Packing dimension profiles and Lévy processes

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Abstract

1. Introduction
Let $X := \{X(t)\}_{t \geq 0}$ be a Lévy process in $\mathbb{R}^d$; that is, $X(0) = 0$, $X$ has stationary and independent increments, and the random function $t \mapsto X(t)$ is almost surely (a.s.) right continuous with left limits [1, 2, 7, 18, 22].

Let $F$ be a nonrandom Borel subset of $\mathbb{R}_+ := [0, \infty)$. It has been known for a long time that the random image set $X(F)$ frequently exhibits fractal structure. And there is a substantial literature that computes the Hausdorff dimension $\dim_H X(F)$ of $X(F)$; see [11] and its extensive bibliography.

Let $\dim_p$ denote the packing dimension. The main goal of the present paper is to evaluate $\dim_p X(F)$ in terms of the geometry of $F$. In order to accomplish this, we shall introduce and study a new family of dimensions related to the set $F$. Those dimensions are inherently probabilistic, but they have analytic significance as well. In fact, they can be viewed as an extension of the notion of packing dimension profiles, as introduced by Falconer and Howroyd [6] to study the packing dimension of orthogonal projections; see also Howroyd [10].

There exists an extensive body of literature related to the Hausdorff dimension $\dim_H X(F)$, but only a few papers study the packing dimension $\dim_p X(F)$. Let us point out two noteworthy cases where $\dim_p X(F)$ has been computed successfully in different settings:

Case 1. When $\dim_H F = \dim_p F$, covering arguments can frequently be used to compute $\dim_p X(F)$. In those cases, the packing and Hausdorff dimensions of $X(F)$ generally agree.

Case 2. When $X$ has statistical self-similarities, one can sometimes appeal to scaling arguments in order to compute $\dim_p X(F)$ solely in terms of $\dim_p F$.

For an example of the more interesting Case 2, consider the situation where $X$ is an isotropic stable Lévy process on $\mathbb{R}^d$ with index $\alpha \in (0, 2]$, that is, the case where the characteristic function is $E \exp(i \xi \cdot X(t)) = \exp(-\text{const} \cdot t \|\xi\|^\alpha)$ simultaneously for all $t > 0$ and $\xi \in \mathbb{R}^d$. Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Then a theorem of Perkins and Taylor [21] implies that if $d \geq \alpha$, then

$$\dim_p X(F) = \alpha \dim_p F \quad \text{a.s.}$$

(1.1)

For related works see also [20, 26]. Up to now, the case $d < \alpha$ has remained open, except when $\alpha = 2$ and $d = 1$, that is, $X$ is linear Brownian motion. In that case, a more general theorem of Xiao [25] implies that $\dim_p X(F)$ cannot, in general, be described by $\dim_p F$; in fact, Xiao’s
Theorem implies that, when \( X \) is linear Brownian motion,
\[
\dim P X(F) = 2 \dim_{1/2}^{FH} F \quad \text{a.s.,}
\]
where \( \dim_{s}^{FH} F \) denotes the \( s \)-dimensional packing dimension profile of Falconer and Howroyd [6]. The complexity of the preceding formula can be appreciated better in light of an example of Talagrand and Xiao [23] which shows that there are sets \( F \) such that:

(i) \( \dim P X(F) \neq \dim H F \) and
(ii) \( \dim P X(F) \) cannot be described solely in terms of simple-to-understand quantities such as \( \dim H F \) and \( \dim P F \).

The main goal of this paper is to introduce a new family of dimension profiles; this family extends the packing dimension profiles of Falconer and Howroyd [6]; see Remark 2.8. We use the dimension profiles of this paper to compute \( \dim P X(F) \) for a general Lévy process \( X \) and an arbitrary nonrandom Borel set \( F \subset \mathbb{R}^+ \). En route we also establish a novel formula for \( \dim M X(F) \), where \( \dim M \) denotes the upper Minkowski dimension.

In order to understand our forthcoming identities better, let us mention three corollaries of the general results of this paper.

**Corollary 1.1.** Let \( X \) denote a \( d \)-dimensional isotropic stable Lévy process with index \( \alpha \in (0, 2] \) and \( F \subseteq \mathbb{R}^+ \) be nonrandom and Borel measurable. Then
\[
\dim P X(F) = \alpha \dim_{d/\alpha}^{FH} F \quad \text{a.s.}
\]

The preceding includes both (1.1) and (1.2) as special cases. Indeed, one obtains (1.1) because \( \dim_{s}^{FH} F = \dim P F \) when \( s \geq 1 \) (see [6]). And one obtains (1.2) by setting \( d = 1 \) and \( \alpha = 2 \).

In order to describe our next two corollaries, let us recall that a stochastic process \( S := \{S(t)\}_{t \geq 0} \) is a subordinator if \( S \) is a one-dimensional Lévy process such that the random function \( t \mapsto S(t) \) is nondecreasing. Also recall that the Laplace exponent \( \Phi \) of \( S \) is given by \( E e^{-\lambda S(t)} = e^{-t \Phi(\lambda)} \) for every \( t, \lambda > 0 \); see Bertoin [3] for more detailed information about subordinators and their remarkable properties.

**Corollary 1.2.** Suppose that \( S \) is a subordinator with Laplace exponent \( \Phi \), and \( F \subseteq \mathbb{R}^+ \) is nonrandom and Borel measurable. Then, a.s.,
\[
\overline{\dim}_M S(F) = \sup \left\{ \eta > 0 : \lim_{\lambda \to \infty} \lambda^{\eta} \inf_{\nu \in \mathcal{P}_{c}(F)} \left[ \int_{[0, \infty)} e^{-|t-s|\Phi(\lambda)} \nu(ds)\nu(dt) = 0 \right] \right\},
\]
where \( \mathcal{P}_{c}(F) \) denotes the collection of all compactly supported Borel probability measures \( \nu \) such that \( \nu(F) = 1 \).

