negative definite symbol \(p(x,\xi)-i.e.,\ a\ locally\ bounded\ function\ p: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}\ satisfying\ a\ \textit{Levy-Khinchine}\ representation\)

\[ p(x,\xi) = a(x) - i\ell(x)\xi + \xi \cdot Q(x)\xi + \int_{h \neq 0} (1 - e^{-ih\xi} - i\chi(h)h\xi) \nu(x, dh), \]

for some centering function \(\chi(h)\) and measurable ‘coefficients’ \(a(x) \geq 0, \ell(x) \in \mathbb{R}^n,\) the positive semidefinite \(Q(x) \in \mathbb{R}^{n \times n}\) and the Levy kernel \(\nu(x, dh)\) which is such that \(\int_{h \neq 0} |h|^2 \wedge 1 \nu(x, dh) < \infty.\)

In order to use the methods developed in [33] we need to know that the process \((X_t, P_x)\) from Section 2 is a \(P_x\)-semimartingale for all \(x \notin N\) and some capacity-zero set \(N.\) This follows from a modification of Lemma 3.2 of [33] and the Fukushima decomposition of additive functionals, see [12, Theorem 5.2.2], in particular (5.2.26): from this we know that

\[ M_{t[u]} := u(X_t) - u(X_0) - \int_0^t Au(X_s) ds, \quad u \in \mathcal{D}(A), \]

is a \(P_x\)-martingale for all \(x \in \mathbb{R}^n \setminus M\) where \(M = M_u\) is an exceptional set that may depend on \(u.\)

\textbf{Lemma 3.1.} Let \((\mathcal{E}, \mathcal{F})\) and \(A\) be as in Theorem 2.2 and assume that \((A')-(C')\) hold true. (In particular, the symbol \(p(x,\xi)\) has bounded coefficients in the sense of (9).) Denote by \(N'\) the exceptional set outside of which the process \((X_t, P_x),\) which is properly associated with \((\mathcal{E}, \mathcal{F}),\) is uniquely defined. Then \((X_t, P_x)\) is a \(P_x\)-semimartingale for all \(x\) outside some (possibly larger) exceptional set \(N \supset N'.\)

\textbf{Proof.} The proof is similar to the argument from [33, Lemma 3.2] and we only sketch the main differences. Recall that our assumptions imply \(C_0^2(\mathbb{R}^n) \subset \mathcal{D}(A).\) Pick \(\phi_k^j \in C_0^2(\mathbb{R}^n)\) such that \(\phi_k^j(x) = x_j\ on \ B_k(0)\) and \(\phi_k^j(x) = 0\ on \ B_k^c(0)\) where \(j = 1, 2, \ldots, n\) and \(k \in \mathbb{N}.\) Using (15) we see that \(M_t := M_{t[u]}^{[u]}, u = \phi_k^j(\cdot - x),\) is a \(P_x\)-martingale for all \(x\) outside
some exceptional set $M_{j,k}$. We set $N := \bigcup_{j,k} M_{j,k} \cup N'$ which is again a capacity-zero set. We can now follow the argument of [33]: \( \int_0^t A\phi_k^j(x_s - x)\,ds \) is a bounded-variation process, and therefore $\phi_k^j(X_t - x)$ are $\mathbb{P}_x$-semimartingales (outside of $N$). Moreover, by pre-stopping

$$\phi_k^j(X_t - x)_{t^\tau_k} = (X_0^{(j)} - X_0^{(j)})_{t^\tau_k}, \quad X_0 = x \notin N,$$

with $\tau_k := \tau^k := \inf\{s \geq 0 : |X_s - x| > k\}$ and where $Y_{t^\tau} := Y_t\mathbf{1}_{[0,\tau)}(t) + Y_{\tau-}\mathbf{1}_{[\tau,\infty)}(t)$ denotes the pre-stopped process. Since a sequence of pre-stopped semimartingales define a semimartingale, see Protter [29, Theorem II.6] or [33, Lemma 2.1], the claim follows.

Since all proofs of [33] only involve stopping techniques (at a sequence of countably many stopping times), we can use all arguments of that paper in the present situation, possibly at the expense of a yet larger exceptional set.

**Theorem 3.2.** Let \( \{(X_t, \mathbb{P}_x), \ t \geq 0, \ x \in \mathbb{R}^n \setminus N\} \) be the Hunt process associated with the regular Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) and assume that the generator of $\mathcal{E}$ is (on the test functions $C_0^\infty(\mathbb{R}^n)$) a pseudo differential operator $-p(x, D)$ with negative definite symbol $-p(x, \xi)$.

If the symbol has bounded coefficients in the sense that $\sup_x |p(x, \xi)| \leq K (1 + |\xi|^2)$ for all $\xi \in \mathbb{R}^n$ and satisfies the following sector condition

$$|\text{Im} \ p(x, \xi)| \leq c_0 \text{Re} \ p(x, \xi), \quad x \in \mathbb{R}^n, \ |\xi| > \rho$$

with absolute constants $c_0, \rho > 0$, then we have for all starting points $x \in \mathbb{R}^n \setminus N, \ t \geq 0$ and $r > 0$

$$\mathbb{P}_x \left( \sup_{s \leq t} |X_s - x| \geq r \right) \leq c_n t \sup_{|x-y| < 2r} \sup_{|e| \leq 1} \text{Re} \ p(y, \frac{e}{4\kappa r})$$

and for all $x \in \mathbb{R}^n \setminus N, \ t \geq 0$ and $0 < r < \rho^{-1}$

$$\mathbb{P}_x \left( \sup_{s \leq t} |X_s - x| < r \right) \leq \frac{c_\kappa}{t} \inf_{|x-y| < 2r} \sup_{|e| = 1} \text{Re} \ p(y, \frac{e}{4\kappa r})$$

$$\mathbb{P}_x \left( \sup_{s \leq t} |X_s - x| < \frac{r}{2} \right) \leq \frac{c_n^2}{t^2} \inf_{|x-y| < 2r} \sup_{|e| = 1} \left[ \text{Re} \ p(y, \frac{e}{4\kappa r}) \right]^2$$

with absolute constants $\kappa^{-1} := 4 \arctan \frac{1}{2c_0}$ and $c_n, c_\kappa > 0$.

Since for all $r > 0$ and $t > 0$

$$\left\{ \sup_{s \leq t} |X_s - x| < r \right\} \subset \left\{ \sigma^x_r > t \right\} \subset \left\{ \sup_{s \leq t} |X_s - x| \leq r \right\},$$

Theorem 3.2 gives also upper bounds for the sojourn probabilities $\mathbb{P}_x(\sigma^x_r > t)$ and $\mathbb{P}_x(\sigma^x_r \leq t)$. As in [33, Theorem 4.7, Remark 4.8] these lead to the following estimates for the mean sojourn time for small balls.
Corollary 3.3. Let $\{(X_t, P_x), \ t \geq 0, \ x \in \mathbb{R}^n \setminus N\}$ be as in Theorem 3.2 and assume that the symbol of its generator $-p(x, D)$ has bounded coefficients and satisfies the sector condition (16). Then we have for all $x \in \mathbb{R}^n \setminus N$ and $0 < r < \rho^{-1}$

$$\sup_{|x - y| < 2r} \sup_{|e| \leq 1} \text{Re} p(y, \frac{e}{r}) \leq C_\kappa \inf_{|x - y| < 2r} \text{Re} p(y, \frac{e}{r})$$

(20)

with absolute constants $\kappa = 4 \arctan \frac{1}{2c_0}$ and $c_\kappa, C_\kappa > 0.$

The proofs of Theorem 3.2 and Corollary 3.3 can be taken almost literally from [33], in particular Lemma 4.1, Lemma 4.2 and Theorem 4.7, Remark 4.8. In [33] we assumed the sector condition (16) for all $\xi \in \mathbb{R}^n$ and obtained estimates for all $r > 0.$ If (16) holds only for $|\xi| > \rho$ for some fixed $\rho > 0,$ the argument [33, p. 607, line 6 from below] shows that the estimates are still valid if we restrict ourselves to small values $r < \rho^{-1}.$

If $z \in B_{r/2}(x),$ then it is clear that $B_{r/2}(z) \subset B_r(x) \subset B_{r/2+r}(z).$ Hence, $\sigma_{r/2}^z \leq \sigma_r \leq \sigma_{3r/2}^z.$ This gives

Corollary 3.4. Under the assumptions made in Corollary 3.3 we have

$$\sup_{|x - y| < 3r/2} \sup_{|e| \leq 1} \text{Re} p(y, \frac{e}{3r/2}) \leq C_\kappa \inf_{|x - y| < 7r/2} \text{Re} p(y, \frac{e}{6r/2})$$

(21)

for all $x \in \mathbb{R}^n,$ $r < \rho^{-1}$ and all $z \in B_{r/2}(x) \setminus N.$

In order to make Theorem 3.2 and Corollaries 3.3, 3.4 work in the setting of Section 2, we have to verify the conditions on the symbol $p(x, \xi).$ From Theorem 2.2 and Corollary 2.4 we know already that

$$(A') - (C') \implies p(x, \xi) \text{ exists and satisfies } \sup_{x} |p(x, \xi)| \leq K(1 + |\xi|^2).$$

For the sector condition (16) we need some preparations.

