EXACT ASYMPTOTIC FORMULAS FOR THE HEAT KERNELS OF SPACE AND TIME-FRACTIONAL EQUATIONS

CHANG-SONG DENG AND RENÉ L. SCHILLING

ABSTRACT. We study the asymptotic behaviour of the fundamental solutions (heat kernels) of non-local (partial and pseudo differential) equations with fractional operators in time and space. In particular, we obtain exact asymptotic formulas for the heat kernels of time-changed Brownian motions and Cauchy processes.

1. INTRODUCTION

We are interested in the asymptotic behaviour at zero and at infinity of time- and space-fractional evolution equations. The simplest examples of such equations are

$$\text{(a)} \quad \frac{\partial u}{\partial t} = -(-\Delta_x)^\beta u \qquad \text{and} \qquad \text{(b)} \quad \frac{\partial^\beta u}{\partial t^\beta} = \Delta_x u, \tag{1.1}$$

where $\Delta = \Delta_x = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $(-\Delta)^\beta$ is the fractional power of the Laplacian of order $0 < \beta < 1$,

$$-(-\Delta_x)^\beta u(x) = \lim_{\epsilon \downarrow 0} \frac{\beta 4^\beta \Gamma\left(\beta + \frac{n}{2}\right)}{\pi^{n/2}\Gamma(1-\beta)} \int_{|y|>\epsilon} \frac{u(x+y) - u(x)}{|y|^{n+2\beta}} \,\mathrm{d}y,$$

and $\frac{\partial^\beta}{\partial t^\beta}$ is the Caputo derivative of order $0 < \beta < 1$, i.e.

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(s) - f(0)}{(t-s)^\beta} \,\mathrm{d}s.$$

Our standard references for fractional derivatives in time and space are Samko *et al.* [17] and Jacob [8]. If $\beta = 1$, (1.1) becomes the classical heat equation whose fundamental solution is the Gauss kernel

$$p(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left[-\frac{|x-y|^2}{4t}\right], \quad t > 0,\ x, y \in \mathbb{R}^n, \tag{1.2}$$

which is also the transition probability density of a Brownian motion $X = (X_t)_{t\geq 0}$ in $\mathbb{R}^n$.

Both equations in (1.1) have interesting probabilistic interpretations. Denote by $S = (S_t)_{t\geq 0}$ a β-stable subordinator ($0 < \beta < 1$), i.e. a non-decreasing Lévy process on $[0,\infty)$ with Laplace transform

$$\mathbb{E}\, \mathrm{e}^{-rS_t} = \mathrm{e}^{-tr^\beta}, \quad r > 0,\ t \geq 0.$$

2010 *Mathematics Subject Classification. Primary:* 60J35. *Secondary:* 60G51; 60K99; 35R11; 35K08.

Key words and phrases. Heat kernel; asymptotic formula; space-fractional equation; time-fractional equation; subordinator; inverse subordinator.

Financial support through the National Natural Science Foundation of China (11401442) (for Chang-Song Deng) is gratefully acknowledged.

If S and X are stochastically independent, the time-changed process $X_S = (X_{S_t})_{t\geq 0}$ is a rotationally symmetric 2β-stable Lévy process. By independence, the transition probability density of X_{S_t} is given by

$$p^S(t,x,y) = \int_0^\infty p(s,x,y)\,\mathrm{d}_s\mathbb{P}(S_t \leq s), \tag{1.3}$$

and Bochner [4] observed that this is the fundamental solution to the space-fractional equation (1.1.a). Here and in what follows, d_s stands for the (generalized) derivative w.r.t. s. This type of time-change is usually called subordination and the process X_S is said to be subordinate to X, cf. [19].

If we perform a time-change with the generalized right-continuous inverse of S,

$$S_t^{-1} = \inf\{s \geq 0 \,:\, S_s > t\} = \sup\{s \geq 0 \,:\, S_s \leq t\}, \quad t \geq 0,$$

we get a stochastic process $X_{S^{-1}} = (X_{S_t^{-1}})_{t\geq 0}$ which is trapped whenever $t \mapsto S_t^{-1}$ is constant. Note that the jumps of $t \mapsto S_t$ correspond to flat pieces of $t \mapsto S_t^{-1}$. These traps slow down the original diffusion process X, and in the physics literature $X_{S^{-1}}$ is commonly referred to as subdiffusion, see e.g. [11, 14, 15] for some applications, [12, 9, 10] for sample path properties and [13] for a representation as scaling limit of a continuous time random walk with heavy-tailed waiting times between the steps.

Since the length of the trapping periods are, in general, not exponentially distributed, we cannot expect that $X_{S^{-1}}$ is a Markov process. Nevertheless, the transition probability density of each random variable $X_{S_t^{-1}}$, $t > 0$, can be expressed as

$$p^{S^{-1}}(t,x,y) = \int_0^\infty p(s,x,y)\,\mathrm{d}_s\mathbb{P}\left(S_t^{-1} \leq s\right), \tag{1.4}$$

and it is not hard to see, using the Fourier–Laplace transform, that $p^{S^{-1}}(t,x,y)$ is the fundamental solution to the time-fractional heat equation (1.1.b), see e.g. [13, Theorem 5.1] or [1, 14].