Consider Corollary 1.2 in the case where \( F \) is an interval; say \( F := [0, 1] \). Then it is intuitively plausible, and possible to prove rigorously, that the minimizing measure \( \nu \) in the infimum ‘\( \inf_{\nu \in \mathcal{P}_{c}(F)} \)’ is the Lebesgue measure on \([0, 1]\). A direct calculation then implies that the convergence condition of (1.4) holds if, and only if, \( \lambda^{\eta} = o(\Phi(\lambda)) \) as \( \lambda \uparrow \infty \). Therefore, Corollary 1.2 yields the following elegant a.s. identity:
\[
\overline{\dim}_M S([0,1]) = \sup \left\{ \eta > 0 : \lim_{\lambda \uparrow \infty} \frac{\Phi(\lambda)}{\lambda^{\eta}} = \infty \right\} \quad \text{a.s.}
\]
And a Baire-category argument can be used to prove that the same formula holds if we replace \( \dim M_S([0,1]) \) by \( \dim P_S([0,1]) \); see also (2.32). The preceding formulas for \( \dim M_S([0,1]) \) and \( \dim P_S([0,1]) \) were derived earlier, using covering arguments; see Fristedt and Taylor \[8\] and Bertoin \[3, Lemma 5.2, p. 41\].

We are not aware of any nontrivial examples of deterministic sets with explicitly known Falconer–Howroyd packing dimension profiles. Remarkably, our third and final corollary computes the Falconer–Howroyd packing dimension profiles of a quite general ‘Markov random set’ in the sense of Krylov and Juškevič \[16, 17\]; see also Hoffmann-Jørgensen \[9\] and Kingman \[15\]. According to a deep result of Maisonneuve \[19\], a Markov random set is the closure of \( S(\mathbb{R}_+) \), where \( S \) is a subordinator. It is not hard to see that any reasonable dimension of the closure of \( S(\mathbb{R}_+) \) is a.s. the same as the same dimension of \( S([0,1]) \). Therefore, our next corollary concentrates on computing many of the Falconer–Howroyd packing dimension profiles of \( S([0,1]) \).

**Corollary 1.3.** If \( S \) is a subordinator with Laplace exponent \( \Phi \), then for every \( s \geq \frac{1}{2} \),

\[
\dim_{FH}^s S([0,1]) = s(1 - \theta) \quad \text{a.s.,} \tag{1.6}
\]

where

\[
\theta := \lim_{\lambda \to \infty} \frac{1}{\log \lambda} \log \left( \int_1^\lambda \frac{dx}{\Phi(x^{1/s})} \right). \tag{1.7}
\]

**Remark 1.4.** The preceding should be compared with the following: With probability 1:

\[
\dim M S([0,1]) = \lim_{\lambda \to \infty} \frac{\log \Phi(\lambda)}{\log \lambda} \quad \text{and} \quad \dim H S([0,1]) = \lim_{\lambda \to \infty} \frac{\log \Phi(\lambda)}{\log \lambda}. \tag{1.8}
\]

See, for example, Fristedt and Taylor \[8\], as well as Bertoin \[3, Lemma 5.2, p. 41, Corollary 5.3, p. 42\]. See also (1.5) and see \[13\] for very general results.

## 2. Analytic preliminaries and the main result

Throughout, we assume that the reader is familiar with the upper Minkowski dimension and the packing dimension, as described in the excellent book by Falconer \[5\], for example. Furthermore, we will not recall the definitions of packing dimension profiles of Falconer and Howroyd \[6\] and those of Howroyd \[10\]. For a brief review of those definitions, we instead refer the reader to Khoshnevisan and Xiao \[12\], where it is also shown that these two packing dimension profiles coincide. See Howroyd \[10\] for a special case of the latter result.

In this section, we introduce a family of generalized packing dimension profiles that are associated to Lévy processes in a natural way. As we shall see later, our profiles include the packing dimension profiles of Falconer and Howroyd \[6\] and Howroyd \[10\].

### 2.1. Packing dimension profiles

Recall that \( X := \{X(t)\}_{t \geq 0} \) is an arbitrary Lévy process on \( \mathbb{R}^d \). If \( |y| := \max_{1 \leq j \leq d} |y_j| \) designates the \( \ell^\infty \) norm of a vector \( y \in \mathbb{R}^d \), then we may consider the family

\[
\kappa := \{\kappa_\epsilon\}_{\epsilon \geq 0} \tag{2.1}
\]

of functions that are defined by

\[
\kappa_\epsilon(t) := P\{X(t) \in B(0,\epsilon)\} \quad \text{for all } \epsilon, t \geq 0. \tag{2.2}
\]
Here and throughout $B(x, r) := \{z \in \mathbb{R}^d : |z - x| < r\}$ denotes the open $\ell^\infty$ ball of radius $r > 0$ about $x \in \mathbb{R}^d$.

One can see at once that $\kappa_\epsilon(t)$ is continuous in $t$ for every fixed $\epsilon$, and nondecreasing in $\epsilon$ for every fixed $t$.

**Definition 2.1.** We define the box-dimension profile $\overline{\dim}_\kappa F$ of a Borel set $F \subseteq \mathbb{R}$ with respect to the family $\kappa$ as follows:

$$\overline{\dim}_\kappa F := \sup \left\{ \eta > 0 : \lim_{\epsilon \to 0} \inf_{\nu \in \mathcal{P}(F)} \int \frac{\kappa_\epsilon(|t - s|)}{\epsilon^\eta} \nu(ds)\nu(dt) = 0 \right\},$$

(2.3)

where $\nu \in \mathcal{P}(F)$ if, and only if, $\nu$ is a Borel probability measure on $\mathbb{R}$ such that $\nu(F) = 1$.

It is possible to express $\overline{\dim}_\kappa F$ in potential-theoretic terms. Indeed,

$$\overline{\dim}_\kappa F = \lim_{\epsilon \to 0} \frac{\log Z_\kappa(\epsilon)}{\log \epsilon},$$

(2.4)

where $Z_\kappa(\epsilon)$ is the minimum $\kappa_\epsilon$-energy of $\nu \in \mathcal{P}(F)$; namely,

$$Z_\kappa(\epsilon) := \inf_{\nu \in \mathcal{P}(F)} \int \kappa_\epsilon(|t - s|) \nu(ds)\nu(dt).$$

(2.5)

Equation (2.4) is reminiscent of, but not the same as, Howroyd’s upper box-dimension with respect to a kernel [10].

The box-dimension profiles $\overline{\dim}_\kappa$ can be regularized in order to produce a proper family of packing-type dimensions.

