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A Probabilistic Inequality Related to Negative Definite Functions

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Abstract. We prove that for any pair of i.i.d. random vectors X, Y in $\mathbb{R}^n$ and any real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ the inequality

$$\mathbb{E}\,\psi(X-Y) \leqslant \mathbb{E}\,\psi(X+Y).$$

holds. In particular, for $\alpha \in (0,2]$ and the Euclidean norm $\|\cdot\|_2$ one has

$$\mathbb{E}\|X-Y\|_2^\alpha \leqslant \mathbb{E}\|X+Y\|_2^\alpha.$$

The latter inequality is due to A. Buja et al. [4] where it is used for some applications in multivariate statistics. We show a surprising connection with bifractional Brownian motion and provide some related counter-examples.

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1. Introduction

Let X, Y be i.i.d. random variables with finite expectations. Then one has

$$\mathbb{E}|X-Y| \leqslant \mathbb{E}|X+Y|. \tag{1.1}$$

The inequality (1.1) appeared recently in an analytic context (properties of integrable functions) [8]. Since (1.1) is a nice fact in itself and since it seems not to be well known in the probabilistic community, it is desirable to search for adequate proofs and to explore possible extensions of it. For instance, for which values of α do we have

$$\mathbb{E}|X-Y|^\alpha \leqslant \mathbb{E}|X+Y|^\alpha\,? \tag{1.2}$$

As before, we assume that X and Y are i.i.d. and $\mathbb{E}|X|^\alpha < \infty$.

Proving (1.1) is a non-trivial exercise for a probability course. If X, Y are real valued, one way to see this inequality is to use the identity

$$\mathbb{E}|X+Y| - \mathbb{E}|X-Y| = 2\int_0^\infty \left[\mathbb{P}(X>r) - \mathbb{P}(X<-r)\right]^2 dr.$$

For (1.2) we are, however, not aware of a similar elementary approach. On the other hand, A. Buja et al. prove in [4] even a multivariate version of (1.2): for any pair of i.i.d. random vectors X, Y in $\mathbb{R}^n$, any $\alpha \in (0,2]$ and for a class of norms $\|\cdot\|$ on $\mathbb{R}^n$ including the Euclidean norm $\|\cdot\|_2$ the estimate

$$\mathbb{E}\|X-Y\|^\alpha \leqslant \mathbb{E}\|X+Y\|^\alpha \tag{1.3}$$

holds true. The elegance of this inequality is obvious; at the same time we stress that it arises from statistical applications. In any case it merits to be better known in the probabilistic community!

In Section 2 we give an extension of (1.3) by replacing the norm with an arbitrary negative definite function. Moreover, we show how this fact extends to an arbitrary number of i.i.d. random vectors. In Sections 3 and 4 we establish a surprising connection to some recent advances in the theory of random processes related to bifractional Brownian motion. A counterexample to (1.2) with $\alpha \in (2,\infty)$ is given in Section 5.

2. Main result

Consider the class of *continuous real-valued negative definite functions*, i.e., characteristic exponents of symmetric Lévy processes. The notion of negative definite function goes back to Schoenberg; good sources are the books [3] and [11]. Recall that a continuous real-valued negative definite function is uniquely given by its Lévy-Khintchine representation

$$\psi(\xi) = a + \frac{1}{2}\langle Q\xi,\xi\rangle + \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos\langle\xi,u\rangle)\,\nu(du), \qquad \xi\in\mathbb{R}^n, \tag{2.1}$$

where $a \geqslant 0$ is a constant, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix and ν is the Lévy measure, i.e., a measure on $\mathbb{R}^n \setminus \{0\}$ satisfying the integrability condition

$$\int_{\mathbb{R}^n\setminus\{0\}} \min\{\|u\|_2^2, 1\}\nu(du) < \infty. \tag{2.2}$$

Without loss of generality, we will always assume that $a = 0$, i.e., $\psi(0) = 0$. For our discussion it is worth noticing that $(\xi,\eta) \mapsto \sqrt{\psi(\xi-\eta)}$ is always a metric. A deep theorem of Schoenberg states that a metric space $(\mathbb{R}^n, d)$ can be isometrically embedded into an (in general infinite-dimensional) Hilbert space $\mathcal{H}$ if, and only if, $d(\xi,\eta)$ is of the form $d_\psi(\xi,\eta) = \sqrt{\psi(\xi-\eta)}$, cf. [12], [2, p. 187] as well as [7] for a discussion of metric measure spaces related to the metric d_ψ.