Lemma 3.5. Let $(E, \mathcal{F})$ be as in Theorem 2.1 and assume that (A'), (B') and the following condition

$$(D') \sup_{x} \int_{|h| < 1} |h| \log \frac{1}{|h|} |j(x, x + h) - j(x, x - h)| \, dh < \infty$$

hold. If for all $|h| < 1$

$$|j(x, x + h) - j(x, x - h)| \leq c |h| \log \frac{1}{|h|} (j(x, x + h) + j(x, x - h)),$$

(22)

then the symbol $p(x, \xi)$ satisfies the following estimate

$$|\text{Im} p(x, \xi)| \leq c \sqrt{\text{Re} p(x, \xi)} \quad \text{for all } x, \xi \in \mathbb{R}^n.$$
Proof. Obviously, \( (D') \) is stronger than \( (C') \) which means that the form \( \mathcal{E} \) is generated by a pseudo differential operator with symbol \( p(x, \xi) \). Define \( \theta(h) := (|h| \wedge 1) \log \frac{e}{|h| \wedge 1}; \ e = 2.71828\ldots \) Then we see using the Cauchy-Schwarz inequality
\[
|\text{Im} \ p(x, \xi)| = \left| \frac{1}{2} \int_{h \neq 0} \sin(h \xi) \left( j(x, x + h) - j(x, x - h) \right) \, dh \right| \\
\leq \int_{h \neq 0} \left| \frac{\sin(h \xi)}{\theta(h)^{1/2}} \right| \theta(h)^{1/2} \left| j(x, x + h) - j(x, x - h) \right| \, dh \\
\leq \left( \int_{h \neq 0} \sin^2(h \xi) \left| \frac{j(x, x + h) - j(x, x - h)}{\theta(h)} \right| \, dh \right)^{1/2} \times \\
\times \left( \int_{h \neq 0} \theta(h) \left| j(x, x + h) - j(x, x - h) \right| \, dh \right)^{1/2}.
\]
Because of \( (B') \) and \( (D') \) the second factor is uniformly bounded for all \( x \); using (22) and the elementary estimate \( \sin^2 t \leq 2(1 - \cos t) \) we arrive at
\[
|\text{Im} \ p(x, \xi)| \leq C \left( \int_{h \neq 0} (1 - \cos h \xi)(j(x, x + h) + j(x, x - h)) \, dh \right)^{1/2}
\]
which is but the assertion. \( \blacksquare \)

Lemma 3.6. Let \( p(x, \xi) \) be as in (6), (7) of Corollary 2.3. If for some absolute constant \( c > 0 \)
\[
(23) \quad \liminf_{|\xi| \to \infty} \frac{j(x, x + \frac{h}{|\xi|})}{|\xi|^n} \geq c > 0, \quad x \in \mathbb{R}^n, \ |h| < 1,
\]
then \( \liminf_{|\xi| \to \infty} |p(x, \xi)| \geq \frac{c}{12} \frac{\pi^{n/2}}{\Gamma(n/2 + 2)}. \)

Proof. Since \( 1 - \cos t \geq t^2/3 \) for \( |t| \leq 1 \) and since \( j(x, x + \frac{h}{|\xi|})|\xi|^{-n} \geq c/2 \) for large values of \( |\xi| \), we find for \( |\xi| \gg 1 \)
\[
|p(x, \xi)| \geq \text{Re} \ p(x, \xi) = \int_{h \neq 0} \left( 1 - \cos(h \cdot \xi) \right) j(x, x + h) \, dh \\
\geq \frac{1}{3} \int_{|h| |\xi| \leq 1} (h \cdot \xi)^2 j(x, x + h) \, dh \\
= \frac{1}{3} \int_{|h| \leq 1} \left( \frac{w \cdot \xi}{|\xi|} \right)^2 j(x, x + \frac{w}{|\xi|}) \frac{dy}{|\xi|^n} \\
\geq \frac{c}{6} \int_{|h| \leq 1} \left( \frac{w \cdot \xi}{|\xi|} \right)^2 \, dy.
\]
From Sonin’s integral formula we know that \( \int_{|y| \leq 1} (y \cdot a)^2 dy = \frac{1}{2} \pi^{n/2} |a|^2, \ a \in \mathbb{R}^n \), so that for large \( |\xi| \gg 1 \)

\[
|p(x, \xi)| \geq \frac{c}{6} \int_{|y| \leq 1} \left( \frac{y \cdot \xi}{|\xi|^2} \right)^2 dy = \frac{c}{12} \pi^{n/2} \Gamma\left(\frac{n}{2} + 2\right) |\xi| |\xi|^{n} + \alpha |\xi|^{-n} = c |\xi|^{n-\alpha} \geq c
\]

Since the right-hand side is independent of \( \xi \), the claim follows as \( |\xi| \to \infty \).

If \( \mathcal{E} \) is as in Section 2 we have the alternative representation \( j(x, y) = |x - y|^\alpha(y, x) + |x - y|^\alpha(x, y) \) for the jump density.

**Proposition 3.7.** Let \( (\mathcal{E}, \mathcal{F}) \) be as in Theorem 2.1, assume that \( (A')-(D') \) and \( (23) \) hold and that \( \alpha(x, y) \) is Lipschitz continuous. Then \( p(x, \xi) \) satisfies the sector condition \( (16) \) for large \( |\xi| \).

**Proof.** The assertion follows directly from Lemmata 3.5 and 3.6. Only \( (22) \) needs proof. For this we use the elementary formula

\[
|t^{-a} - t^{-b}| = \left| \int_{a}^{b} t^{-u} \log t \, du \right| \leq |b - a| \log |t^{-a} + t^{-b}|
\]

with \( a = \alpha(x, x + h) \) resp. \( \alpha(x + h, x) \), \( b = \alpha(x, x - h) \) resp. \( \alpha(x - h, x) \) and \( t = |h| \).

**Example 3.8.** We have seen in Example 2.7 that \( (10), (11), \) and \( (12) \) imply \( (A')-(C') \) and even \( (D') \). If we also assume that \( 0 \leq \alpha \leq \beta < 2 \), we find for \( |\xi| > 1 \) and \( |h| < 1 \)

\[
\frac{j(x, x + h)}{|\xi|^n} \geq \frac{c}{|h|^n + \alpha} |\xi|^{n+\alpha} |\xi|^{-n} = \frac{c}{|h|^n + \alpha} \geq \frac{c}{|h|^n + \alpha} \geq c
\]

which shows that condition \( (23) \) from Lemma 3.6 is satisfied. Thus, Proposition 3.7 holds.

The above assumptions are more far-reaching. Let \( p(x, \xi) \) and \( j(x, y) \) be as before and define a new symbol by

(24)

\[
p_1(x, \xi) := \text{p.v.} \int_{h \neq 0} (1 - e^{ih\xi}) j_1(x, x + h) \, dh,
\]

with the modified jump measure

(25)

\[
j_1(x, y) := j(x, y) 1_{B_1(0)}(x - y) + e^{-|x - y|} 1_{B_1(0)}(x - y).
\]

The corresponding pseudo differential operator with symbol \( p_1 \) can then be written in the following form

(26)

\[
-p_1(x, D)u(x) = \text{p.v.} \int_{0 < |h| < 1} (u(x + h) - u(x)) j_1(x, x + h) \, dh + \int_{|h| \geq 1} (u(x + h) - u(x)) e^{-|h|} \, dh.
\]
Note that \( j_1 \) has the same small jumps as \( j \) but that large jumps occur at a different—possibly much smaller—rate than before. Using (10) we see

\[
c \int_{|h| < 1} (1 - \cos \xi h) \frac{dh}{|h|^{n+\alpha}} \leq \text{Re} p_1(x, \xi) \leq C \int_{|h| < 1} (1 - \cos \xi h) \frac{dh}{|h|^{n+\beta}} \leq C \int_{h \neq 0} (1 - \cos \xi h) \frac{dh}{|h|^{n+\beta}} = c_\beta |\xi|^\beta.
\]