Already in the simple setting (1.1), the densities p^S and $p^{S^{-1}}$ are often not explicitly known – a notable exception is p^S for the $\beta = \frac{1}{2}$-stable subordinator: In this case X_S is the symmetric Cauchy process and its transition probability density is the Poisson kernel on $\mathbb{R}^n$,

$$p(t,x,y) = \frac{c(n)}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}} = \frac{c(n)t}{(t^2 + |x-y|^2)^{(n+1)/2}}, \tag{1.5}$$

where $c(n) := \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)$. Therefore, it is important to know the asymptotic behaviour of p^S and $p^{S^{-1}}$ at zero and infinity. For the fundamental solution to (1.1.a) the asymptotics of p^S at infinity is known to be

$$p^S(t,x,y) \sim \frac{c(n,\beta)t}{(|x-y|^2 + t^{1/\beta})^{(n+2\beta)/2}} \quad \text{as } |x-y|^2 t^{-1/\beta} \to \infty, \tag{1.6}$$

where $c(n,\beta) := \beta 4^\beta \pi^{-1-n/2}\sin(\pi\beta)\Gamma\left(\frac{n+2\beta}{2}\right)\Gamma(\beta)$. For $d = 1$ this formula is due to Pólya [16] who used Fourier methods, and the case $d \geq 1$ can be found in Blumenthal and Getoor [3]. A beautiful short proof is due to Bendikov [2]. Our approach is similar to Bendikov's and we show that this method also yields the asymptotics at zero.

2. Results

Let $\mathcal{L}$ be the infinitesimal generator of a Feller process $X = (X_t)_{t\geq 0}$ which takes values in a locally compact separable metric space (M, ρ). We assume that X has a transition probability density $p(t,x,y)$; note that p is the fundamental solution to the Kolmogorov backward equation $\frac{\partial u}{\partial t} = \mathcal{L}_x u$. We may replace in (1.1) the Laplace operator Δ by the generator $\mathcal{L}$. The resulting equations

$$\text{(2.1)} \qquad \text{(a)} \quad \frac{\partial u}{\partial t} = -(-\mathcal{L}_x)^\beta u \qquad \text{and} \qquad \text{(b)} \quad \frac{\partial^\beta u}{\partial t^\beta} = \mathcal{L}_x u,$$

are the Kolmogorov backward equation (2.1.a), resp., the master equation (2.1.b) of the time-changed processes X_S and $X_{S^{-1}}$, respectively. As before, $S = (S_t)_{t\geq 0}$ is a β-stable subordinator, and the fundamental solutions to the problems (2.1) are still given by the formulas (1.3) and (1.4), with $p(t,x,y)$ being the probability density of X_t.

We assume that $p(t,x,y)$ is of the following form

$$\text{(2.2)} \qquad p(t,x,y) = \frac{C_1}{t^{n/\alpha}} F\left(C_2 \frac{\rho(x,y)}{t^{1/\alpha}}\right), \quad t > 0,\ x,y \in M,$$

where $n, \alpha, C_1, C_2 > 0$ are constants, and $F : [0,\infty) \to (0,\infty)$ is a non-increasing profile function. Typically, profiles are either of exponential type

$$\text{(2.3)} \qquad F(r) = \exp\left[-r^{\alpha/(\alpha-1)}\right] \quad \text{for some } \alpha \geq 2,$$

or of polynomial type

$$\text{(2.4)} \qquad F(r) = (1+r^2)^{-(d+\alpha)/2} \quad \text{for some } \alpha > 0.$$

Indeed, (1.2) is exponential with $M = \mathbb{R}^n$, $\rho(x,y) = |x-y|$, $\alpha = 2$, $C_1 = (4\pi)^{-n/2}$, $C_2 = 1/2$, and $F(r) = \mathrm{e}^{-r^2}$, while (1.5) is of polynomial type with $M = \mathbb{R}^n$, $\rho(x,y) = |x-y|$, $\alpha = 1$, $C_1 = c(n)$, $C_2 = 1$, and $F(r) = (1+r^2)^{-(n+1)/2}$.

The exponential-polynomial dichotomy is not artificial. This becomes clear from a deep result by Grigor'yan and Kumagai on two-sided heat kernel estimates [7, Theorem 4.1]: Under some reasonable conditions, $p(t,x,y)$ satisfies

$$\text{(2.5)} \qquad p(t,x,y) \asymp \frac{C_1}{t^{n/\alpha}} F\left(C_2 \frac{\rho(x,y)}{t^{1/\alpha}}\right),$$

where F is either of exponential (2.3) or polynomial (2.4) type and with constants $n, \alpha, C_1, C_2 > 0$; the symbol $\asymp$ means that $p(t,x,y)$ can be estimated from above and below by the expression on the right-hand side, possibly with different constants C_1, C_2 in each direction.

Recently, Chen *et al.* [5] have established two-sided heat kernel estimates for $p^{S^{-1}}(t,x,y)$ where S is a (not necessarily stable) subordinator and under the assumption that the original heat kernel $p(t,x,y)$ satisfies two-sided estimates of the form (2.5).

Our aim is to investigate the exact asymptotic behaviour of the heat kernels $p^S(t,x,y)$ and $p^{S^{-1}}(t,x,y)$ at zero and at infinity. The setting is as described above, throughout we assume that S is a β-stable subordinator.