**Definition 2.2.** We define the packing dimension profile $\dim_\kappa F$ of an arbitrary set $F \subseteq \mathbb{R}$ with respect to the family $\kappa$ as follows:

$$\dim_\kappa F := \inf \sup_{n \geq 1} \overline{\dim}_\kappa F_n,$$

(2.6)

where the infimum is taken over all countable coverings of $F$ by bounded Borel sets $F_1, F_2, \ldots$.

One can verify from this definition that $\dim_\kappa$ has the following properties that are expected to hold for any reasonable notion of ‘fractal dimension’:

(i) $\dim_\kappa$ is monotone; namely, $\dim_\kappa F \leq \dim_\kappa G$ whenever $F \subseteq G$;

(ii) $\dim_\kappa$ is $\sigma$-stable; namely, $\dim_\kappa \bigcup_{n=1}^{\infty} G_n = \sup_{n \geq 1} \dim_\kappa G_n$.

We skip the verification of these properties, as they require routine arguments.

2.2. A relation to a family of packing measures

Next we outline how $\dim_\kappa$ can be associated to a packing measure with respect to the family $\kappa$, where $\kappa$ was defined in (2.1) and (2.2).

**Definition 2.3.** Fix a set $F \subseteq \mathbb{R}$ and a number $\delta > 0$. We say that a sequence $\{(w_j, t_j, \epsilon_j) : j \geq 1\}$ of triplets is a $(\kappa, \delta)$-packing of $F$ if, for all $j \geq 1$: (a) $w_j \geq 0$; (b) $t_j \in F$; (c) $\epsilon_j \in (0, \delta)$ and (d) $\sum_{i=1}^{\infty} w_i \kappa_\epsilon(|t_i - t_j|) \leq 1$. 
The preceding general definition is modelled after the ideas of Howroyd [10], and leads readily to packing measures. Indeed, we have the following.

**Definition 2.4.** For a given constant $s > 0$, we define the $s$-dimensional packing measure $\mathcal{P}^{s,\kappa}(F)$ of $F \subset \mathbb{R}$ with respect to the family $\kappa$ as

$$\mathcal{P}^{s,\kappa}(F) := \inf_{n \geq 1} \sup \mathcal{P}^{s,\kappa}_0(F_n),$$

(2.7)

where the infimum is taken over all bounded Borel sets $F_1, F_2, \ldots$ such that $F \subseteq \bigcup_{n=1}^{\infty} F_n$, and $\mathcal{P}^{s,\kappa}_0$ denotes a so-called premeasure that is defined by

$$\mathcal{P}^{s,\kappa}_0(F) := \lim_{\delta \downarrow 0} \left( \sup \sum_{j=1}^{\infty} w_j \epsilon_j^s \right),$$

(2.8)

where the supremum is taken over all $(\kappa, \delta)$-packings $\{(w_j, t_j, \epsilon_j) : j \geq 1\}$ of $F$ and $\sup \emptyset := 0$, as usual.

**Definition 2.5.** We define the packing dimension $\text{P-dim}_\kappa F$ of $F \subset \mathbb{R}$ with respect to the family $\kappa$ as

$$\text{P-dim}_\kappa F := \inf \{s > 0 : \mathcal{P}^{s,\kappa}(F) = 0\}.$$

(2.9)

It is possible to adapt the proof of Theorem 26 of Howroyd’s paper [10] and show that our two packing dimension profiles coincide. That is,

$$\dim_\kappa F = \text{P-dim}_\kappa F \quad \text{for all} \quad F \subset \mathbb{R}. \quad (2.9)$$

We omit the proof, as it requires only an adaptation of ideas of Howroyd [10, Proof of Theorem 26] to the present, more general, setting.

### 2.3. A relation to harmonic analysis

There are time-honoured, as well as deep, connections between Hausdorff measures and harmonic analysis. In this section, we establish a useful harmonic-analytic result about the packing measures of this section.

Let $\Psi$ denote the characteristic exponent of $X$, normalized so that

$$E e^{iz \cdot X(t)} = e^{-t \Psi(z)} \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad z \in \mathbb{R}^d. \quad (2.10)$$

For every Borel probability measure $\nu$ on $\mathbb{R}$, and, for all $z \in \mathbb{R}$, define the energy form:

$$\mathcal{E}_\nu(z) := \iint \exp(-|t - s|\Psi(\text{sgn}(t - s)z))\nu(ds)\nu(dt). \quad (2.11)$$

Note that

$$0 \leq \mathcal{E}_\nu(z) \leq 1 \quad \text{for all} \quad z \in \mathbb{R}^d \quad \text{and} \quad \nu \in \mathcal{P}(F). \quad (2.12)$$

This can be seen from the following computation:

$$\mathcal{E}_\nu(z) = \iint E|e^{iz \cdot (X(t) - X(s))}|\nu(ds)\nu(dt) = E \left( \iint |e^{iz \cdot X(t)}\nu(dt)|^2 \right), \quad (2.13)$$

where we used Fubini’s theorem to interchange the order of the integrals. This proves that $\mathcal{E}_\nu$ is real-valued and positive; the fact that $\mathcal{E}_\nu(z) \leq 1$ is now obvious.
THEOREM 2.6. For every compact set \( F \subset \mathbb{R}_+ \),

\[
\overline{\text{Dim}}_\kappa F = \sup \left\{ \eta > 0 : \lim_{\epsilon \downarrow 0} \inf_{\nu \in \mathcal{P}(F)} \int_{\mathbb{R}^d} \frac{e^{-\eta \mathcal{E}_\nu(z/\epsilon)}}{\prod_{j=1}^{d}(1 + z_j^2)} \, dz = 0 \right\}. \tag{2.14}
\]

Proof. We apply a variation of the Cauchy semigroup argument of Khoshnevisan and Xiao [13, Proof of Theorem 1.1]. Define, for all \( \epsilon > 0 \), the (scaled) Pólya distribution

\[
P_\epsilon(x) := \prod_{j=1}^{d} \left(1 - \cos(2\pi x_j)\right) 
\tag{2.15}
\]