An important subclass of continuous negative definite functions are the spherically symmetric negative definite functions. These are of the form

$$\xi \mapsto f(\|\xi\|_2^2) \qquad \text{where } f \text{ is a Bernstein function.} \tag{2.3}$$

Recall that a *Bernstein function* is a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which admits the following Lévy-Khintchine representation

$$f(\lambda) = a + b\lambda + \int_0^\infty \left(1 - e^{-t\lambda}\right) \mu(dt);$$

here $a, b \geqslant 0$ are constants and μ is a measure on $(0, \infty)$ satisfying the integrability condition $\int_0^\infty \min\{t, 1\}\, \mu(dt) < \infty$. In probability theory Bernstein functions arise as the characteristic exponents of the Laplace transform of subordinators, i.e., increasing one-dimensional Lévy processes. Bernstein functions, many examples and their connections to various fields of mathematics are discussed in the monograph [11]. It is easy to see that Bernstein functions are infinitely many times differentiable, increasing, concave; moreover, they grow at most linearly. Typical examples are $\lambda \mapsto \log(1 + \lambda)$ and $\lambda \mapsto f_\beta(\lambda) := \lambda^\beta$ for $0 < \beta \leqslant 1$. Note that the composition $f \circ \psi$ of a Bernstein function f with a continuous real-valued negative definite function ψ is again a continuous real-valued negative definite function. At the level of stochastic processes this corresponds to *Bochner's subordination* of the Lévy process with characteristic exponent ψ by the subordinator with the Laplace exponent f.

Using the Bernstein functions f_β with $\beta = \alpha/2$ and $0 < \alpha \leqslant 2$ we obtain

$$\begin{aligned} \xi &\mapsto \|\xi\|_2^\alpha = f_{\alpha/2}(\|\xi\|^2), && 0 < \alpha \leqslant 2, \\ \xi &\mapsto d_\psi(\xi, 0)^\alpha = \sqrt{\psi(\xi)}^{\,\alpha} = f_{\alpha/2}(\psi(\xi)), && 0 < \alpha \leqslant 2, \end{aligned}$$

as examples for real-valued continuous negative definite functions. Note that the functions defined by (2.3) are characteristic exponents of subordinate Brownian motions.

We prove the following result extending (1.3).

Theorem 2.1. *Let ψ be a real-valued continuous negative definite function on $\mathbb{R}^n$. For any pair of i.i.d. random vectors X, Y in $\mathbb{R}^n$ it is true that*

$$\mathbb{E}\, \psi(X - Y) \leqslant \mathbb{E}\, \psi(X + Y). \tag{2.4}$$

Proof. Without loss of generality we may assume that $a = 0$ and $Q = 0$ – in both cases the inequality (2.4) is elementary.

Using the Lévy-Khintchine representation of ψ we get

$$\begin{aligned} \mathbb{E}\, \psi(X + Y) &= \mathbb{E} \int_{\mathbb{R}^n \setminus \{0\}} \big(1 - \cos\langle X + Y, u\rangle\big)\, \nu(du) \\ &= \mathbb{E} \int_{\mathbb{R}^n \setminus \{0\}} \big(1 - \operatorname{Re} \exp(i\langle X + Y, u\rangle)\big)\, \nu(du) \\ &= \int_{\mathbb{R}^n \setminus \{0\}} \big(1 - \operatorname{Re}\, \mathbb{E} \exp(i\langle X + Y, u\rangle)\big)\, \nu(du) \end{aligned}$$

$$= \int_{\mathbb{R}^n\setminus\{0\}} \Big(1 - \operatorname{Re}\,[\mathbb{E}\exp(i\langle X,u\rangle)]^2\Big)\,\nu(du).$$

A similar calculation yields

$$\begin{aligned}\mathbb{E}\,\psi(X-Y) &= \mathbb{E}\int_{\mathbb{R}^n\setminus\{0\}} \big(1-\cos\langle X-Y,u\rangle\big)\,\nu(du)\\ &= \mathbb{E}\int_{\mathbb{R}^n\setminus\{0\}} \big(1-\operatorname{Re}\exp(i\langle X-Y,u\rangle)\big)\,\nu(du)\\ &= \int_{\mathbb{R}^n\setminus\{0\}} \big(1-\operatorname{Re}\,\mathbb{E}\exp(i\langle X-Y,u\rangle)\big)\,\nu(du)\\ &= \int_{\mathbb{R}^n\setminus\{0\}} \Big(1-|\mathbb{E}\exp(i\langle X,u\rangle)|^2\Big)\,\nu(du).\end{aligned}$$