Theorem 2.1 (Asymptotics for subordination)**.** *Assume that $p(t,x,y)$ is given by (2.2) and $(S_t)_{t\geq 0}$ is a β-stable subordinator for some $0 < \beta < 1$.*

a) *If* $\int_1^\infty s^{n+\alpha\beta-1}F(s)\,\mathrm{d}s<\infty$, *then as* $\rho(x,y)t^{-1/(\alpha\beta)}\to\infty$,

$$p^S(t,x,y)\;\sim\;C_1\frac{\alpha\beta}{\Gamma(1-\beta)}\,C_2^{-n-\alpha\beta}\int_0^\infty s^{n+\alpha\beta-1}F(s)\,\mathrm{d}s\cdot\rho(x,y)^{-n-\alpha\beta}t.$$

b) *As* $\rho(x,y)t^{-1/(\alpha\beta)}\to 0$,

$$p^S(t,x,y)\;\sim\;C_1F(0+)\frac{\Gamma\left(\frac{n}{\alpha\beta}\right)}{\beta\Gamma\left(\frac{n}{\alpha}\right)}\,t^{-n/(\alpha\beta)}.$$

Theorem 2.2 (Asymptotics for inverse subordination)**.** *Assume that* $p(t,x,y)$ *is of the form* (2.2) *and* $(S_t^{-1})_{t\geq 0}$ *is an inverse* β*-stable subordinator for some* $0<\beta<1$.

a) *If* $\int_1^\infty s^{n-\alpha-1}F(s)\,\mathrm{d}s<\infty$, *then as* $\rho(x,y)^{-\alpha/\beta}t\to\infty$,

$$p^{S^{-1}}(t,x,y)\;\sim\;\begin{cases} C_1F(0+)\dfrac{\Gamma\left(1-\frac{n}{\alpha}\right)}{\Gamma\left(1-\frac{\beta n}{\alpha}\right)}\,t^{-\beta n/\alpha}, & \text{if } n<\alpha,\\[2ex] C_1\dfrac{\beta}{\Gamma(1-\beta)}\,F(0+)\,t^{-\beta}\log\left[\rho(x,y)^{-\alpha/\beta}t\right], & \text{if } n=\alpha,\\[2ex] C_1\dfrac{C_2^{\alpha-n}\alpha}{\Gamma(1-\beta)}\displaystyle\int_0^\infty s^{n-\alpha-1}F(s)\,\mathrm{d}s\cdot\rho(x,y)^{\alpha-n}t^{-\beta}, & \text{if } n>\alpha.\end{cases}$$

b) *If* F *is of polynomial type* (2.4), *then as* $\rho(x,y)^{-\alpha/\beta}t\to 0$,

$$p^{S^{-1}}(t,x,y)\;\sim\;C_1\frac{C_2^{-n-\alpha}}{\beta\Gamma(\beta)}\,\rho(x,y)^{-n-\alpha}t^{\beta}.$$

c) *If* F *is of exponential type* (2.3), *then as* $\rho(x,y)^{-\alpha/\beta}t\to 0$,

$$p^{S^{-1}}(t,x,y)\;\sim\;K_1\rho(x,y)^{-\frac{n(1-\beta)}{\alpha-\beta}}t^{-\frac{n(\alpha-1)\beta}{\alpha(\alpha-\beta)}}\exp\left[-K_2\rho(x,y)^{\frac{\alpha}{\alpha-\beta}}t^{-\frac{\beta}{\alpha-\beta}}\right]$$

with the constants

$$K_1:=C_1C_2^{-\frac{n(1-\beta)}{\alpha-\beta}}\sqrt{\frac{\alpha-1}{(\alpha-\beta)\beta}}\,(\alpha-1)^{\frac{n(\alpha-1)(1-\beta)}{\alpha(\alpha-\beta)}}\beta^{\frac{n(\alpha-1)\beta}{\alpha(\alpha-\beta)}},$$

$$K_2:=C_2^{\frac{\alpha}{\alpha-\beta}}(\alpha-\beta)(\alpha-1)^{-\frac{\alpha-1}{\alpha-\beta}}\beta^{\frac{\beta}{\alpha-\beta}}.$$

Remark 2.3. We can state Theorem 2.2.c) in the following way:

$$p^{S^{-1}}(t,x,y)\;\sim\;K_1t^{-\frac{\beta n}{\alpha}}A^{-\frac{n(1-\beta)}{\alpha-\beta}}\exp\left[-K_2A^{\frac{\alpha}{\alpha-\beta}}\right]\quad\text{as}\;\;A:=\rho(x,y)t^{-\beta/\alpha}\to\infty.$$

This shows that there exist constants $c_i=c_i(n,\alpha,\beta)$, $i=1,2,3,4$, such that

$$c_1t^{-\frac{\beta n}{\alpha}}\exp\left[-c_2A^{\frac{\alpha}{\alpha-\beta}}\right]\leq p^{S^{-1}}(t,x,y)\leq c_3t^{-\frac{\beta n}{\alpha}}\exp\left[-c_4A^{\frac{\alpha}{\alpha-\beta}}\right]\quad\text{for all } A\geq 1.$$

This is in line with the two-sided estimates for $p^{S^{-1}}(t,x,y)$ derived by Chen *et al.* in [5, Corollary 1.5 (i)].

If $p(t,x,y)$ is the Gaussian (1.2) or Cauchy (1.5) density, Theorems 2.1 and 2.2 take the following special form.

Corollary 2.4. *Assume that* $p(t,x,y)$ *is the Gauss kernel* (1.2) *and* $(S_t)_{t\geq 0}$ *is a* β*-stable subordinator for some* $0<\beta<1$.