Since \( \int_{h \neq 0} (1 - \cos \xi h) |h|^{-n-\alpha} dh = c_\alpha |\xi|^\alpha \) and since

\[
\int_{|h| \geq 1} (1 - \cos \xi h) \frac{dh}{|h|^{n+\alpha}} \leq 2 \int_{|h| \geq 1} \frac{dh}{|h|^{n+\alpha}} < \infty,
\]

we find that for a suitable constant \( c_{\alpha,\rho} \) not depending on \( x \) and \( |\xi| > \rho \)

(27)

\[
c_{\alpha,\rho} |\xi|^\alpha \leq \text{Re} p_1(x, \xi) \leq c_\beta |\xi|^\beta, \quad x \in \mathbb{R}^n, \ |\xi| \geq \rho.
\]

Since by assumption (B′)

\[
\int_{|h| \geq 1} (1 - \cos \xi h) j(x, x + h) dh \leq 2 \sup_x \int_{|h| \geq 1} j(x, x + h) dh < \infty,
\]

we conclude that \( \text{Re} p(x, \xi) \sim \text{Re} p_1(x, \xi) \) for large \( |\xi| \) and that for some \( c_{\beta,\rho} \)

(28)

\[
c_{\alpha,\rho} |\xi|^\alpha \leq \text{Re} p(x, \xi) \leq c_{\beta,\rho} |\xi|^\beta, \quad x \in \mathbb{R}^n, \ |\xi| > \rho.
\]

Substituting (28) into (17), (18) and (21) yields

**Corollary 3.9.** Let \( p(x, \xi) \) and \( p_1(x, \xi) \) be as in Example 3.8, i.e. satisfying (10)–(12). Then the following estimates hold for the sojourn time \( \sigma^x_r, x \in \mathbb{R}^n, r < 1 \), of the process belonging to either symbol:

\[
\mathbb{P}_x(\sigma^x_r \leq t) \leq c_n t r^{-\beta}, \quad \forall r < 1
\]

\[
\mathbb{P}_x(\sigma^x_r > t) \leq c_n t^{-1} r^\alpha, \quad \forall r < 1/\rho,
\]

\[
c_{n, \beta, \rho} r^\beta \leq \mathbb{E}_x \sigma^x_r \leq c_{n, \alpha, \rho} r^\alpha, \quad \forall r < 1/\rho, \ z \in B_{r/2}(x),
\]

(the constant \( \rho \) is that of (16) and (27), (28)).

**Remark 3.10.** (i) The arguments used in Example 3.8 show that we can replace in Theorem 3.2 and Corollary 3.3 the full symbol \( \text{Re} p \) by the modification \( \text{Re} p_1 \) whenever

\[
\liminf_{|\xi| \to \infty} \text{Re} p(x, \xi) \geq c > 0 \text{ and } \sup_{x, \xi} |p(x, \xi) - p_1(x, \xi)| < \infty; \text{ the last is always implied by (even equivalent to) conditions of the type (B′)}.
\]

(ii) The exponents \( \alpha, \beta \) in the estimates in Corollary 3.9 come from the comparison (28) of the symbol with symbols of \( \alpha \)- resp. \( \beta \)-stable Lévy processes. This is somewhat ad
hoc; what really matters is the asymptotic behaviour of \( p(x, \xi) \) as \(|\xi| \to \infty\). Exploiting this idea one can define various Blumenthal-Getoor-type indices and get estimates where \( \alpha \) and \( \beta \) are obtained as asymptotic growth exponents of \( p(x, \xi) \). Exact definitions and more details can be found in [33, §§4,5].

4 A perturbation result

In Example 3.8 we have modified the (large-jump part of the) symbol \( p(x, \xi) \) given by (7) to become

\[
p_1(x, \xi) = \text{p.v.} \int (1 - e^{ih\xi}) j_1(x, x + h) \, dh
\]

\[
= \text{p.v.} \int_{|h|<1} (1 - e^{ih\xi}) j(x, x + h) \, dh + \int_{|h|\geq 1} (1 - e^{ih\xi}) e^{-|h|} \, dh.
\]

The jump measure \( j(x, y) \) was assumed to satisfy the conditions (10)–(12) which meant, in particular, that \(-p(x, D)\) and \(-p_1(x, D)\) (defined on, say, \( C_0^\infty(\mathbb{R}^n) \)), have extensions \( A \) and \( A_1 \) to generators of Dirichlet forms and that for the corresponding stochastic processes all results of §§1–3 hold true.

From now on we will always work under these assumptions.

Setting \( q_1(x, \xi) := p(x, \xi) - p_1(x, \xi) \) we have

\[
q_1(x, \xi) = \int_{|h|\geq 1} (1 - e^{ih\xi}) \left( j(x, x + h) - e^{-|h|} \right) \, dh
\]

so that

\[
-q_1(x, D)u(x) = \int_{|h|\geq 1} (u(x + h) - u(x)) \left( j(x, x + h) - e^{-|h|} \right) \, dh.
\]

It is now easy to see that under (B’)

\[
\| -q_1(\cdot, D)u\|_\infty \leq 2 \left( \sup_x \int_{|h|\geq 1} j(x, x + h) \, dh + c_n \right) \|u\|_\infty,
\]

which shows that \(-q_1(x, D)\) extends naturally to a continuous operator \( B \) on \( L^\infty(\mathbb{R}^n) \), resp., \( B_0(\mathbb{R}^n) \). From \( A|_{C_0^\infty(\mathbb{R}^n)} = -p(x, D) = -p_1(x, D) - q_1(x, D) \), we see that \(-p_1(x, D)\) has an extension \( A_1 \) on \( D(A) \).

Since \( A \) and \( A_1 \) generate Dirichlet forms, we can associate with both of them sub-Markovian semigroups and resolvent operators on \( L^2(\mathbb{R}^n) \) which we will denote by \( T^A_t \), \( T^A_1 \) and \( R^A_\lambda \), \( R^A_{\lambda_1} \), respectively.
Recall that a sub-Markovian operator $T$ is said to be Feller, if $T$ maps the set $C_\infty(\mathbb{R}^n)$ into itself; $T$ is called a strong Feller operator, if $T$ maps $B_b(\mathbb{R}^n)$ into $C_b(\mathbb{R}^n)$. Note that a sub-Markovian operator that is defined on $L^2(\mathbb{R}^n)$ has a canonical extension onto $L^\infty(\mathbb{R}^n)$, while a Feller operator can be canonically extended to $B_b(\mathbb{R}^n)$, see the proof of Lemma 1.6.4 in Fukushima et al. [12] and Lemma 1 in [35].

Since we work both in $L^2$, $L^\infty$ (made up of equivalence classes of functions) and in $B_b(\mathbb{R}^n)$ (comprising everywhere defined functions), it is helpful to have the following result.

**Lemma 4.1.** Let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a continuous operator which (has an extension which) is strongly Fellerian. Then $T$ maps $L^\infty(\mathbb{R}^n)$ into $C_b(\mathbb{R}^n)$.

**Proof.** The point is to show that any two representatives $f, \phi \in B_b(\mathbb{R}^n)$ of some equivalence class $[f] \in L^\infty(\mathbb{R}^n)$ have the same image under $T$. By assumption, $N := \{x : f(x) \neq \phi(x)\}$ is a Lebesgue null set and, therefore, $f - \phi \in L^2(\mathbb{R}^n)$ and so

$$\|T(f - \phi)\|_{L^2} = 0.$$  

Thus, $Tf = T\phi$ almost everywhere, hence, everywhere since $Tf, T\phi \in C_b(\mathbb{R}^n)$.

**Proposition 4.2.** Let $A, A_1$ be infinitesimal generators of sub-Markovian semigroups on $L^2(\mathbb{R}^n)$ such that $B := A - A_1$ is a bounded operator on $L^\infty(\mathbb{R}^n)$. If the resolvent operators $(R^A_{\lambda})_{\lambda > 0}$ are strong Feller operators, then the resolvent $(R^A_{\lambda})_{\lambda > 0}$ is also strongly Fellerian.

**Proof.** Note that $R^A_{\lambda}, R^A_{\lambda}$ are a priori defined on $L^2(\mathbb{R}^n)$. Since $\lambda R^A_{\lambda}, \lambda R^A_{\lambda}$ are sub-Markovian operators, they have an extension to $L^\infty(\mathbb{R}^n)$; the assumption says that $R^A_{\lambda}$ can be defined on $B_b(\mathbb{R}^n)$. Using the second resolvent equation

$$R^A_{\lambda} - R^A_{\lambda} = R^A_{\lambda}(A_1 - A)R^A_{\lambda}$$

we get for $u \in B_b(\mathbb{R}^n)$ or $L^\infty(\mathbb{R}^n)$

$$R^A_{\lambda}u = R^A_{\lambda}u + R^A_{\lambda}BR^A_{\lambda}u.$$  

Since $BR^A_{\lambda}$ is bounded on $L^\infty(\mathbb{R}^n)$, the assertion follows from Lemma 4.1 and the strong Feller property of $R^A_{\lambda}$.