Using the elementary estimate $\operatorname{Re}(z^2) \leqslant |z^2| = |z|^2$ we obtain (2.4). □

Remark 2.2. Let $X_1,\dots,X_{2m}$ be i.i.d. random variables in $\mathbb{R}^n$ and $\varepsilon_j = \pm 1$ (non-random, or even random but independent of the $X_1,\dots,X_{2m}$) constants satisfying $\sum_{j=1}^{2m}\varepsilon_j = 0$. Then

$$\mathbb{E}\,\psi\left(\sum_{j=1}^{2m}\varepsilon_j X_j\right) \leqslant \mathbb{E}\,\psi\left(\sum_{j=1}^{2m} X_j\right). \tag{2.5}$$

This follows if we use Theorem 2.1 for $X = \sum_{j=1}^{2m}\varepsilon_j^+ X_j$ and $Y = \sum_{j=1}^{2m}\varepsilon_j^- X_j$.

Remark 2.3. Essentially the same calculations as in the proof of Theorem 2.1 show that we also have

$$\mathbb{E}\psi(X) \leqslant \mathbb{E}\psi(X+Y). \tag{2.6}$$

This follows from the elementary inequality $\operatorname{Re}(z^2) \leqslant \operatorname{Re} z$ for $|z| \leqslant 1$ and the fact that

$$\mathbb{E}\psi(X) = \int_{\mathbb{R}^n\setminus\{0\}} \Big(1-\operatorname{Re}\,\mathbb{E}\exp(i\langle X,u\rangle)\Big)\,\nu(du).$$

A special case of the inequality (2.6) with $\psi(\xi) = |\xi|$ and $\nu(du) = \frac{1}{\pi}\,u^{-2}\,du$ appeared in the 2003 Putnam competition, cf. [10, Problem B6, p. 783 and p. 790] where the task was to show that

$$\int_0^1\int_0^1 |f(x)+f(y)|\,dx\,dy \geqslant \int_0^1 |f(x)|\,dx$$

for a continuous real-valued function f defined on the interval $[0,1]$.

Using the distance function $d_\psi(\xi,\eta) := \sqrt{\psi(\xi-\eta)}$ related to a real-valued continuous negative definite function ψ we get the following counterpart of (1.3).

Corollary 2.4. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function, $d_\psi(\xi,\eta) = \sqrt{\psi(\xi-\eta)}$ the associated metric and $0 < \alpha \leqslant 2$. For any pair of i.i.d. random vectors X, Y in $\mathbb{R}^n$ it is true that*

$$\mathbb{E}\, d_\psi^\alpha(X-Y) \leqslant \mathbb{E}\, d_\psi^\alpha(X+Y). \tag{2.7}$$

Remark 2.5. Assume that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that $\psi(0) = 0$ and $\psi(\xi) = \psi(-\xi)$. If (2.4) holds for this ψ and *any* random variable X (and an independent copy Y of X), then one can show that the kernel $K_\psi(\xi,\eta) := \psi(\xi+\eta) - \psi(\xi-\eta)$ is positive definite. We wonder whether this already entails that ψ is a continuous negative definite function.

3. A relation to random processes

We will show now that the inequality (2.4) has an interesting relation to Gaussian processes. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function defined on $\mathbb{R}^n$.

Lemma 3.1. *The kernel $K^\psi(\xi,\eta) = \psi(\xi+\eta) - \psi(\xi-\eta)$ is positive definite.*

Proof. By the Lévy-Khintchine formula (2.1) we get

$$K^\psi(\xi,\eta) = 2\langle Q\xi,\eta\rangle + \int_{\mathbb{R}^n\setminus\{0\}} \left(\cos(\langle \xi-\eta,u\rangle) - \cos(\langle \xi+\eta,u\rangle)\right) \nu(du).$$

Using the elementary trigonometric identity

$$\cos\langle \xi-\eta,u\rangle - \cos\langle \xi+\eta,u\rangle = 2\sin\langle \xi,u\rangle \sin\langle \eta,u\rangle,$$

we see that

$$K^\psi(\xi,\eta) = 2\langle Q\xi,\eta\rangle + 2\int_{\mathbb{R}^n\setminus\{0\}} \sin\langle \xi,u\rangle \sin\langle \eta,u\rangle\, \nu(du).$$