Then it is well known, as well as elementary, that

\[
\hat{P}_\epsilon(\xi) = \prod_{j=1}^{d} \left(1 - \frac{|\xi_j|}{2\epsilon}\right)^+ \quad \text{for all } \xi \in \mathbb{R}, \tag{2.16}
\]

where \( a^+ := \max(a, 0) \) for all real numbers \( a \), and \( \hat{f} \) denotes the Fourier transform of \( f \) normalized so that \( \hat{f}(z) = \int_{\mathbb{R}^d} f(x) \exp(i x \cdot z) \, dx \) for all integrable functions \( f : \mathbb{R}^d \to \mathbb{R} \). If \( z \in B(0, \epsilon) \), then \( 1 - (2\epsilon)^{-1}|z_j| \geq \frac{1}{2} \). Consequently, \( 1_{B(0,\epsilon)}(z) \leq 2^d \hat{P}_\epsilon(z) \) for all \( z \in \mathbb{R}^d \). Set \( z := X(t) \) and take expectations in the preceding inequality to find that, for all \( \epsilon > 0 \) and \( t \geq 0 \),

\[
\kappa_\epsilon(t) \leq 2^d E[\hat{P}_\epsilon(X(t))] = 2^d \int_{\mathbb{R}^d} \hat{P}_\epsilon(z) P(X(t) \in dz). \tag{2.17}
\]

We can apply Plancherel’s identity to the right-hand side of this inequality and deduce that

\[
\kappa_\epsilon(t) \leq 2^d \int_{\mathbb{R}^d} P_\epsilon(y) e^{-t\Psi(y)} \, dy \quad \text{for all } \epsilon > 0, \ t \geq 0. \tag{2.18}
\]

On the other hand, if \( t < 0 \), then we use the Lévy process \(-X(-t)\) in place of \( X(t) \) to deduce that \( \kappa_\epsilon(-t) \leq 2^d E[\hat{P}_\epsilon(-X(-t))] \), whence it follows that

\[
\kappa_\epsilon(-t) \leq 2^d \int_{\mathbb{R}^d} P_\epsilon(y) e^{t\Psi(-y)} \, dy \quad \text{for all } \epsilon > 0 \text{ and } t < 0. \tag{2.19}
\]

Consequently, the following holds for all \( \epsilon > 0 \) and \( t \in \mathbb{R} \):

\[
\kappa_\epsilon(|t|) \leq 2^d \int_{\mathbb{R}^d} P_\epsilon(y) \exp(-|t|\Psi(\text{sgn}(t)y)) \, dy. \tag{2.20}
\]

Define \( f_C \) to be the standard Cauchy density on \( \mathbb{R}^d \); that is,

\[
f_C(z) := \pi^{-d} \prod_{j=1}^{d} (1 + z_j^2)^{-1} \quad \text{for every } z := (z_1, \ldots, z_d) \in \mathbb{R}^d. \tag{2.21}
\]

Because of the elementary inequality

\[
\frac{1 - \cos(2u)}{2\pi u^2} = \frac{\sin^2 u}{\pi u^2} \leq \frac{1}{1 + u^2}, \tag{2.22}
\]

valid for all nonzero \( u \), it follows that \( P_\epsilon(y) \leq (\pi \epsilon)^d f_C(y) \) for all \( y \in \mathbb{R}^d \). Thus, (2.20) and a change of variables imply that

\[
\int \kappa_\epsilon(|t - s|) \nu(dt) \nu(ds) \leq (2\pi)^d \int_{\mathbb{R}^d} f_C(z) \mathcal{E}_\nu \left(\frac{z}{\epsilon}\right) \, dz. \tag{2.23}
\]

Thus, every \( \eta \) that is smaller than the right-hand side of (2.14) also satisfies \( \eta < \overline{\text{Dim}}_\kappa F \). Consequently, \( \overline{\text{Dim}}_\kappa F \) is larger than or equal to the supremum that appears in (2.14).
Let us now establish the converse estimate. After enlarging the underlying probability space if necessary, we can introduce a Cauchy process $C := \{C(t)\}_{t \geq 0}$, independent of $X$, whose coordinate processes $C_1, \ldots, C_d$ are independent identically distributed standard symmetric Cauchy processes on the line. For every $\epsilon > 0$, $k \geq 1$ and $x \in \mathbb{R}^d$,
\[ E[x^* \cdot C(k/\epsilon)] - e^{-k} = e^{-k|x|/\epsilon} - e^{-k} \leq e^{-k}(e^{-k(|x|/\epsilon^2) - 1}) \leq 1_{B(0,\epsilon)}(x). \] (2.24)
In the above, $|x| = \sum_{j=1}^{d} |x_j|$ is the $\ell^1$ norm of $x \in \mathbb{R}^d$.

If $t \geq 0$, then we set $x := X(t)$ in (2.24) and take expectations to find that
\[ \kappa_{\epsilon}(t) \geq E[\exp(iX(t) \cdot C(k/\epsilon))] - e^{-k} = E[\exp(-t\Psi(C(k/\epsilon))) - e^{-k}]
= \int_{\mathbb{R}^d} f_{C}(z) e^{-t\Psi(kz/\epsilon)} \, dz - e^{-k}. \] (2.25)

If $t < 0$, then a similar calculation with $x := -X(-t)$ in place of $X(t)$ yields
\[ \kappa_{\epsilon}(-t) \geq E[\exp(-iX(-t) \cdot C(k/\epsilon))] - e^{-k} = \int_{\mathbb{R}^d} f_{C}(z) e^{t\Psi(-kz/\epsilon)} \, dz - e^{-k}. \] (2.26)

Thus,
\[ \kappa_{\epsilon}(|t|) \geq \int_{\mathbb{R}^d} f_{C}(z) \exp(-|t|\Psi(\text{sgn}(t)kz/\epsilon)) \, dz - e^{-k} \quad \text{for all } t \in \mathbb{R}. \] (2.27)

We choose $k := \epsilon^{-\delta}$, where $\delta$ is positive but arbitrarily small, replace $|t|$ by $|t-s|$ and integrate with respect to $\nu(dt)\nu(ds)$ to find that
\[ \left\| \kappa_{\epsilon}(|t-s|)\nu(ds)\nu(dt) \right\| \geq \int_{\mathbb{R}^d} f_{C}(z) \mathcal{E}_\nu \left( \frac{z}{\epsilon^{1+\delta}} \right) \, dz - \exp \left( - \left[ \frac{1}{\epsilon^{1+\delta}} \right]^{1+\delta} \right). \] (2.28)

If $\eta < \text{Dim}_\kappa F$, then
\[ \lim_{\epsilon \downarrow 0} \inf_{\nu \in \mathcal{P}(F)} \left\| \kappa_{\epsilon}(|t-s|)\nu(ds)\nu(dt) \right\| = 0, \] (2.29)
and hence the preceding discussion implies that
\[ \lim_{h \downarrow 0} \inf_{\nu \in \mathcal{P}(F)} h^{-\eta/(1+\delta)} \int_{\mathbb{R}^d} f_{C}(z) \mathcal{E}_\nu (z/h) \, dz = 0. \] (2.30)
That is, $\eta/(1+\delta)$ is smaller than the supremum that appears on the right-hand side of (2.14). This implies that the right-hand side of (2.14) is less than or equal to $\text{Dim}_\kappa F/(1+\delta)$, which entails the theorem because $\delta$ is arbitrary.