Since $R^A_{\lambda} = \int_0^\infty e^{-\lambda t}T^A_t dt$, it is clear from Lebesgue’s dominated convergence theorem that $R^A_{\lambda}$ inherits the (strong) Feller property from $T^A_t$. Conversely, if $(R^A_{\lambda})_{\lambda > 0}$ is strongly Fellerian, $T_t$ need not have the strong Feller property; consider, e.g., the shift semigroup $T_t u(x) := u(x+t)$ which is certainly not strongly Feller, but has a strong Feller resolvent. Things are different if we consider the Feller property. The following result seems to be
part of mathematical folklore, see e.g., Sharpe [36, pp. 44-51], but as we could not find a quotable reference we decided to include the (not completely trivial) proof. As usual, we call a resolvent \((R_\lambda)_{\lambda > 0}\) sub-Markovian, if all operators \(\lambda R_\lambda\) are sub-Markovian.

**Theorem 4.3.** Let \((R_\lambda)_{\lambda > 0}\) be a sub-Markovian resolvent and assume that it corresponds to a sub-Markovian semigroup \((T_t)_{t \geq 0}\). If the operators \(R_\lambda\) have the Feller property and if \(\lim_{\lambda \to \infty} \lambda R_\lambda u(x) = u(x)\) for all \(u \in C_\infty(\mathbb{R}^n)\), then the semigroup operators \(T_t, t > 0\) are Fellerian and, as operators on \(C_\infty(\mathbb{R}^n)\), strongly continuous at \(t = 0\).

**Proof.** Because of the Feller property of \(R_\lambda\) we can always extend the operators \(R_\lambda\) onto \(B_b(\mathbb{R}^n)\).

Denote by \(S^\alpha, \alpha > 0\), the \(\alpha\)-supermedian functions, i.e., functions \(u \in B_b(\mathbb{R}^n)\) such that
\[
u \geq 0 \quad \text{and} \quad \lambda R_{\lambda + \alpha} u \leq u, \quad \lambda > 0.
\]
From the resolvent equation \(R_\alpha - R_{\lambda + \alpha} = \lambda R_{\lambda + \alpha} R_\alpha\) it is easy to see that \(R_\alpha u\) is \(\alpha\)-supermedian for any nonnegative \(u \in B^+_\alpha(\mathbb{R}^n)\). Moreover, if \(\beta \geq \alpha\) and \(u \in B_b(\mathbb{R}^n), u \geq 0\), we have
\[
R_{\lambda + \alpha} u - R_{\lambda + \beta} u = (\beta - \alpha) R_{\lambda + \alpha} R_{\lambda + \beta} u \geq 0
\]
so that for \(u \in S^\alpha\)
\[
\lambda R_{\lambda + \beta} u \leq \lambda R_{\lambda + \alpha} u \leq u
\]
implies that \(S^\alpha \subset S^\beta\) whenever \(\beta > \alpha\). Denote by \(\mathbb{R}^n_\Delta\) the one-point compactification of \(\mathbb{R}^n\); since \(C(\mathbb{R}^n_\Delta)\) can be identified with \(C_\infty(\mathbb{R}^n) \oplus \mathbb{R}\), the operators \(R_\lambda\) and \(T_t\) and the notion of a function being supermedian can easily be extended to \(C(\mathbb{R}^n_\Delta)\) and it makes sense to consider the space
\[
S := \bigcup_{\alpha > 0} (S^\alpha \cap C(\mathbb{R}^n_\Delta)).
\]
For any two \(u, v \in S\) we find some \(\alpha > 0\) with \(u, v \in S^\alpha \cap C(\mathbb{R}^n_\Delta)\). Since \(R_\lambda\) is positivity preserving, it is monotone and we have
\[
\lambda R_{\lambda + \alpha}(u \land v) \leq \lambda (R_{\lambda + \alpha} u \land R_{\lambda + \alpha} v) \leq u \land v \quad \implies \quad u \land v \in S^\alpha.
\]
Moreover, \(1 \in S\) as \(\lambda R_{\lambda + \alpha} 1 \leq \lambda \frac{1}{\lambda + \alpha} \leq 1\) by the sub-Markov property of \(\mu R_\mu\). Finally, for \(x \in \mathbb{R}^n, y \in \mathbb{R}^n_\Delta\) and \(x \neq y\) we find some \(u \in C_\infty(\mathbb{R}^n)\) with \(u(x) = 1 \neq 0 = u(y)\). Since \(\alpha R_{\alpha} u \underset{a \to \infty}{\longrightarrow} u\), we find for every \(\epsilon\) some \(\alpha = \alpha(\epsilon) > 0\) such that
\[
\alpha R_{\alpha} u(y) < \epsilon < 1 - \epsilon < \alpha R_{\alpha} u(x).
\]
Since \(R_{\alpha} u \in S\), we conclude that \(S\) separates points in \(\mathbb{R}^n_\Delta\).
By the Stone-Weierstraß theorem \( S - S \) is dense in \( C(\mathbb{R}^n) \) under uniform convergence and, in particular, \( (S - S) \cap C_\infty(\mathbb{R}^n) \) is dense in \( C_\infty(\mathbb{R}^n) \).

Using the resolvent equation we get for all \( \mu > \lambda, \alpha \geq 0 \) and all measurable functions \( u \geq 0 \)
\[(30) \quad (\mu R_{\mu+\alpha} - \lambda R_{\lambda+\alpha})u = (\mu - \lambda)R_{\lambda+\alpha}(1 - \mu R_{\mu+\alpha})u \geq 0; \]
for \( u \in S^\alpha \) we find therefore \( \lim_{\lambda \to \infty} \lambda R_{\lambda+\alpha} u = \sup_{\lambda > 0} \lambda R_{\lambda+\alpha} u \). Since
\[ \lambda R_{\lambda+\alpha} u = (\lambda + \alpha)R_{\lambda+\alpha} u - \alpha R_{\lambda+\alpha} u \]
and since \( \|\alpha R_{\lambda+\alpha} u\|_{\infty} \leq \frac{\alpha}{\lambda + \alpha} \|u\|_{\infty} \xrightarrow{\lambda \to \infty} 0 \), we conclude that \( \lambda R_{\lambda+\alpha} u \) increases towards \( u \) as \( \lambda \to \infty \). From Dini’s theorem we know that the convergence is uniform. Thus we have shown that \( D := \bigcup_{\lambda > 0} R_{\lambda}(S - S) \) is dense in \( S - S \) which, in turn, is dense in \( C_\infty(\mathbb{R}^n) \).

Since \( S - S \) is a vector space and since \( R_{\lambda} \) preserves \( S - S \), the resolvent equation also shows that \( D = R_{\mu}(S - S) \) for any \( \mu > 0 \).

If \( A \) is the infinitesimal generator of \( R_{\lambda} \), we have thus shown that both \( D = \mathcal{D}(A) \) and \( \text{Range}(\lambda - A), \lambda > 0 \), are dense in \( C_\infty(\mathbb{R}^n) \). Since \( \lambda R_{\lambda} \) is sub-Markovian, it follows that \( \| (\lambda - A)u \|_{\infty} \geq \lambda \|u\|_{\infty} \), i.e., \( A \) is dissipative. The Hille-Yosida-Ray theorem now shows that \( A \) generates a strongly continuous contraction semigroup on \( C_\infty(\mathbb{R}^n) \), and since \( A \) is also the generator of \( T_t \), we can infer that \( T_t \) has (a modification satisfying) the Feller property.

Let us now see how we can apply Proposition 4.2 and Theorem 4.3 in the situation described at the beginning of this section.

**Corollary 4.4.** Let \( A \) and \( A_1 \) be the extensions of \( -p(x, D) \) and \( -p_1(x, D) \) described at the beginning of §4; in particular we know that \( (A') \) and \( (B') \) hold. If the \( L^2 \) sub-Markovian contraction resolvent \( R_{\lambda}^{A_1} \) has the strong Feller property, then the \( L^2 \)-semigroup \( (T_t^A)_{t \geq 0} \) generated by \( A \) is a Feller semigroup, i.e., a strongly continuous sub-Markovian contraction semigroup on \( C_\infty(\mathbb{R}^n) \).