a) *As* $|x-y|t^{-1/(2\beta)} \to \infty$,

$$p^S(t,x,y) \sim \frac{\beta 4^\beta \Gamma\left(\frac{n}{2}+\beta\right)}{\pi^{n/2}\Gamma(1-\beta)} |x-y|^{-n-2\beta} t.$$

b) *As* $|x-y|t^{-1/(2\beta)} \to 0$,

$$p^S(t,x,y) \sim \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2\beta}\right)}{2\beta\pi^{(n+1)/2}\Gamma(n)} t^{-n/(2\beta)}.$$

c) *As* $|x-y|^{-2/\beta} t \to \infty$,

$$p^{S^{-1}}(t,x,y) \sim \begin{cases} \dfrac{1}{2\Gamma\left(1-\frac{\beta}{2}\right)} t^{-\beta/2}, & \textit{if } n=1, \\ \dfrac{\beta}{4\pi\Gamma(1-\beta)} t^{-\beta} \log\left[|x-y|^{-2/\beta} t\right], & \textit{if } n=2, \\ \dfrac{\Gamma\left(\frac{n}{2}-1\right)}{4\pi^{n/2}\Gamma(1-\beta)} |x-y|^{2-n} t^{-\beta}, & \textit{if } n\geq 3. \end{cases}$$

d) *As* $|x-y|^{-2/\beta} t \to 0$,

$$p^{S^{-1}}(t,x,y) \sim K_1 |x-y|^{-\frac{n(1-\beta)}{2-\beta}} t^{-\frac{n\beta}{2(2-\beta)}} \exp\left[-K_2 |x-y|^{\frac{2}{2-\beta}} t^{-\frac{\beta}{2-\beta}}\right],$$

with the constants

$$K_1 := \frac{1}{\sqrt{\beta(2-\beta)}} \pi^{-\frac{n}{2}} 2^{-\frac{n}{2-\beta}} \beta^{\frac{n\beta}{2(2-\beta)}} \quad \textit{and} \quad K_2 := (2-\beta) 2^{-\frac{2}{2-\beta}} \beta^{\frac{\beta}{2-\beta}}.$$

Proof. Corollary 2.4 follows directly from the respective cases in Theorem 2.1 and 2.2. For Part b) we use Legendre's doubling formula for the Gamma function

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \quad z>0, \tag{2.6}$$

with $z = n/2$. □

Note that Euler's reflection formula for the Gamma function, $\Gamma(\beta)\Gamma(1-\beta) = \frac{\pi}{\sin \pi\beta}$, shows that Corollary 2.4.a) coincides with (1.6).

Corollary 2.5. *Assume that* $p(t,x,y)$ *is the Cauchy kernel* (1.5) *and* $(S_t)_{t\geq 0}$ *is a* β*-stable subordinator for some* $0<\beta<1$.

a) *As* $|x-y|t^{-1/\beta} \to \infty$,

$$p^S(t,x,y) \sim \frac{\beta 2^{\beta-1}\Gamma\left(\frac{n+\beta}{2}\right)}{\pi^{n/2}\Gamma\left(1-\frac{\beta}{2}\right)} |x-y|^{-n-\beta} t.$$

b) *As* $|x-y|t^{-1/\beta} \to 0$,

$$p^S(t,x,y) \sim \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{\beta}\right)}{\beta\pi^{(n+1)/2}\Gamma(n)} t^{-n/\beta}.$$

c) *As* $|x-y|^{-1/\beta} t \to \infty$,

$$p^{S^{-1}}(t,x,y) \sim \begin{cases} \dfrac{\beta}{\pi\Gamma(1-\beta)} t^{-\beta} \log\left[|x-y|^{-1/\beta} t\right], & \textit{if } n=1, \\ \dfrac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{(n+1)/2}\Gamma(1-\beta)} |x-y|^{1-n} t^{-\beta}, & \textit{if } n\geq 2. \end{cases}$$

d) *As* $|x-y|^{-1/\beta}t \to 0$,

$$p^{S^{-1}}(t,x,y) \sim \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}\beta\Gamma(\beta)}\,|x-y|^{-n-1}t^{\beta}.$$

Proof. All assertions follow from the respective cases in Theorems 2.1 and 2.2. For the proof of a) and c), we use the well-known integral formula for Euler's Beta function

$$B(r,s)=\int_0^\infty z^{r-1}(1+z)^{-r-s}\,\mathrm{d}z=\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)},\quad r,s>0.$$

Part a) also needs (2.6) with $z=\frac12(1-\beta)$.

Alternatively, we can obtain a) and b) from Corollary 2.4.a), b) if we replace in these formulas β by $\beta/2$. This follows from the observation that the Cauchy kernel can be obtained from the Gaussian kernel by subordination with a $\frac12$-stable subordinator. Since the composition of a $\frac12$-stable and a β-stable subordinator coincides (in law) with a $\frac{\beta}{2}$-stable subordinator, a) and b) are special cases of Corollary 2.4.a), b). □

3. Proof of Theorem 2.1 and 2.2

For the proof of our main results we need some preparations. Let $(S_t)_{t\geq 0}$ be a β-stable subordinator, $\beta\in(0,1)$. It is well known that S_1 has a density $p_\beta(s)$, $s>0$, with respect to Lebesgue measure; moreover, p_β is of class $C^\infty(0,\infty)$, bounded, unimodal and it has the following asymptotics at zero and infinity, cf. [20, Theorem 4.7.1 (4.7.13) and Theorem 5.4.1],

$$p_\beta(s)\sim\begin{cases}\dfrac{1}{\sqrt{2\pi(1-\beta)}}\,\beta^{\frac{1}{2(1-\beta)}}s^{-\frac{2-\beta}{2(1-\beta)}}\exp\left[-(1-\beta)\left(\beta s^{-1}\right)^{\frac{\beta}{1-\beta}}\right], & \text{as } s\to 0,\\[2ex] \dfrac{\beta}{\Gamma(1-\beta)}\,s^{-\beta-1}, & \text{as } s\to\infty.\end{cases}$$