2.4. The main result and proofs of corollaries

**Theorem 2.7.** Let $X := \{X(t)\}_{t \geq 0}$ denote a Lévy process in $\mathbb{R}^d$ and let $\kappa$ be defined by (2.1) and (2.2). Then, for all nonrandom bounded Borel sets $F \subseteq \mathbb{R}_+$,
\[ \overline{\text{dim}}_M X(F) = \overline{\text{dim}}_\kappa F \quad \text{a.s. and} \]
\[ \text{dim}_P X(F) = \text{dim}_\kappa F \quad \text{a.s.} \] (2.31) (2.32)

Theorem 2.7 is proved in the Section 3. In the remaining part of this section, we apply Theorem 2.7 in order to verify the three corollaries (Corollaries 1.1–1.3) that were mentioned earlier in Section 1.

**Proof of Corollary 1.1.** It is well known that, for all $T > 0$, there exist positive and finite constants $0 < A_1 \leq A_2 < \infty$ such that, uniformly for all $t \in [0, T]$ and $\epsilon \in (0, 1),
\[ A_1 \left( \frac{\epsilon}{t^{1/\alpha}} \wedge 1 \right)^d \leq \kappa_{\epsilon}(t) \leq A_2 \left( \frac{\epsilon}{t^{1/\alpha}} \wedge 1 \right)^d. \] (2.33)
It follows from the very definition of $\overline{\dim}_\kappa$ that $\overline{\dim}_\kappa F$ is equal to the supremum of all $\eta > 0$ such that
\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{\eta/\alpha}} \inf_{\nu \in \mathcal{P}(F)} \left\{ \left( \frac{\varepsilon}{|t-s|} \right)^{d/\alpha} \land 1 \right\} \nu(ds)\nu(dt) = 0. \tag{2.34}
\]
Using an earlier result [12, Theorem 4.1] of two of the authors, we see that this implies
\[
\overline{\dim}_\kappa F = \alpha \overline{\dim}_{d/\alpha} F, \tag{2.35}
\]
where $\overline{\dim}_{d/\alpha} F$ denotes the $s$-dimensional box-dimension profile of Howroyd [10]. We regularize this equality (cf. Definition 2.2) and then apply Theorem 1.1 in [12] to deduce that $\dim_\kappa F = \alpha \overline{\dim}_{d/\alpha} F$, where $\overline{\dim}_{d/\alpha} F$ denotes the $s$-dimensional packing dimension profile of Falconer and Howroyd [6]. Finally, we can use (2.32) to see that $\overline{\dim}_\kappa F = \dim_F X(F)$. This concludes the proof.

\[ \square \]

Remark 2.8. For all $s \geq \frac{1}{2}$, we take $\alpha = 1/s$ and let $\kappa$ be the family of functions in (2.2) that correspond to a symmetric $\alpha$-stable Lévy process in $\mathbb{R}$. Our proof of Corollary 1.1 shows that the packing dimensional profile $\overline{\dim}_\kappa$ is a constant multiple of the $s$-dimensional packing profiles of Falconer and Howroyd [6].

\textbf{Proof of Corollary 1.2.} Recall that $\Phi$ is the Laplace exponent of the subordinator $S$ and write $\Psi$ for its characteristic Lévy exponent; that is, $E e^{i\xi S(t)} = e^{-t\Phi(\xi)}$. We may introduce an independent real-valued symmetric Cauchy process $X$ and denote, respectively, by $E_S$ and $E_X$ the expectations corresponding to $S$ and $X$. In this way, we find that, for all $\lambda \geq 0$,
\[
e^{-t\Phi(\lambda)} = E_S e^{-\lambda S(t)} = E_S E_X e^{i\lambda (S(t) \cdot X(\lambda))} = E_X e^{iX(\lambda)S(t)} = E_X e^{-t\Psi(\lambda)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t\Psi(\lambda z)}}{1 + z^2} dz. \tag{2.36}
\]
This and a symmetry argument together show that, for all $s, t, \lambda \geq 0$,
\[
e^{-|t-s|\Phi(\lambda)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-|t-s|\Psi(\text{sgn}(t-s)\lambda z)}}{1 + z^2} dz. \tag{2.37}
\]
Let $\lambda := 1/\epsilon$, and integrate both sides with respect to $\nu(ds)\nu(dt)$, for an arbitrary $\nu \in \mathcal{P}_c(F)$, in order to find that
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} E_\nu(z/\epsilon) \frac{1}{1 + z^2} dz = \int e^{-|t-s|\Phi(1/\epsilon)} \nu(ds)\nu(dt). \tag{2.38}
\]
Thus, we obtain the corollary immediately from Theorems 2.6 and 2.7.

\[ \square \]

\textbf{Proof of Corollary 1.3.} We can write $s = 1/\alpha$, where $\alpha \in (0, 2]$. By enlarging the underlying probability space, if need be, we introduce an independent, linear, symmetric stable Lévy process $X_\alpha$ with index $\alpha$. The subordinate process $X_\alpha \circ S$ is itself a Lévy process, and its characteristic exponent is $z \mapsto \Phi(|z|^{\alpha})$ for $z \in \mathbb{R}$. According to [13, Theorem 1.1], the following holds a.s.:
\[
dim_F X_\alpha(S([0,1])) = \lim_{r \downarrow 0} \frac{1}{\log r} \log \left( \int_{0}^{\infty} \frac{dx}{(1 + x^2)(1 + \Phi((x/r)^\alpha))} \right). \tag{2.39}
\]
We analyse the integral by splitting it over three regions. Without loss of generality, we may assume that $0 < r < \frac{1}{2}$.