Now let S be a finite set and $(\lambda_\xi,\ \xi\in S)$ be complex numbers. Then

$$\begin{aligned}
&\sum_{\xi,\eta\in S} K^\psi(\xi,\eta)\lambda_\xi\overline{\lambda_\eta}\\
&\quad = 2\sum_{\xi,\eta\in S}\lambda_\xi\overline{\lambda_\eta}\langle Q\xi,\eta\rangle + 2\int_{\mathbb{R}^n\setminus\{0\}}\left(\sum_{\xi,\eta\in S}\lambda_\xi\sin\langle\xi,u\rangle\,\overline{\lambda_\eta\sin\langle\eta,u\rangle}\right)\nu(du)\\
&\quad = 2\left\langle Q\sum_{\xi\in S}\lambda_\xi\,\xi, \sum_{\xi\in S}\lambda_\xi\,\xi\right\rangle + 2\int_{\mathbb{R}^n\setminus\{0\}}\left|\sum_{\xi\in S}\lambda_\xi\sin\langle\xi,u\rangle\right|^2\nu(du) \geqslant 0,
\end{aligned}$$

which means that $K^\psi(\cdot,\cdot)$ is positive definite. □

Remark 3.2. A special case of Lemma 3.1 for powers of ℓ_p-norms is proved in [4].

Probabilistic proof of Theorem 2.1. Since $K^\psi(\xi,\eta)$ is positive definite, there is a centered Gaussian process $\left(G_\xi^\psi,\ \xi\in\mathbb{R}^n\right)$ whose covariance function is $K^\psi(\xi,\eta)$.

For given i.i.d. random vectors $X, Y \in \mathbb{R}^n$ set

$$Z^\psi := \int_{\mathbb{R}^n} G_\xi^\psi \, P(d\xi),$$

where P stands for the common distribution of X and Y. Then

$$\begin{aligned} 0 \leqslant \operatorname{Var}(Z^\psi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K^\psi(\xi, \eta) \, P(d\xi) \, P(d\eta) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\psi(\xi + \eta) - \psi(\xi - \eta)) \; P(d\xi) \, P(d\eta) \\ &= \mathbb{E} \, \psi(X + Y) - \mathbb{E} \, \psi(X - Y), \end{aligned}$$

and we obtain again $\mathbb{E} \, \psi(X - Y) \leqslant \mathbb{E} \, \psi(X + Y)$. □

4. Relation to bifractional Brownian motion

In some most important cases it is possible to identify the Gaussian process $\left(G_\xi^\psi, \ \xi \in \mathbb{R}^n\right)$ of Section 3 with *bifractional Brownian motion* (bBm). The latter process was introduced by Houdré and Villa in [6] as a centered Gaussian process $B^{H,K} = \left(B_t^{H,K}, \ t \in \mathbb{R}^n\right)$ with covariance function

$$R^{H,K}(t, s) := \mathbb{E} \left(B_t^{H,K} B_s^{H,K} \right) = 2^{-K} \left((||t||_2^{2H} + ||s||_2^{2H})^K - ||t - s||_2^{2HK} \right),$$

where $s, t \in \mathbb{R}^n$. For $n = 1$, $K = 1$ we get the usual fractional Brownian motion B^H with Hurst index H. Originally, the process was defined for the parameters $H \in (0, 1]$ and $K \in (0, 1]$. Bardina and Es-Sebaiy [1] recently proved that $B^{H,K}$ exists for all $(H, K) \in \mathcal{D}$, where

$$\mathcal{D} := \{H, K : 0 < H \leqslant 1, 0 < K \leqslant 2, H \cdot K \leqslant 1\}.$$

(The possibility of such an extension was already indicated in the earlier work by Lei and Nualart [9] who established an integral representation relating $B^{H,K}$ with fractional Brownian motion B^{HK}.)

For $\psi(\xi) := |\xi|^\alpha$, $0 < \alpha \leqslant 2$, and

$$G_\xi^\psi := 2^{\alpha/2} \operatorname{sgn}(\xi) \, B_{|\xi|}^{\frac{1}{2}, \alpha}, \qquad \xi \in \mathbb{R},$$

it is trivial to see that

$$\mathbb{E} \left(G_\xi^\psi G_\eta^\psi \right) = \operatorname{sgn}(\xi\eta) \, 2^\alpha \, \mathbb{E} \left(B_{|\xi|}^{\frac{1}{2}, \alpha}, B_{|\xi|}^{\frac{1}{2}, \alpha} \right) = |\xi + \eta|^\alpha - |\xi - \eta|^\alpha = K^\psi(\xi, \eta).$$

Therefore, we are led to a probabilistic interpretation of the inequality (1.2) through $B^{\frac{1}{2}, \alpha}$.