This allows us to rewrite $p_\beta(s)$ for $s>0$ in the following way

$$p_\beta(s)=\frac{1}{\sqrt{2\pi(1-\beta)}}\,\beta^{\frac{1}{2(1-\beta)}}s^{-\frac{2-\beta}{2(1-\beta)}}\exp\left[-(1-\beta)\left(\beta s^{-1}\right)^{\frac{\beta}{1-\beta}}\right](1+\phi_\beta(s)),\tag{3.1}$$

$$p_\beta(s)=\frac{\beta}{\Gamma(1-\beta)}\,s^{-\beta-1}\,(1+\psi_\beta(s)),\tag{3.2}$$

where $\phi_\beta,\psi_\beta:(0,\infty)\to(-1,\infty)$ are continuous functions satisfying

$$\lim_{s\downarrow 0}\phi_\beta(s)=0\quad\text{and}\quad\lim_{s\to\infty}\psi_\beta(s)=0$$

and

$$\lim_{s\to\infty}s^{-\frac{\beta(2\beta-1)}{2(1-\beta)}}(1+\phi_\beta(s))=\frac{\sqrt{2\pi(1-\beta)}}{\Gamma(1-\beta)}\,\beta^{\frac{1-2\beta}{2(1-\beta)}}.$$

Let us denote by G_β the distribution function of S_1, i.e.

$$G_\beta(x)=\mathbb{P}(S_1\leq x)=\int_0^x p_\beta(s)\,\mathrm{d}s,\quad x\geq 0.$$

Because of the scaling property of a β-stable subordinator, we have for $s,t>0$,

$$\begin{aligned}\mathbb{P}\left(S_t^{-1}\le s\right)=\mathbb{P}\left(S_s\ge t\right)=\mathbb{P}\left(s^{1/\beta}S_1\ge t\right)&=1-\mathbb{P}\left(S_1<s^{-1/\beta}t\right)\\&=1-G_\beta\left(s^{-1/\beta}t\right).\end{aligned}$$

Combining this with (1.4), we have

$$p^{S^{-1}}(t,x,y)=-\int_0^\infty p(s,x,y)\,\mathrm{d}_sG_\beta\big(s^{-1/\beta}t\big)=\int_0^\infty p\left(t^\beta s^{-\beta},x,y\right)\,\mathrm{d}G_\beta(s), \tag{3.3}$$

and, similarly,

$$p^{S}(t,x,y)=\int_0^\infty p(s,x,y)\,\mathrm{d}_sG_\beta\big(t^{-1/\beta}s\big)=\int_0^\infty p\left(t^{1/\beta}s,x,y\right)\,\mathrm{d}G_\beta(s). \tag{3.4}$$

Proof of Theorem 2.1. Set $A:=\rho(x,y)t^{-1/(\alpha\beta)}$; from (3.4) and the assumption on $p(s,x,y)$ we get

$$\begin{aligned}p^S(t,x,y)&=C_1t^{-n/(\alpha\beta)}\int_0^\infty s^{-n/\alpha}F\left(C_2\frac{\rho(x,y)}{\left(t^{1/\beta}s\right)^{1/\alpha}}\right)\mathrm{d}G_\beta(s)\\&=C_1t^{-n/(\alpha\beta)}\int_0^\infty s^{-n/\alpha}F\left(C_2As^{-1/\alpha}\right)\,\mathrm{d}G_\beta(s).\end{aligned}$$

a) Since F is bounded, $\int_1^\infty s^{n+\alpha\beta-1}F(s)\,\mathrm{d}s<\infty$ implies that $\int_0^\infty s^{n+\alpha\beta-1}F(s)\,\mathrm{d}s<\infty$. From (3.2) and the definition of G_β we see

$$\begin{aligned}p^S(t,x,y)&=\frac{C_1\beta}{\Gamma(1-\beta)}t^{-n/(\alpha\beta)}\int_0^\infty s^{-n/\alpha-\beta-1}F\left(C_2As^{-1/\alpha}\right)\left[1+\psi_\beta(s)\right]\mathrm{d}s\\&=\frac{C_1\alpha\beta}{\Gamma(1-\beta)}C_2^{-n-\alpha\beta}\frac{t}{\rho(x,y)^{n+\alpha\beta}}\int_0^\infty s^{n+\alpha\beta-1}F(s)\left[1+\psi_\beta\left(C_2^\alpha A^\alpha s^{-\alpha}\right)\right]\,\mathrm{d}s.\end{aligned}$$

Since ψ_β is bounded on $(0,\infty)$, we may use the dominated convergence theorem to get

$$\lim_{A\to\infty}t^{-1}\rho(x,y)^{n+\alpha\beta}p^S(t,x,y)=\frac{C_1\alpha\beta}{\Gamma(1-\beta)}C_2^{-n-\alpha\beta}\int_0^\infty s^{n+\alpha\beta-1}F(s)\,\mathrm{d}s.$$

b) Using the dominated convergence theorem once again, we see

$$\begin{aligned}\lim_{A\to0}t^{n/(\alpha\beta)}p^S(t,x,y)&=C_1\lim_{A\to0}\int_0^\infty s^{-n/\alpha}F\left(C_2As^{-1/\alpha}\right)\,\mathrm{d}G_\beta(s)\\&=C_1F(0+)\int_0^\infty s^{-n/\alpha}\,\mathrm{d}G_\beta(s)\\&=C_1F(0+)\mathbb{E}S_1^{-n/\alpha}\\&\overset{(*)}{=}C_1F(0+)\frac{\Gamma\left(1+\frac{n}{\alpha\beta}\right)}{\Gamma\left(1+\frac{n}{\alpha}\right)}=C_1F(0+)\frac{\Gamma\left(\frac{n}{\alpha\beta}\right)}{\beta\Gamma\left(\frac{n}{\alpha}\right)};\end{aligned}$$

in the equality marked by an asterisk (*) we use Lemma 4.1 in the appendix with $\kappa=-n/\alpha$ and $t=1$. The last equality follows from the functional equation $z\Gamma(z)=\Gamma(1+z)$ for the Gamma function. $\square$