If $x \in (0, r)$, then $0 \leq \Phi((x/r)^\alpha) \leq \Phi(1)$, and hence
\[
\int_{0}^{r} \frac{dx}{(1 + x^2)(1 + \Phi((x/r)^\alpha))} \asymp r \quad \text{as} \ r \downarrow 0, \tag{2.40}
\]
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where \( f(r) \asymp g(r) \) as \( r \downarrow 0 \) means that \( f(r)/g(r) \) is bounded above and below by constants that do not depend on \( r \) as \( r \downarrow 0 \).

Similarly,

\[
\int_r^1 \frac{dx}{(1 + x^2)(1 + \Phi((x/r)^\alpha))} \asymp \frac{1}{r} \int_{1/r}^1 \frac{dx}{\Phi(x^\alpha)} =: f(r) \quad \text{as} \quad r \downarrow 0
\] (2.41)

and

\[
\int_1^\infty \frac{dx}{(1 + x^2)(1 + \Phi((x/r)^\alpha))} \asymp \frac{1}{r} \int_{1/r}^\infty \frac{dx}{x^2 \Phi(x^\alpha)} =: g(r) \quad \text{as} \quad r \downarrow 0.
\] (2.42)

We first observe that

\[
\int_{1/r}^1 \frac{dx}{\Phi(x^\alpha)}
\]

is bounded away from zero for all \( r \in (0, 1/2) \). This proves that \( r = O(f(r)) \) as \( r \downarrow 0 \), and hence the integral in (2.40) does not contribute to the limit in (2.39). In addition, \( f(r) \geq 1/\Phi(r^{-\alpha}) \), and hence

\[
g(r) \leq \frac{1}{r \Phi(r^{-\alpha})} \int_{1/r}^\infty \frac{dx}{x^2} \leq f(r).
\] (2.43)

Because \( \alpha = 1/s \), the preceding observations together prove that

\[
dim_P X_\alpha(S([0, 1])) = \lim_{r \to 0} \frac{\log f(r)}{\log r} = 1 - \theta, \quad \text{a.s.} \quad (2.44)
\]

On the other hand, we can apply (2.32), conditionally on the process \( S \), in order to deduce that

\[
dim_P X_\alpha(S([0, 1])) = \alpha \dim_{FH}^{1/\alpha} S([0, 1]) \quad \text{a.s.;} \quad (2.45)
\]

see also Corollary 1.1. Corollary 1.3 follows upon combining (2.44) and (2.45).

3. Proof of Theorem 2.7

Here and throughout, we define a measure \( P_{\lambda_d} \) by

\[
P_{\lambda_d}(\cdot) := \int_{\mathbb{R}^d} P^x(\cdot) \, dx.
\] (3.1)

It is easy to see that \( P_{\lambda_d} \) is a \( \sigma \)-finite measure on the underlying measurable space \((\Omega, \mathcal{F})\). The corresponding expectation operator will be denoted by \( E_{\lambda_d} \); that is, \( E_{\lambda_d}(Z) := \int_{\mathbb{R}^d} E^x(Z) \, dx \) for every nonnegative measurable random variable \( Z \).

Let \( \{\mathcal{F}_t\}_{t \geq 0} \) denote the filtration generated by \( X \), augmented in the usual way. In order to prove (2.31), we make use of the following strong Markov property for \( P_{\lambda_d} \).

**Lemma 3.1.** If \( f : \mathbb{R}^d \to \mathbb{R} \) is a bounded measurable function and \( T \) is a stopping time such that \( P\{T < \infty\} = 1 \), then

\[
E_{\lambda_d}[f(X(t)) \mid \mathcal{F}_T] = E^{X(T)}[f(X(t - T))], \quad (3.2)
\]

\( P_{\lambda_d} \)-a.s. on \( \{T < t\} \).

**Proof.** This is well known, particularly for Brownian motion; see, for example, Chung [4, Theorem 3, p. 58]. We include an elementary self-contained proof.

If \( T \) is nonrandom, say \( T = s \) a.s., then (3.2) follows directly from [14, Proposition 3.2]. It can be verified by elementary computations that (3.2) holds also when \( T \) is a discrete stopping time.

In general, there exists a sequence of discrete stopping times \( T_n \) such that \( T_n \downarrow T \). For any event \( A \in \mathcal{F}_T \), we have \( A \in \mathcal{F}_{T_n} \). It follows that, for all bounded and continuous functions \( f \)
and $g$ on $\mathbb{R}^d$ ($g \in L^1(\mathbb{R}^d)$),
\[ E_{\lambda_d}[f(X(t))g(X(T_n))1_{A \cap (T_n < t)}] = E_{\lambda_d}[E^{X(T_n)}(f(X(t-T_n)))g(X(T_n))]1_{A \cap (T_n < t)}. \] (3.3)
Since the function $E^x[f(X(s))]$ is continuous in the variables $(x, s)$, the integrand in the last expression tends to $E^x[f(X(t-T))]g(X(T))1_{A \cap (T < t)}$ a.s. as $n \to \infty$. Hence, we can apply the dominated convergence theorem to derive (3.2) from (3.3).

**Proof of Theorem 2.7 (Equation (2.31)).** Given $\mu \in \mathcal{P}(F)$ and $\epsilon > 0$, let us define
\[ \ell_{\epsilon,\mu} := \int \frac{1_{B(0,\epsilon)}(X(s))}{(2\epsilon)^d} \mu(ds). \] (3.4)
Note that $E_{\lambda_d}(\ell_{\epsilon,\mu}) = 1$.