Remark 4.1. In higher dimensions bi-fractional Brownian motion does not show up in the context of our inequalities (nor do we rely on bBm with $H \neq \frac{1}{2}$); therefore it becomes natural to search for the extensions of bBm based upon general negative definite functions. This will be done elsewhere.

5. A counterexample

The inequality (1.2) trivially extends to the case $\alpha = \infty$ in the following sense. Let

$$M = \sup\{r : \mathbb{P}(X < r) < 1\} = \operatorname{ess\,sup} X;$$
$$m = \sup\{r : \mathbb{P}(X < r) = 0\} = \operatorname{ess\,inf} X.$$

Then

$$\|X - Y\|_\infty = M - m \leqslant 2\max\{|M|, |m|\} = \|X + Y\|_\infty.$$

Without further assumptions the inequality (1.2) will, in general, not hold, for $2 < \alpha < \infty$. To see this, fix $\alpha \in (2, \infty)$ and $c > 0$. For any $M \geqslant c$ set $q := c/M$ and $p := 1 - q$. Let X_M, Y_M be i.i.d. random variables such that

$$\mathbb{P}(X_M = 1) = \mathbb{P}(Y_M = 1) = p;$$
$$\mathbb{P}(X_M = -M) = \mathbb{P}(Y_M = -M) = q.$$

If $M \geqslant 1$, then

$$\begin{aligned}\mathbb{E}|X_M - Y_M|^\alpha - \mathbb{E}|X_M + Y_M|^\alpha &\\ &= 2pq\left[(M+1)^\alpha - (M-1)^\alpha\right] - 2^\alpha M^\alpha q^2 - 2^\alpha p^2\\ &\geqslant 4pq\alpha M^{\alpha-1} - 2^\alpha M^\alpha q^2 - 2^\alpha p^2\\ &= M^{\alpha-2}(4p\alpha c - 2^\alpha c^2) - 2^\alpha p^2.\end{aligned}$$

Hence, whenever $c < 2^{2-\alpha}\alpha$ and M is large enough,

$$\mathbb{E}|X_M - Y_M|^\alpha - \mathbb{E}|X_M + Y_M|^\alpha > 0,$$

and (1.2) fails.

Remark 5.1. Further counterexamples are presented in [4].

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References

[1] X. Bardina, K. Es-Sebaiy (2011), An extension of bifractional Brownian motion, *Commun. Stochast. Analysis* **5** 333–340. [arXiv:1002.3680]

[2] Y. Benyamini, J. Lindenstrauss (2000), *Geometric Nonlinear Functional Analysis. Vol.* 1, American Mathematical Society, AMS Colloquium Publications vol. **48**, Providence (RI).

[3] C. Berg, G. Frost (1975), *Potential Theory on Locally Compact Abelian Groups*, Springer, Berlin, Ergebnisse der Mathematik und ihrer Grenzgebiete **87**.

[4] A. Buja, B.F. Logan, J.A. Reeds, L.A. Shepp (1994), Inequalities and positive-definite functions arising from a problem in multidimensional scaling, *Ann. Statist.* **22**, 406–438.

[5] C. Dellacherie, P.-A. Meyer (1983), *Probabilités et potentiel, Chapitres IX à XI: Théorie discrète du potentiel*, Hermann, Paris.

[6] C. Houdré, J. Villa (2003), An example of infinite dimensional quasi-helix, in: J.M. González-Barrios, J.A. León, A. Meda (eds.), *Stochastic Models*, American Mathematical Society, Contemporary Mathematics vol. **336**, 195–201, Provicence (RI).

[7] N. Jacob, V. Knopova, S. Landwehr, R.L. Schilling (2012), A geometric interpretation of the transition density of a symmetric Lévy process. To appear in: *Sci. China: Mathematics*.

[8] H. Kempka, A. Nekvinda (2011), private communication.

[9] P. Lei, D. Nualart (2009), A decomposition of the bifractional Brownian motion and some applications, *Statist. Probab. Letters* **79** 619–624. [`arXiv:0803.2227`]

[10] Putnam Competition (2004) Sixty-fourth Annual William Lowell Putnam Mathematical Competition, Saturday December 6, 2003, *Amer. Math. Monthly* **111** 780–790.

[11] R. Schilling, R. Song, Z. Vondraček (2010), *Bernstein Functions*, de Gruyter, Berlin, Studies in Mathematics **37**.

[12] I.J. Schoenberg (1938), Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.* **44**, 522–536.

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