Proof of Theorem 2.2. Define $A := \rho(x,y)^{-\alpha/\beta}t$. Using (3.3) we get

$$
\begin{aligned}
p^{S^{-1}}(t,x,y) &= C_1 t^{-\beta n/\alpha}\int_0^\infty s^{\beta n/\alpha} F\left(C_2\frac{\rho(x,y)}{(t^\beta s^{-\beta})^{1/\alpha}}\right)\mathrm{d}G_\beta(s)\\
&= C_1 t^{-\beta n/\alpha}\int_0^\infty s^{\beta n/\alpha} F\left(C_2\left(\frac{s}{A}\right)^{\beta/\alpha}\right)\mathrm{d}G_\beta(s).
\end{aligned}
\tag{3.5}
$$

a) We begin with the asymptotics for $A\to\infty$.

Case 1: $n<\alpha$: If we use in (3.5) the monotone convergence theorem and Lemma 4.1 from the appendix, we obtain

$$
\begin{aligned}
\lim_{A\to\infty} t^{\beta n/\alpha} p^{S^{-1}}(t,x,y) &= C_1 \lim_{A\to\infty}\int_0^\infty s^{\beta n/\alpha} F\left(C_2\left(\frac{s}{A}\right)^{\beta/\alpha}\right)\mathrm{d}G_\beta(s)\\
&= C_1 F(0+)\int_0^\infty s^{\beta n/\alpha}\,\mathrm{d}G_\beta(s)\\
&= C_1 F(0+)\mathbb{E}S_1^{\beta n/\alpha}\\
&= C_1 F(0+)\frac{\Gamma\left(1-\frac{n}{\alpha}\right)}{\Gamma\left(1-\frac{\beta n}{\alpha}\right)}.
\end{aligned}
$$

Case 2: $n=\alpha$: We know that $\int_1^\infty s^{-1}F(s)\,\mathrm{d}s<\infty$. Inserting (3.2) into (3.5) yields

$$
\begin{aligned}
\frac{t^\beta}{\log A} p^{S^{-1}}(t,x,y) &= \frac{C_1}{\log A}\int_0^\infty s^\beta F\left(C_2\left(\frac{s}{A}\right)^{\beta/\alpha}\right)\mathrm{d}G_\beta(s)\\
&=: \frac{C_1}{\log A}\big(I_1(A,\alpha,\beta)+I_2(A,\alpha,\beta)+I_3(A,\alpha,\beta)\big)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(A,\alpha,\beta) :=& \int_0^1 s^\beta F\left(C_2\left(\frac{s}{A}\right)^{\beta/\alpha}\right)p_\beta(s)\,\mathrm{d}s\\
\le& F(0+)\int_0^1 s^\beta p_\beta(s)\,\mathrm{d}s \;\le\; F(0+),
\end{aligned}
$$

$$
\begin{aligned}
I_2(A,\alpha,\beta) :=& \frac{\beta}{\Gamma(1-\beta)}\int_1^\infty s^{-1} F\left(C_2\left(\frac{s}{A}\right)^{\beta/\alpha}\right)\mathrm{d}s\\
=& \frac{\alpha}{\Gamma(1-\beta)}\int_{C_2A^{-\beta/\alpha}}^\infty s^{-1}F(s)\,\mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3(A,\alpha,\beta) :=& \frac{\beta}{\Gamma(1-\beta)}\int_1^\infty s^{-1} F\left(C_2\left(\frac{s}{A}\right)^{\beta/\alpha}\right)\psi_\beta(s)\,\mathrm{d}s\\
=& \frac{\alpha}{\Gamma(1-\beta)}\int_{C_2A^{-\beta/\alpha}}^\infty s^{-1}F(s)\psi_\beta\left(C_2^{-\alpha/\beta}As^{\alpha/\beta}\right)\,\mathrm{d}s.
\end{aligned}
$$

Now we can use Lemma 4.2 from the appendix to get

$$\lim_{A\to\infty} \frac{t^\beta}{\log A} p^{S^{-1}}(t,x,y) = C_1 \left(0 + \frac{\alpha}{\Gamma(1-\beta)} \cdot \frac{\beta}{\alpha} F(0+) + \frac{\alpha}{\Gamma(1-\beta)} \cdot 0\right)$$
$$= \frac{C_1\beta}{\Gamma(1-\beta)} F(0+).$$

Case 3: $n > \alpha$: We know that $\int_1^\infty s^{n-\alpha-1}F(s)\,\mathrm{d}s < \infty$. Since F is bounded, this yields $\int_0^\infty s^{n-\alpha-1}F(s)\,\mathrm{d}s < \infty$. Using (3.5) and (3.2) we get

$$\rho(x,y)^{n-\alpha}t^\beta p^{S^{-1}}(t,x,y)$$
$$= \frac{C_1\beta}{\Gamma(1-\beta)} A^{\beta(1-n/\alpha)} \int_0^\infty s^{\beta n/\alpha-1-\beta} F\left(C_2\left(\frac{s}{A}\right)^{\beta/\alpha}\right)\left[1+\psi_\beta(s)\right]\mathrm{d}s$$
$$= \frac{C_1C_2^{\alpha-n}\alpha}{\Gamma(1-\beta)} \int_0^\infty s^{n-\alpha-1}F(s)\left[1+\psi_\beta\left(C_2^{-\alpha/\beta}As^{\alpha/\beta}\right)\right]\mathrm{d}s.$$

Since ψ_β is bounded on $(0,\infty)$, the dominated convergence theorem gives

$$\lim_{A\to\infty}\int_0^\infty s^{n-\alpha-1}F(s)\psi_\beta\left(C_2^{-\alpha/\beta}As^{\alpha/\beta}\right)\mathrm{d}s = 0.$$