**Proof.** Note that \( B := A - A_1 \) is a bounded operator on \( L^\infty(\mathbb{R}^n) \). Theorem 4.2 therefore shows that \( R_{\lambda}^A \) inherits the strong Feller property from \( R_{\lambda}^{A_1} \). This means, in particular, that \( R_{\lambda}^A : C_\infty(\mathbb{R}^n) \to C_b(\mathbb{R}^n) \). To get the Feller property, it remains to show that
\[ \lim_{|x| \to \infty} R_{\lambda}^A u(x) = 0 \]
for all \( \lambda > 0 \).
From Theorem 2.2 we know that $C_0^\infty(\mathbb{R}^n) \subset \mathcal{D}(A)$ (note that $\mathcal{D}(A)$ is the $L^2$ domain of the operator $A$). From the estimate (8) we find for all $v \in C_0^\infty(\mathbb{R}^n)$
\[ \|\mu R^A_\mu v - v\|_\infty = \|R^A_\mu Av\|_\infty \leq \frac{1}{\mu} \|Av\|_\infty \leq \frac{C}{\mu} \sum_{|\alpha| \leq 2} \|\partial^\alpha v\|_\infty. \]
Since $C_0^\infty(\mathbb{R}^n)$ is dense in $C^\infty(\mathbb{R}^n)$ and since $\|\mu R^A_\mu\|_\infty \leq 1$, we get with a routine $\epsilon$-argument that
\[
\lim_{\mu \to \infty} \|\mu R^A_\mu u - u\|_\infty = 0 \text{ for all } u \in C_\infty(\mathbb{R}^n).
\]

In the proof of Theorem 4.3 we showed in (31) that $\lambda \mapsto \lambda R^A_\lambda v$ is increasing for measurable functions $v \geq 0$. Thus, we find for $u \in C_\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and all $\mu > \lambda$
\[ |\lambda R^A_\lambda u(x)| \leq \lambda R^A_\lambda |u|(x) \leq \mu R^A_\mu |u|(x) \leq \|\mu R^A_\mu|u| - |u||_\infty + |u(x)|. \]
Letting first $\mu \to \infty$ and then $|x| \to \infty$ proves $\lim_{|x| \to \infty} |\lambda R^A_\lambda u(x)| = 0$.

The assertion follows now from Theorem 4.3.

### 5 A Harnack inequality

Let $p(x, \xi), j(x, y), A$ and $p_1(x, \xi), j_1(x, y), A_1$ be as described at the beginning of Section 4. We write $(X_t)_{t \geq 0}$ resp. $(Y_t)_{t \geq 0}$ for the Hunt processes generated by $A$ resp. $A_1$; without loss of generality we can assume that the exceptional set $N$ is the same for both processes. Note that this ensures that all assumptions of Corollary 3.9 are satisfied.

In this section we will prove the Harnack inequality for the process generated by the modified operator $A_1$. Our argument follows closely the methods developed by Bass and co-authors [3, 2], see also Song and Vondracek [38].

Let $D$ be a domain in $\mathbb{R}^n$. A function $h$ defined on $\mathbb{R}^n \setminus N$ is said to be harmonic (relative to $A_1$, for short: $A_1$-harmonic) in $D \setminus N$ if it is not identically infinite in $D \setminus N$ and if for all bounded open sets $U \subset \overline{U} \subset D$
\[ h(x) = \mathbb{E}_x[h(Y_{\sigma_U})], \quad x \in U \setminus N, \]
where $\sigma_U$ is the first exit time from $U$ of the process $(Y_t)_{t \geq 0}$ belonging to $A_1$.

We say that the Harnack inequality holds for the process if for every domain $D \subset \mathbb{R}^n$ and every compact set $K \subset D$ there exists a constant $C = C_{D,K} > 0$ such that for all positive harmonic functions $h$ in $D \setminus N$ the inequality
\[ \sup_{x \in K \setminus N} h(x) \leq C \inf_{x \in K \setminus N} h(x) \]
holds.

We begin with a simple auxiliary estimate.
Lemma 5.1. For all $x \in \mathbb{R}^n$, $r < 1$ and all $v, y, z \in \mathbb{R}^n$ satisfying $|v - x| > r$ and $|y - x| < \frac{r}{2}$, $|z - x| < \frac{r}{2}$ the following estimate holds

$$j_1(y, v) \leq \gamma r^{\alpha - \beta} j_1(z, v)$$

with an absolute constant $0 < \gamma < \infty$.

Proof. Let $r$ and $v, x, y, z$ be as in the statement. From (10) we know that for suitable constants $0 < c, C < \infty$

$$\frac{c}{|y - v|^\alpha + n} \leq j_1(y, v) \leq \frac{C}{|y - v|^{\beta + n}}, \quad |y - v| < 1.$$  

Since $j_1(y, v) = e^{[y-v]}$ for all $|y - v| \geq 1$, a simple continuity argument extends the above estimate to all $|y - v| < 2$:

$$\frac{c'}{|y - v|^\alpha + n} \leq j_1(y, v) \leq \frac{C'}{|y - v|^{\beta + n}}, \quad |y - v| < 2.$$  

In order to prove our claim we distinguish between three cases:

Case 1: $|y - v| < 2$ and $|z - v| < 2$. First we note that

$$|y - v| \geq |v - x| - |x - y| > r - \frac{r}{2} = \frac{r}{2}.$$  

Moreover,

$$j_1(y, v) \leq \frac{C'}{|y - v|^{\beta + n}} = \frac{C'}{|z - v|^\alpha + n} \left(\frac{|z - v|}{|y - v|}\right)^{\alpha + n} \frac{1}{|y - v|^{\beta - \alpha}} \leq \frac{C'}{c'} j_1(z, v) \left(\frac{|z - y| + |y - v|}{|y - v|}\right)^{\alpha + n} \left(\frac{2}{r}\right)^{\beta - \alpha} \leq \frac{C'}{c'} j_1(z, v) \left(1 + \frac{r}{r/2}\right)^{\alpha + n} 2^{\beta - \alpha} r^{\alpha - \beta} = \gamma r^{\alpha - \beta} j_1(z, v).$$  

Case 2: $|y - v| > 1$ and $|z - v| > 1$. Then

$$j_1(y, v) = e^{-|y-v|} \leq e^{-|z-v|} = e^{-|y-z|} j_1(z, v) \leq e^r j_1(z, v) \leq 3 r^{\alpha - \beta} j_1(z, v),$$  

where we used that $r < 1$.

Case 3: $|y - v| > 1$ and $|z - v| \leq 1$, or, $|y - v| \leq 1$ and $|z - v| > 1$. Since

$$|y - v| \leq |y - z| + |z - v| \leq r + 1 < 2,$$

we are back in the first case.  

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Proposition 5.2. Let $p_1(x, \xi)$ and $j_1(x, y)$ be as above, $x \in \mathbb{R}^n \setminus N$ and $0 < r < 1 \land \rho^{-1}$. Then there exists a constant $c$ such that for all $H \in B_0^+(\mathbb{R}^n)$ with support $\text{supp} \ H \subset B_r^c(x)$ the inequality

$$
\mathbb{E}_y H(Y_{\sigma_{r/2}}) \leq c r^{2(\alpha - \beta)} \mathbb{E}_z H(Y_{\sigma_{r/2}}), \quad y, z \in B_{r/4}(x) \setminus N,
$$

holds.

Proof. We consider first functions $\phi \in C^2_c(\mathbb{R}^n) \subset \mathcal{D}(A_1)$, with $\phi \geq 0$ and $\text{supp} \ \phi \subset B_{c \cdot r}(x)$. To simplify notation we write $\sigma := \sigma_{r/2}$. For all $z \in B_{r/4}(x)$ we find using Dynkin’s formula

$$
\mathbb{E}_z \phi(Y_{\sigma}) = \mathbb{E}_z \phi(Y_{\sigma}) - \phi(z) = \mathbb{E}_z \left( \int_0^\sigma A_1 \phi(Y_s) \ ds \right)
$$

$$
= \mathbb{E}_z \left( \int_0^\sigma \text{p.v.} \int (\phi(Y_s + h) - \phi(Y_s)) \ j_1(Y_s, Y_s + h) \ dh \ ds \right)
$$

$$
= \mathbb{E}_z \left( \int_0^\sigma \int \phi(v) \ j_1(Y_s, v) \ dv \ ds \right)
$$

$$
\leq \left( \mathbb{E}_z \sigma \right) \int \phi(v) \sup_{y \in B_{r/2}(x)} j_1(y, v) \ dv
$$

From Corollary 3.9 we get for $r < 1/\rho$ and $y, z \in B_{r/2}(x)$

$$
\mathbb{E}_z \sigma \leq C \ r^{\alpha - \beta} \mathbb{E}_y \sigma,
$$

$C = C_{\kappa, \alpha, \beta, \rho}$, and with Lemma 5.1 we conclude

$$
\mathbb{E}_z \phi(Y_{\sigma}) \leq C \ r^{\alpha - \beta} \left( \mathbb{E}_y \sigma \right) \int \phi(v) \gamma \inf_{y \in B_{r/2}(x)} j_1(y, v) \ dv
$$

$$
\leq C' \ r^{2(\alpha - \beta)} \mathbb{E}_y \left( \int_0^\sigma \int \phi(v) \ j_1(Y_s, v) \ dv \ ds \right)
$$

$$
= C' \ r^{2(\alpha - \beta)} \mathbb{E}_y \left( \int_0^\sigma A_1 \phi(Y_s) \ ds \right)
$$

$$
= C' \ r^{2(\alpha - \beta)} \mathbb{E}_y \phi(Y_{\sigma}).
$$

Since we can approximate any positive measurable function $H$ with $C^2_{\infty}(\mathbb{R}^n)$-functions, the claim follows.