Now, let $T := T_F(\epsilon) := \inf\{t \in F : |X(t)| \leq \epsilon\}$, where $\inf \emptyset := \infty$. Then $T$ is a stopping time and by Lemma 3.1, for all $n \geq 1$,
\[ E_{\lambda_d}(\ell_{2\epsilon,\mu} \mid \mathcal{F}_{T \wedge n}) \geq \frac{1}{(4\epsilon)^d} \int_{T \wedge n} P_{\lambda_d}(|X(s)| \leq 2\epsilon \mid \mathcal{F}_{T \wedge n}) \mu(ds) \geq \frac{1}{(4\epsilon)^d} \int_T \kappa_\epsilon(s - T) \mu(ds) \cdot 1_{\{T < n\}}. \] (3.5)
We have used the triangle inequality, together with the fact that $|X(T \wedge n)| \leq \epsilon$, $P_{\lambda_d}$-a.s. on $\{T < n\}$. Since $E_{\lambda_d}(\ell_{2\epsilon,\mu}) = 1$, we find that
\[ 1 = E_{\lambda_d}[E_{\lambda_d}(\ell_{2\epsilon,\mu} \mid \mathcal{F}_{T \wedge n})] \geq \frac{1}{(4\epsilon)^d} E_{\lambda_d} \left[ \int_T \kappa_\epsilon(s - T) \mu(ds) \cdot 1_{\{T < n\}} \right]. \] (3.6)
We can let $n \to \infty$ and deduce the following from the monotone convergence theorem:
\[ 1 \geq \frac{1}{(4\epsilon)^d} E_{\lambda_d} \left[ \int_T \kappa_\epsilon(s - T) \mu(ds) \cdot 1_{\{T < \infty\}} \right] = \frac{1}{(4\epsilon)^d} \int_0^\infty \mu(ds) \kappa_\epsilon(s - t) \cdot P_{\lambda_d}\{T < \infty\}. \] (3.7)
This holds for all probability measures $\mu$. We now consider the particular choice $\mu(dt) := P_{\lambda_d}(T \in dt \mid T < \infty)$ to find that, for all $\epsilon > 0$,
\[ P_{\lambda_d} \left( \inf_{t \in F} |X(t)| \leq \epsilon \right) \leq \frac{2^{d+1}(2\epsilon)^d}{\inf_{t \in F} \int \kappa_\epsilon(|t - s|) \nu(ds) \nu(dt)}. \] (3.8)
On the other hand, by the Fubini–Tonelli theorem,
\[ P_{\lambda_d} \left( \inf_{t \in F} |X(t)| \leq \epsilon \right) = \int_{\mathbb{R}^d} P \left( \inf_{t \in F} |x + X(t)| \leq \epsilon \right) dx = E\{\lambda_d[(X(F)]\}. \] (3.9)
where $G^\epsilon := G + B(0,\epsilon)$ denotes the closed $\epsilon$-enlargement of $G$ in the $\ell^\infty$-metric of $\mathbb{R}^d$.

Let $K_G$ denote the Kolmogorov capacity of the set $G$. That is, $K_G(r)$ is the maximal number $m$ of points $x_1, \ldots, x_m$ in $G$ with $\min_{j \neq k} |x_j - x_k| \geq r$.

The following three observations will be important for our argument.

(i) We have $r^d K_G(r) \leq \lambda_d(G^r)$ for every $r > 0$, where $G^r$ denotes the closed $r$-enlargement of $G$. Indeed, we can find $k := K_G(r)$ points $x_1, \ldots, x_k \in G$ such that $B(x_1, r/2), \ldots, B(x_k, r/2)$ are disjoint. Since $G^r$ contains $\bigcup_{j=1}^k B(x_j, r/2)$ as a subset, the claim follows from the monotonicity of the Lebesgue measure.

(ii) For every bounded set $G \subseteq \mathbb{R}$, the [upper] Minkowski dimension of $G$ is defined by $\dim_M G := \lim_{r \to 0} K_G(r)/\log(1/r)$. And Tricot [24] (see also Falconer [5]) has proved that the packing dimension can be defined by regularizing $\dim_M$. Namely, for every set $F \subseteq \mathbb{R}$,
\[ \dim_F f = \inf_{n \geq 1} \dim_M F_n. \] (3.10)
where the infimum is taken over all bounded Borel sets \( F_1, F_2, \ldots \) such that \( F \subseteq \bigcup_{n=1}^{\infty} F_n \).

(iii) For every analytic set \( G \subseteq \mathbb{R} \),

\[
\text{dim}_M G = \sup \left\{ \eta > 0 : \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon^\eta} \inf_{\nu \in \mathcal{P}(G)} \int \mathbf{1}_{|t-s| \leq \epsilon} \nu(ds)\nu(dt) = 0 \right\}. \tag{3.11}
\]

This follows from [12, Theorem 4.1], which extended in turn an earlier result of Howroyd [10].

Consequently, we use (3.8) and (3.9) to obtain

\[
E[K_X(F)(\epsilon)] \leq \frac{1}{\epsilon^\eta} E[\lambda_\nu((X(F))^\epsilon)] \leq \frac{2^{2d+1}}{\inf_{\nu \in \mathcal{P}(F)} \int \kappa_\nu(|s-t|)\nu(ds)\nu(dt)}. \tag{3.12}
\]

Fix a number \( s > \text{Dim}_\kappa F \). By (2.3) there exists a finite constant \( c > 0 \) such that

\[
\inf_{\nu \in \mathcal{P}(F)} \int \kappa_\nu(|t-s|)\nu(ds)\nu(dt) \geq c^s \tag{3.13}
\]

for all sufficiently small \( \epsilon > 0 \). It follows from (3.12) and (3.13) that, for all \( q \in (0,1) \),

\[
P\{K_X(F)(\epsilon) \geq \epsilon^{-(q+s)}\} = O(\epsilon^q) \quad (\epsilon \downarrow 0). \tag{3.14}
\]

We apply the preceding with \( \epsilon := 2^{-n} \), and use the Borel–Cantelli lemma together with a standard monotonicity, in order to obtain the following:

\[
K_X(F)(\epsilon) = O(\epsilon^{-(q+s)}) \quad (\epsilon \downarrow 0) \quad \text{a.s.} \tag{3.15}
\]

This proves that \( \text{dim}_M X(F) \leq s + q \) a.s. Now we first let \( q \downarrow 0 \) and then \( s \downarrow \text{Dim}_\kappa F \) (along countable sequences) to deduce the almost sure inequality \( \text{dim}_M X(F) \leq \text{Dim}_\kappa F \).