Therefore, we have

$$\lim_{A\to\infty}\rho(x,y)^{n-\alpha}t^\beta p^{S^{-1}}(t,x,y) = C_1\frac{C_2^{\alpha-n}\alpha}{\Gamma(1-\beta)}\int_0^\infty s^{n-\alpha-1}F(s)\,\mathrm{d}s.$$

b) Now we consider the limit $A \to 0$ for the profile $F(r) = (1+r^2)^{-(n+\alpha)/2}$ where $\alpha > 0$. Applying in (3.5) the dominated convergence theorem and Lemma 4.1 from the appendix gives

$$\lim_{A\to 0}\rho(x,y)^{n+\alpha}t^{-\beta}p^{S^{-1}}(t,x,y)$$
$$= C_1\lim_{A\to 0} A^{-\frac{\beta n}{\alpha}-\beta}\int_0^\infty s^{\beta n/\alpha}\left(1+C_2^2\left(\frac{s}{A}\right)^{2\beta/\alpha}\right)^{-(n+\alpha)/2}\mathrm{d}G_\beta(s)$$
$$= C_1\lim_{A\to 0}\int_0^\infty s^{\beta n/\alpha}\left(A^{2\beta/\alpha}+C_2^2s^{2\beta/\alpha}\right)^{-(n+\alpha)/2}\mathrm{d}G_\beta(s)$$
$$= C_1C_2^{-n-\alpha}\int_0^\infty s^{-\beta}\,\mathrm{d}G_\beta(s)$$
$$= C_1C_2^{-n-\alpha}\mathbb{E}S_1^{-\beta} = C_1C_2^{-n-\alpha}\frac{\Gamma(2)}{\Gamma(1+\beta)} = \frac{C_1C_2^{-n-\alpha}}{\beta\Gamma(\beta)}.$$

c) Finally we consider $A \to 0$ for the profile $F(r) = \exp\left[-r^{\alpha/(\alpha-1)}\right]$ with $\alpha \geq 2$. Combining (3.5) and (3.1) yields

$$p^{S^{-1}}(t,x,y) = C_1t^{-\frac{\beta n}{\alpha}}\int_0^\infty s^{\frac{\beta n}{\alpha}}\exp\left[-C_2^{\frac{\alpha}{\alpha-1}}A^{-\frac{\beta}{\alpha-1}}s^{\frac{\beta}{\alpha-1}}\right]p_\beta(s)\,\mathrm{d}s$$
$$= \frac{C_1}{\sqrt{2\pi(1-\beta)}}\beta^{\frac{1}{2(1-\beta)}}t^{-\frac{\beta n}{\alpha}}\int_0^\infty s^{\frac{\beta n}{\alpha}-\frac{2-\beta}{2(1-\beta)}}\times$$
$$\times\exp\left[-C_2^{\frac{\alpha}{\alpha-1}}A^{-\frac{\beta}{\alpha-1}}s^{\frac{\beta}{\alpha-1}}-(1-\beta)\beta^{\frac{\beta}{1-\beta}}s^{-\frac{\beta}{1-\beta}}\right]\left(1+\phi_\beta(s)\right)\mathrm{d}s.$$

The claim follows with Lemma 4.4 from the appendix. □

4. Appendix

Although the following moment formula for stable subordinators is known, see e.g. Sato [18, Eq. (25.5), p. 162], we include for the readers' convenience the short proof; this simple argument seems to be new.

Lemma 4.1. *The moments of order $\kappa \in (-\infty, \beta)$ of a β-stable subordinator $(S_t)_{t\geq 0}$ exist and are given by*

$$\mathbb{E}S_t^\kappa = \frac{\Gamma\left(1-\frac{\kappa}{\beta}\right)}{\Gamma(1-\kappa)}\, t^{\kappa/\beta}, \quad t>0.$$

Proof. Since $S_t \overset{\text{law}}{=} t^{1/\beta} S_1$, it is enough to consider $t=1$. Recall that the Laplace transform of S_1 is $\mathbb{E}e^{-tS_1} = e^{-t^\beta}$, $t>0$. Substituting $\lambda = S_1$ in the well-known formula [19, p. vii]

$$\lambda^{-r} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-\lambda t} t^{r-1}\, dt, \quad \lambda>0,\ r>0,$$

and taking expectations yields, because of Tonelli's theorem,

$$\mathbb{E}S_1^{-r} = \frac{1}{\Gamma(r)} \int_0^\infty \mathbb{E}e^{-tS_1} t^{r-1}\, dt = \frac{1}{\Gamma(r)} \int_0^\infty e^{-t^\beta} t^r\, \frac{dt}{t}.$$

Now we change variables according to $x=t^\beta$ and $\frac{dx}{x} = \beta\frac{dt}{t}$, and get

$$\mathbb{E}S_1^{-r} = \frac{1}{\Gamma(r)}\frac{1}{\beta} \int_0^\infty e^{-x} x^{\frac{r}{\beta}}\, \frac{dx}{x} = \frac{1}{r\Gamma(r)}\cdot\frac{r}{\beta}\,\Gamma\left(\frac{r}{\beta}\right) = \frac{\Gamma\left(1+\frac{r}{\beta}\right)}{\Gamma(1+r)}.$$