Proposition 5.3. There exists a constant $c > 0$ such that for all $x \in \mathbb{R}^n$, $0 < r < \frac{1}{2} \land \frac{1}{\rho}$, $D \subset B_r(x)$ and $y \in B_{r/2}(x)$ the following inequality holds:

$$
\mathbb{P}_y(\tau_D < \sigma_{r}) \geq c r^{\beta - \alpha} \left| \frac{|D|}{|B_r(x)|} \right|,
$$

where $\tau_D$ is the first entrance time into $D$ and $\sigma_{r} = \sigma_{B_r(x)}$ is the first exit time from $B_r(x)$.

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Proof. Fix \( y \in B_{r/2}(x) \setminus N \). If \( \mathbb{P}_y(\tau_D < \sigma^x_r) \geq \frac{1}{4} \), the claim is obviously true. This happens, in particular, if \( y \in D \setminus N \). Let us, therefore, assume that \( y \notin D \) and that \( \mathbb{P}_y(\tau_D < \sigma^x_r) < \frac{1}{4} \). Pick a sequence \( \phi_j \in C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{D}(A_1) \) such that \( 0 \leq \phi_j \uparrow 1_D \) and \( \phi_j(y) = 0 \). Then we get from Dynkin’s formula and the form (26) of the generator \( A_1 \)

\[
\mathbb{P}_y(\tau_D < \sigma^x_r) \geq \mathbb{E}_y1_D(\tau_D \wedge \sigma^x_r) - 1_D(y) = \sup_j \mathbb{E}_y\left( \int_0^{\tau_D \wedge \sigma^x_r} A_1 \phi_j(Y_s) \, ds \right) \\
= \sup_j \mathbb{E}_y\left( \int_0^{\tau_D \wedge \sigma^x_r} \phi_j(Y_s+h) j_1(Y_s,Y_s+h) \, dh \, ds \right) \\
\geq \sup_j \mathbb{E}_y\left( \int_0^{\tau_D \wedge \sigma^x_r} \phi_j(Y_s+h) \frac{c}{|h|^{\alpha+n}} \, dh \, ds \right) \\
= \mathbb{E}_y\left( \int_0^{\tau_D \wedge \sigma^x_r} \int_0^{\tau_D \wedge \sigma^x_r} 1_D(v) \frac{c}{|Y_s-v|^{\alpha+n}} \, dv \, ds \right) \\
\geq \mathbb{E}_y\left( \int_0^{\tau_D \wedge \sigma^x_r} \int_0^{\tau_D \wedge \sigma^x_r} 1_D(v) \frac{c}{(2r)^{\alpha+n}} \, dv \, ds \right) \\
= c' \mathbb{E}_y(\tau_D \wedge \sigma^x_r)^{r-\alpha} \frac{|D|}{|B_r(x)|}.
\]

In the penultimate step we used that for \( s < \sigma^x_r \)

\[
|Y_s - v| \leq |Y_s - x| + |x - v| \leq 2r.
\]

From the Markov inequality and Corollary 3.9 we get for any \( T > 0 \)

\[
\mathbb{E}_y(\tau_D \wedge \sigma^x_r) \geq T \left( 1 - \mathbb{P}_y(\tau_D < \sigma^x_r) \right) - \mathbb{P}_y(\sigma^x_r < T) \geq T \left( \frac{3}{4} - \mathbb{P}_y(\sigma^x_r < T) \right) \geq T \left( \frac{3}{4} - c_n T r^{-\beta} \right).
\]

The last expression reaches its maximum at \( T = \frac{3r^\beta}{8c_n} \), and we find

\[
\mathbb{E}_y(\tau_D \wedge \sigma^x_r) \geq \frac{9}{64c_n} r^\beta.
\]

Inserting this into the first estimate we finally arrive at

\[
\mathbb{P}_x(\tau_D < \sigma^x_r) \geq c'' r^{\beta-\alpha} \frac{|D|}{|B_r(x)|}.
\]

Remark 5.4. Proposition 5.3 shows that our condition (10) is somewhat ad hoc and that a finer analysis using the full strength of (20) from Corollary 3.3 could improve the lower estimate of \( \mathbb{P}_y(\tau_D < \sigma^x_r) \); in particular one should be able to find conditions for the symbol (or the jump kernel) which reduce or remove the factor \( r^{\beta-\alpha} \) on the right hand side of the estimate. This will eventually force us to assume in the next section that \( \alpha = \beta \).
We can now show the Harnack inequality. If we take into account the exceptional set \( N \), the proof is almost literally the same as Bass and Kaßmann’s proof of Theorem 4.1 in [2]; we will therefore only quote the result.

**Theorem 5.5.** Let \( A_1 = -p_1(x, D), p_1(x, \xi) \) and \( j_1(x, y) \) be as above. For all \( R > 0 \) and \( x_0 \in \mathbb{R}^n \) there exists a constant \( C = C(R, x_0) > 0 \) such that for all positive bounded functions \( h \) on \( \mathbb{R}^n \setminus N \) which are \( A_1 \)-harmonic in \( B_R(x_0) \) the following Harnack inequality holds:

\[
h(x) \leq C h(y), \quad \forall x, y \in B_{R/2}(x_0) \setminus N.
\]

6 The Feller property of the semigroup \( e^{tA} \)

Let \( p(x, \xi), j(x, y) \) and \( j_1(x, y), A_1 \) be as in the previous section, see the description at the beginning of Section 4. We write \( (X_t)_{t \geq 0} \) resp. \( (Y_t)_{t \geq 0} \) for the Hunt processes generated by \( A \) resp. \( A_1 \); without loss of generality we can assume that the exceptional set \( N \) is the same for both processes.

For reasons that were explained in Remark 5.4 we will assume that from now to the end of the paper \( \alpha = \beta \).

**Lemma 6.1.** Let \( 0 < r < \frac{1}{2} \wedge \frac{1}{p} \). Then we have for all \( x \in \mathbb{R}^n \setminus N \) and \( R > 2r \)

\[
\mathbb{P}_x(Y_{\sigma^x} \notin B_R(x)) \leq c'' \frac{r^{\alpha}}{R^\alpha}.
\]

**Proof.** Pick a sequence \( \phi_j \in C^2_0(\mathbb{R}^n) \subset D(A_1) \) with \( 0 \leq \phi_j \uparrow 1_{B_R(x)} \). From Dynkin’s formula and the structure (26) of \( A_1 = -p_1(x, D) \) we get

\[
\mathbb{P}_x(Y_{\sigma^x} \notin B_R(x)) = \sup_j \mathbb{E}_x(\phi_j(Y_{\sigma^x})) = \sup_j \mathbb{E}_x\left( \int_0^{\sigma^x} A_1 \phi_j(Y_s) \, ds \right)
\]

\[
= \sup_j \mathbb{E}_x\left( \int_0^{\sigma^x} \int \phi_j(Y_s + h) j_1(Y_s, Y_s + h) \, dh \, ds \right)
\]

\[
\leq \mathbb{E}_x\left( \int_0^{\sigma^x} \int 1_{B_R(x)}(Y_s + h) j_1(Y_s, Y_s + h) \, dh \, ds \right)
\]

\[
\leq \mathbb{E}_x\left( \int_0^{\sigma^x} \int 1_{B_R(x)}(v) \frac{C}{|Y_s - v|^{n+\alpha}} \, dv \, ds \right).
\]

The last estimate follows from (10) if \( |Y_s - v| \leq 1 \), and from the trivial estimate \( e^{-|Y_s - v|} \leq |Y_s - v|^{-n-\alpha} \), if \( |Y_s - v| > 1 \). For \( s < \sigma^x \) we have \( |Y_s - x| < r \) which means that for \( v \in B_R(x) \)

\[
|Y_s - v| \geq |x - v| - |Y_s - x| \geq R - r \geq \frac{R}{2}
\]

\[
\mathbb{P}_x(Y_{\sigma^x} \notin B_R(x)) \leq c'' \frac{r^{\alpha}}{R^\alpha}.
\]
holds; in particular, \( B^c_R(x) \subset B^c_{R/2}(Y_*) \). Thus,

\[
\mathbb{P}_x(Y_{\sigma x} \notin B_R(x)) \leq \mathbb{E}_x \left( \int_0^{\sigma^x} \int_{B_{R/2}(Y_*)} \frac{C}{|Y_s-v|^\alpha} \, dv \, ds \right) = \mathbb{E}_x \left( \int_0^{\sigma^x} \int_{|v| \geq R/2} \frac{C}{|v|^\alpha} \, dv \, ds \right) = C'_n \mathbb{E}_x \left( \int_{R/2}^{\infty} \frac{du}{|u|^\alpha} \right) \leq C''_n \mathbb{E}_x \left( \int_{R/2}^{\infty} \frac{du}{|u|^\alpha} \right),
\]

where we used Corollary 3.9 for the last estimate.