Next we prove \( \text{dim}_M X(F) \geq \text{Dim}_\kappa F \) a.s. The definition (2.3) of \( \text{Dim}_\kappa \) implies that, for all \( \eta < \text{Dim}_\kappa F \), there exist a sequence of positive numbers \( \epsilon_n \downarrow 0 \) and a sequence of measures \( \nu_1, \nu_2, \ldots \in \mathcal{P}(F) \) such that

\[
\lim_{n \to 0} \int \kappa_\nu(|t-s|)\nu_n(ds)\nu_n(dt) = 0. \tag{3.16}
\]

If \( m_n := \nu_n \circ X^{-1} \), then \( m_n \in \mathcal{P}(X(F)) \) a.s. and

\[
E \left[ \int \frac{1}{\epsilon_n^\eta} m_n(dy)m_n(dx) \right] = \frac{1}{\epsilon_n^\eta} \int \kappa_\nu(|t-s|)\nu_n(ds)\nu_n(dt). \tag{3.17}
\]

This, Fatou’s lemma and (3.16) together imply that

\[
\lim_{\epsilon \downarrow 0} \inf_{m \in \mathcal{P}(X(F))} \int \frac{1}{\epsilon^\eta} m(dy)m(dx) = 0 \quad \text{a.s.} \tag{3.18}
\]

Consequently, it follows from (3.11) that \( \text{dim}_M X(F) \geq \eta \) a.s. Let \( \eta \) tend upward to \( \text{Dim}_\kappa F \) in order to conclude that \( \text{dim}_M X(F) \geq \text{Dim}_\kappa F \) a.s., whence (2.31).

**Proof of Theorem 2.7 (Equation (2.32)).** First we prove the upper bound in (2.32). By the definition (2.6) of \( \text{Dim}_\kappa \), for all \( \gamma > \text{Dim}_\kappa F \), there exists a sequence \( \{F_n\}_{n \geq 1} \) of bounded Borel sets such that

\[
F \subseteq \bigcup_{n=1}^{\infty} F_n \quad \text{and} \quad \sup_{n \geq 1} \text{Dim}_\kappa F_n < \gamma. \tag{3.19}
\]

Since \( X(F) \subseteq \bigcup_{n=1}^{\infty} X(F_n) \), (3.10) and Theorem 2.7, (2.31) together imply that

\[
\text{dim}_P X(F) \leq \sup_{n \geq 1} \text{dim}_M X(F_n) = \sup_{n \geq 1} \text{Dim}_\kappa F_n < \gamma \quad \text{a.s.} \tag{3.20}
\]

Thus, \( \text{dim}_P X(F) \leq \text{Dim}_\kappa F \) a.s.
Next we complete the proof of (2.32) by deriving the complementary lower bound:
\[ \dim_P X(F) \geq \Dim_\kappa F \text{ a.s.} \] (3.21)
It suffices to consider only the case where \( \Dim_\kappa F > 0 \); otherwise, there is nothing to prove.

First, we claim that (2.6) implies that, for every \( 0 < \gamma < \Dim_\kappa F \), there exists a compact subset \( E \subseteq F \) such that \( \Dim_\kappa(E \cap (s, t)) \geq \gamma \) for all \( s, t \in \mathbb{Q}_+ \) and \( s < t \) that satisfy \( E \cap (s, t) \neq \emptyset \). In order to verify this claim, let us note that if, in addition, \( F \) were closed, then we could apply the \( \sigma \)-stability of \( \Dim_\kappa \) in order to construct a compact set \( E \subseteq F \) with the desired property as in the proof of Lemma 4.1 in Talagrand and Xiao [23]. In the general case, we proceed as in Howroyd’s proof of his Theorem 22 [10]. Since this is a lengthy calculation and not essential to the rest of the proof, we omit the details.

We now demonstrate the a.s. lower bound, \( \dim_P X(E) \geq \gamma \). Since \( \overline{X(E)} \) and \( X(E) \) only differ by at most a countable set, it is sufficient to prove \( \dim_P \overline{X(E)} \geq \gamma \) a.s. Observe that there are at most countably many points in \( \overline{X(E)} \) with the following property: each of them corresponds to a \( t \in E \) such that \( X \) has a jump at \( t \), and \( t \) cannot be approached from the right by the elements of \( E \). Removing these isolated points from \( \overline{X(E)} \) yields a closed subset with the same packing dimension as \( \dim_P \overline{X(E)} \). Therefore, without loss of generality, we may and will assume that \( \overline{X(E)} \) is a.s. a closed set and every point in \( \overline{X(E)} \) is the limit of a sequence \( X(t_n) \) with \( t_n \in E \).

Since \( \overline{X(E)} \) is a.s. closed, one can apply Baire’s category theorem as in Tricot’s proof of Theorem 2.7 [24, Proposition 3]. Thus, it suffices to prove that a.s.
\[ \overline{\dim_M}[X(E) \cap B(a, r)] \geq \gamma \text{ for all } a \in \mathbb{Q}^d \text{ and } r \in \mathbb{Q} \cap (0, \infty), \] (3.22)
whenever \( X(E) \cap B(a, r) \neq \emptyset \).

According to the already established first assertion (2.31) of Theorem 2.7,
\[ P\{\overline{\dim_M}X(E \cap (s, t)) = \overline{\dim_M}[E \cap (s, t)] \text{ for all } s < t \in \mathbb{Q}_+\} = 1. \] (3.23)
Fix \( a \in \mathbb{Q}^d \) and \( r \in \mathbb{Q} \cap (0, \infty) \). It follows from (3.23) that
\[ \overline{\dim_M}[X(E) \cap B(a, r)] \geq \sup \overline{\dim_M}[E \cap (s, t)] \text{ a.s.,} \] (3.24)
where the supremum is taken over all rationals \( s, t > 0 \) such that \( X(E \cap (s, t)) \subseteq B(a, r) \). By the aforementioned assumption on \( \overline{X(E)} \) we see that if \( X(E) \cap B(a, r) \neq \emptyset \), then we can always find rationals \( s, t > 0 \) such that \( E \cap (s, t) \neq \emptyset \) and \( X(E \cap (s, t)) \subseteq B(a, r) \). This, together with (3.24), implies that a.s. \( \overline{\dim_M}[X(E) \cap B(a, r)] \geq \gamma \), provided that \( \overline{X(E)} \cap B(a, r) \neq \emptyset \).

Finally, we can choose a \( P \)-null event such that the preceding holds, outside of that null event, simultaneously for all \( a \in \mathbb{Q}^d \) and \( r \in \mathbb{Q} \cap (0, \infty) \). This proves (3.22), whence \( \dim_P X(E) \geq \gamma \) a.s.; (2.31) follows immediately. \( \square \)

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