Theorem 6.2. Let \( 0 < r < \frac{1}{2} \land \frac{1}{\rho} \) and \( x_0 \in \mathbb{R}^n \). There exist constants \( c < \infty \) and \( \kappa > 0 \) such that for all bounded functions \( h \in \mathbb{R}^n \setminus N \) which are \( A_1 \)-harmonic in \( B_r(x_0) \)

\[
|h(x) - h(y)| \leq c \|h\|_\infty |x - y|^{\kappa}, \quad x, y \in B_{r/2}(x_0).
\]

The constants \( c, \kappa \) are independent of \( 0 < r < \frac{1}{2} \land \frac{1}{\rho}, \) \( x_0 \) and \( h \).

Proof. From Proposition 5.3 we know that for all \( D \subset B_r(x) \) with \( |D| \geq \frac{1}{3} |B_r(x)| \) there exists a constant \( c > 0 \) such that

\[
\mathbb{P}_y(\tau_D < \sigma^x_r) \geq \frac{c}{3}.
\]

Take some \( h \) as in the statement of the theorem; by adding a suitable constant we can achieve that \( h \geq 0 \). Set

\[
M := \|h\|_\infty, \quad \eta^2 := 1 - \frac{c}{12}, \quad \text{and} \quad \theta^\alpha := \frac{c}{24 c''} \sqrt{1 - \frac{c}{12}}
\]

where \( c, c'' \) are the constants from Proposition 5.3 and Lemma 6.1, respectively. Without loss of generality we can assume that \( c'' > \rho \), where \( \rho \) is as in Corollary 3.9. Obviously, \( \eta < 1 \) and we can choose \( c'' \) so large that \( \theta^\alpha/\eta < 1 \) and \( \theta < \frac{1}{\rho} \land \frac{1}{2} \).

Following the idea of Bass and Levin [3] we are going to show that

\[
(32) \quad \sup_{j \in \mathbb{Z}} h(B_{\theta^k}(x) \setminus N) - \inf_{j \in \mathbb{Z}} h(B_{\theta^k}(x) \setminus N) \leq M \eta^k \quad \text{for all} \ k \in \mathbb{Z}.
\]

We write for \( j \in \mathbb{Z} \)

\[
B_j := B_{\theta^j}(x) \setminus N, \quad \sigma_j := \sigma^x_{\theta^j}, \quad a_j := \inf_{j \in \mathbb{Z}} h(B_j), \quad \text{and} \quad b_{j} := \sup_{j \in \mathbb{Z}} h(B_j).
\]

Since \( \eta < 1 \), (32) clearly holds for all negative \( k \in \mathbb{Z} \). For \( k \in \mathbb{N} \) we use induction. Assume that (32) is true for \( 0, 1, 2, \ldots, k \). Fix \( \epsilon > 0 \); from the definition of \( a_{k+1} \) and \( b_{k+1} \) we find some \( y, z \in B_{k+1} \) such that

\[
b_{k+1} - a_{k+1} \leq h(y) - h(z) + \epsilon.
\]
Define \( D' := \{ z \in B_k : h(z) \leq \frac{1}{2}(a_k + b_k) \} \). We may assume that \(|D'| \geq \frac{1}{2}|B_k|\)—otherwise we consider \( M - h \) instead of \( h \). Since Lebesgue measure is inner regular, we can find a compact set \( D \subset D' \) such that \(|D| \geq \frac{1}{2}|B_k|\).

Since \( h \) is harmonic, we use the strong Markov property to deduce

\[
 h(y) - h(z) = \mathbb{E}_y(h(Y_{\sigma_k}) - h(z))
 = \mathbb{E}_y(h(Y_{\sigma_k}) - h(z); \sigma_k \geq \tau_D) + \mathbb{E}_y(h(Y_{\sigma_k}) - h(z); \sigma_k < \tau_D)
 = \mathbb{E}_y(h(Y_{\tau_D}) - h(z); \sigma_k \geq \tau_D) + \mathbb{E}_y(h(Y_{\sigma_k}) - h(z); \sigma_k < \tau_D)
 = \mathbb{E}_y(h(Y_{\tau_D}) - h(z); \sigma_k \geq \tau_D) + \mathbb{E}_y(h(Y_{\sigma_k}) - h(z); \sigma_k < \tau_D, Y_{\sigma_k} \in B_{k-1})
 + \sum_{j=1}^{\infty} \mathbb{E}_y(h(Y_{\sigma_k}) - h(z); \sigma_k < \tau_D, Y_{\sigma_k} \in B_{k-j-1} \setminus B_{k-j}) .
\]

By the very definition of the set \( D \), the first term on the right is bounded by

\[
 \left( \frac{1}{2}(a_k + b_k) - a_k \right) \mathbb{P}_y(\tau_D < \sigma_k) = \frac{1}{2}(b_k - a_k) \mathbb{P}_y(\tau_D < \sigma_k).
\]

The second term is less or equal than

\[
 (b_{k-1} - a_{k}) \mathbb{P}_y(\sigma_k \leq \tau_D) \leq (b_{k-1} - a_{k-1}) (1 - \mathbb{P}_y(\sigma_k > \tau_D)) .
\]

Using the induction assumption and Lemma 6.1 we find that the third term is dominated by

\[
 \sum_{j=1}^{\infty} (b_{k-j-1} - a_{k-j-1}) \mathbb{P}_y(Y_{\sigma_k} \notin B_{k-j}) \leq \sum_{j=1}^{\infty} M \eta^{k-j-1} c'' \frac{\theta^\alpha}{\theta^{(k-j)\alpha}} = c'' M^k \sum_{j=1}^{\infty} \left( \frac{\theta^\alpha}{\eta} \right)^j
 = c'' M \frac{\theta^\alpha}{\eta} \frac{1}{1 - \frac{\theta^\alpha}{\eta}} .
\]

Since \( \frac{\theta^\alpha}{\eta} \leq \frac{1}{2} \), the last fraction is less than 2. Bearing in mind that \( \eta \leq 1 \) and that \( \mathbb{P}_y(\tau_D < \sigma_k) > \frac{\epsilon}{3} \), we find altogether

\[
 h(y) - h(z) \leq \frac{1}{2} M \eta^k \mathbb{P}_y(\tau_D < \sigma_k) + M \eta^{k-1} \left( 1 - \mathbb{P}_y(\tau_D < \sigma_k) \right) + 2 c'' M \eta^{k-1} \frac{\theta^\alpha}{\eta}
 \leq M \eta^{k-1} \left( 1 - \frac{\epsilon}{6} \right) + 2 c'' M \eta^{k-1} \frac{\theta^\alpha}{\eta} ,
\]

and inserting the definition of \( \frac{\theta^\alpha}{\eta} \) we get

\[
 h(y) - h(z) \leq M \eta^{k-1} \left( 1 - \frac{\epsilon}{6} + \frac{\epsilon}{12} \right) = M \eta^{k-1} \eta^2 .
\]

This finishes the induction step.

The rest is now routine: if \( x, y \in B_r(x_0) \setminus N \), let \( k \in \mathbb{Z} \) be the smallest integer such that \( \theta^{k+1} \leq |x - y| < \theta^k \). Then \( \log |x - y| \geq (k + 1) \log \theta \), \( y \in B_{\theta^k}(x) \), and

\[
 h(x) - h(y) \leq M \eta^k = Me^{k \log \eta} \leq Me^{(\log |x-y|/|x-y|) \log \eta} = \frac{M}{\eta^2} |x - y|^{\log \eta/\log \rho} .
\]
Recall that the resolvent of $A_1$ is given by

$$R_{\lambda}^{A_1} f(x) = \int_0^\infty e^{-\lambda t} T_t^{A_1} f(x) \, dt = \mathbb{E}_x \left( \int_0^\infty e^{-\lambda t} f(Y_t) \, dt \right), \quad f \in L^\infty(\mathbb{R}^n),$$

where the integrals converge for all $\lambda > 0$.

**Theorem 6.3.** For every compact set $K$ there exist constants $C < \infty$ and $\kappa > 0$ such that for every $\lambda > 0$ the resolvent $R_{\lambda}^{A_1} f, f \in L^\infty(\mathbb{R}^n)$, is Hölder continuous:

$$|R_{\lambda}^{A_1} f(x) - R_{\lambda}^{A_1} f(y)| \leq C \left( 1 + \frac{1}{\lambda} \right) \|f\|_\infty |x - y|^{\kappa};$$

for all $x, y \in K \setminus N$.

**Proof.** Without loss of generality we can assume that $x, y \in K \setminus N$ are so close together that $r := 3|x - y| \leq \frac{1}{3} \wedge \frac{1}{\lambda}$. (If $x$ and $y$ are further apart, we can link them with a finite chain of neighbouring intermediate points.) Fix $\lambda > 0$ and $f \in L^\infty(\mathbb{R}^n)$. An application of the strong Markov property shows for $z = x, x \notin N, y \in B_r(x) \setminus N$

$$R_{\lambda}^{A_1} f(z) = \mathbb{E}_z \left( \int_0^{\sigma^z_x} e^{-\lambda s} f(Y_s) \, ds \right) + \mathbb{E}_z \left( e^{-\lambda \sigma^z_x} R_{\lambda}^{A_1} f(Y_{\sigma^z_x}) \right) + \mathbb{E}_z \left( \int_0^{\sigma^z_x} e^{-\lambda s} f(Y_s) \, ds \right) + \mathbb{E}_z \left( \left( e^{-\lambda \sigma^z_x} - 1 \right) R_{\lambda}^{A_1} f(Y_{\sigma^z_x}) \right) + \mathbb{E}_z \left( R_{\lambda}^{A_1} f(Y_{\sigma^z_x}) \right).$$

A further application of the strong Markov property reveals that the last term, $z \mapsto \mathbb{E}_z \left( R_{\lambda}^{A_1} f(Y_{\sigma^z_x}) \right)$ is $A_1$-harmonic in $B_r(x) \setminus N$. Using the elementary estimates $|e^{-\lambda s} - 1| \leq \lambda s$ and $|e^{-\lambda s}| \leq 1$, we get from Theorem 6.2 and Corollary 3.9 that

$$|R_{\lambda}^{A_1} f(x) - R_{\lambda}^{A_1} f(y)| \leq 2\|f\|_\infty \max_{z = x, y} \mathbb{E}_z \sigma^z_x + 2\lambda \|R_{\lambda}^{A_1} f\|_\infty \max_{z = x, y} \mathbb{E}_z \sigma^z_x + c \|R_{\lambda}^{A_1} f\|_\infty |x - y|^{\kappa}$$

$$\leq 4 \|f\|_\infty r^\alpha + c \|f\|_\infty |x - y|^{\kappa}$$

$$\leq C \left( 1 + \frac{1}{\lambda} \right) (r^\alpha + |x - y|^\kappa) \|f\|_\infty.$$

Since $r = 3|x - y|$ we finally arrive at

$$|R_{\lambda}^{A_1} f(x) - R_{\lambda}^{A_1} f(y)| \leq C' \left( 1 + \frac{1}{\lambda} \right) \|f\|_\infty |x - y|^{\alpha + \kappa},$$

and the theorem follows. 

The following result has been obtained by Komatsu [23] for non-degenerate Lévy-kernels of the form $k(x, y)|x - y|^{-\alpha - n}$ where $0 < c_1 \leq k(x, y) \leq c_2 < \infty$ ($k$ may be even time-dependent) using pseudo-differential operator methods and a smoothing technique for non-smooth kernels $k$. 

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Corollary 6.4. The semigroups \( T_t^{A_1} \) and \( T_t^{A} \) have modifications \( \tilde{T}_t^{A_1} \) and \( \tilde{T}_t^{A} \) which are Feller semigroups. In particular, we can take the exceptional set \( N = \emptyset \) and both \( (X_t)_{t \geq 0} \) and \( (Y_t)_{t \geq 0} \) are everywhere defined processes.

Proof. Theorem 6.3 shows that \( R_\lambda^{A_1} \) has a modification \( \tilde{R}_\lambda^{A_1} \) which has the strong Feller property, i.e., which maps \( B_b(\mathbb{R}^n) \) to \( C_b(\mathbb{R}^n) \). Thus, the Feller property of (a modification of) \( T_t^{A} \) follows from Corollary 4.4. The claim for \( T_t^{A_1} \) follows if we take \( A = A_1 \) in the first place.

Corollary 6.5. If \( h \in L^\infty(\mathbb{R}^n) \cap \mathcal{D}(A) \) is such that \( Ah \in L^\infty(\mathbb{R}^n) \)—e.g., if \( h \) is \( A \)-harmonic on \( \mathbb{R}^n \)—, then \( h \in C_b(\mathbb{R}^n) \) and even Hölder continuous.

Proof. Set \( g = Ah \). Since \( B = A - A_1 \) is a bounded operator in \( L^\infty(\mathbb{R}^n) \), cf. (29), we see that

\[
g = Ah = A_1 h + Bh \implies h - A_1 h = h - g + Bh.
\]

Applying \( R_1^{A_1} \) on both sides of this equality we see that

\[
h = R_1^{A_1}(h - g + Bh);
\]

since \( h - g + Bh \in L^\infty(\mathbb{R}^n) \), the claim follows from the strong Feller property of the resolvent operator \( R_1^{A_1} \). In view of Theorem 6.3 we have even Hölder continuity.

7 Concluding remarks

We have mentioned several times, cf. Remark 3.10 and 5.4, that an obvious extension of the present paper would be to use the full strength of the estimate (21) and to relax the condition \( \alpha = \beta \); this could be achieved by techniques similar to the ones used in [33] (on the level of symbols) or the paper by Pruitt [30] which uses the Lévy characteristics.

Another possibility would be to use the perturbation technique of Section 4. In the present paper we have always assumed that \( A - A_1 \) is a bounded operator in \( L^\infty(\mathbb{R}^n) \); this is equivalent to consider only perturbations of the large jump part of \( j(x, y) \) where \( |x - y| > 1 \), say. In fact, all arguments in §4 and §7 only require that \( (A - A_1)R_\lambda^{A_1} \) is a bounded operator on \( L^\infty(\mathbb{R}^n) \). The following lemma shows what this actually means.

Lemma 7.1. Assume that \( (A, \mathcal{D}(A)) \) and \( (A_1, \mathcal{D}(A_1)) \) are infinitesimal generators of strongly continuous contraction semigroups on the Banach space \( (X, \|\cdot\|) \). Then

\[
(A - A_1)R_\lambda^{A_1} \text{ is bounded in } X \iff \mathcal{D}(A) \subset \mathcal{D}(A_1).
\]
Proof. By the second resolvent equation we have
\[ R_\lambda^{A_1} - R_\lambda^A = R_\lambda^{A_1} (A_1 - A) R_\lambda^A \quad \text{or} \quad R_\lambda^A u = R_\lambda^{A_1} (u + (A_1 - A) R_\lambda^A u). \]

If \((A_1 - A) R_\lambda^A : X \to X\) is bounded, then
\[ \mathcal{D}(A) = R_\lambda^A X \subset R_\lambda^{A_1} X \subset \mathcal{D}(A_1). \]

Conversely, if \(\mathcal{D}(A) \subset \mathcal{D}(A_1)\), we can use a theorem on the comparison of closed operators due to Hörmander, cf. Yosida [42, II.6, Theorem 2], and conclude that
\[ \|A_1 u\| \leq C \left( \|Au\| + \|u\| \right), \quad u \in \mathcal{D}(A). \]

Since \(\|AR_\lambda^A\| = |1 - \lambda R_\lambda^A| \leq 2\), we have
\[ \|A_1 R_\lambda^A u\| \leq C \left( \|AR_\lambda^A u\| + \|R_\lambda^A u\| \right) \leq C \left( 2 + \frac{1}{\lambda} \right) \|u\|, \]
and so,
\[ \|(A - A_1) R_\lambda^A u\| \leq C \left( \|AR_\lambda^A u\| + \|AR_\lambda^{A_1} u\| \right) \leq C_\lambda \|u\|. \]

A typical application of the above lemma would be a perturbation of some \(j(x, y)\) which satisfies conditions (10) with \(\alpha = \beta\) by \(\tilde{j}(x, y)1_{\{|x-y|<1\}}(x - y) \sim |x - y|^{-n-\tilde{\alpha}}\), where \(0 < \tilde{\alpha} < \alpha\). (Note that both the original and the perturbed operator will generate both \(L^2\)- and Feller semigroups.) This covers, e.g., jump kernels considered by Song and Vondraček [38].

References


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