Setting $\kappa=-r$ proves the assertion for $\kappa\in(-\infty,0)$. Note that this formula extends (analytically) to $-r=\kappa<\beta$. Alternatively, use the very same calculation and the formula [19, p. vii]

$$\lambda^r = \frac{r}{\Gamma(1-r)} \int_0^\infty \left(1-e^{-\lambda t}\right) t^{-r-1}\, dt, \quad \lambda>0,\ r\in(0,1),$$

to get the assertion for $\kappa\in(0,\beta)$. □

Lemma 4.2. *Let $\psi : [1,\infty)\to\mathbb{R}$ be a bounded function such that $\lim_{s\to\infty}\psi(s)=0$ and $\omega : [0,\infty)\to(0,\infty)$ a non-increasing function satisfying $\int_1^\infty s^{-1}\omega(s)\, ds<\infty$. For any $c>0$ and $\delta>0$ one has*

$$\lim_{A\to\infty} \frac{1}{\log A} \int_{cA^{-\delta}}^\infty s^{-1}\omega(s)\, ds = \delta\omega(0+)$$

and

$$\lim_{A\to\infty} \frac{1}{\log A} \int_{cA^{-\delta}}^\infty s^{-1}\omega(s)\psi\big(c^{-1/\delta} A s^{1/\delta}\big)\, ds = 0.$$

Proof. The first claim follows easily from l'Hospital's rule. For $n \in \mathbb{N}$, we can use the first part of the lemma and get

$$\begin{aligned}
&\left|\frac{1}{\log A}\int_{cA^{-\delta}}^{\infty} s^{-1}\omega(s)\psi\big(c^{-1/\delta}As^{1/\delta}\big)\,\mathrm{d}s\right| \\
&\quad\le \frac{1}{\log A}\int_{cA^{-\delta}}^{\infty} s^{-1}\omega(s)\left|\psi\big(c^{-1/\delta}As^{1/\delta}\big)\right|\,\mathrm{d}s \\
&\quad\le \|\psi\|_\infty \frac{1}{\log A}\left(\int_{cA^{-\delta}}^{\infty} - \int_{ncA^{-\delta}}^{\infty}\right) s^{-1}\omega(s)\,\mathrm{d}s \\
&\qquad + \|\psi\mathbb{1}_{[n^{1/\delta},\infty)}\|_\infty \frac{1}{\log A}\int_{ncA^{-\delta}}^{\infty} s^{-1}\omega(s)\,\mathrm{d}s \\
&\quad\xrightarrow[A\to\infty]{} \|\psi\|_\infty\big(\delta\omega(0+) - \delta\omega(0+)\big) + \|\psi\mathbb{1}_{[n^{1/\delta},\infty)}\|_\infty\delta\omega(0+) \\
&\quad= \|\psi\mathbb{1}_{[n^{1/\delta},\infty)}\|_\infty\delta\omega(0+) \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

and this completes the proof. □

The following asymptotic formula for integrals can be proved by the Laplace method, see e.g. de Bruijn [6, Section 4.2, pp. 63–65]. For the sake of completeness and our readers' convenience, we give a brief outline of the proof.

Lemma 4.3. *Assume that* $-\infty \le v < w \le \infty$, $h \in C^2(v,w)$, *and* $\int_v^w \mathrm{e}^{-h(r)}\,\mathrm{d}r < \infty$. *Let* $r_0 \in (v,w)$. *If* $h(r_0) \ge 0$, $h''(r_0) > 0$, *and* h *is strictly decreasing on* $(v, r_0]$ *and strictly increasing on* $[r_0, w)$, *then*

$$\int_v^w \mathrm{e}^{-Ch(r)}\,\mathrm{d}r \sim \mathrm{e}^{-Ch(r_0)}\sqrt{\frac{2\pi}{Ch''(r_0)}} \quad \textit{as } C\to\infty.$$

Proof. Since h'' is continuous, we find for any $\epsilon \in (0, h''(r_0)/2)$ some $\delta = \delta(\epsilon) \in (0, (w-r_0)\wedge(r_0-v))$ such that $|h''(\zeta) - h''(r_0)| < \epsilon$ for all $|\zeta - r_0| < \delta$. As $h'(r_0) = 0$, Taylor's theorem implies for all $r \in (r_0-\delta, r_0+\delta)$

$$h(r_0) + \frac{h''(r_0)-\epsilon}{2}(r-r_0)^2 \le h(r) \le h(r_0) + \frac{h''(r_0)+\epsilon}{2}(r-r_0)^2. \tag{4.1}$$

From the second inequality in (4.1), we get for the limit $C\to\infty$

$$\begin{aligned}
\int_v^w \mathrm{e}^{-Ch(r)}\,\mathrm{d}r &\ge \int_{r_0-\delta}^{r_0+\delta} \mathrm{e}^{-Ch(r)}\,\mathrm{d}r \\
&\ge \mathrm{e}^{-Ch(r_0)}\int_{r_0-\delta}^{r_0+\delta}\exp\left[-C\frac{h''(r_0)+\epsilon}{2}(r-r_0)^2\right]\mathrm{d}r \\
&= \frac{\mathrm{e}^{-Ch(r_0)}}{\sqrt{C(h''(r_0)+\epsilon)}}\int_{-\delta\sqrt{C(h''(r_0)+\epsilon)}}^{\delta\sqrt{C(h''(r_0)+\epsilon)}}\exp\left[-\frac{s^2}{2}\right]\mathrm{d}s \\
&\sim \frac{\mathrm{e}^{-Ch(r_0)}}{\sqrt{C(h''(r_0)+\epsilon)}}\int_{-\infty}^{\infty}\exp\left[-\frac{s^2}{2}\right]\mathrm{d}s \\
&= \mathrm{e}^{-Ch(r_0)}\sqrt{\frac{2\pi}{C(h''(r_0)+\epsilon)}},
\end{aligned} \tag{4.2}$$

which implies that

$$\int_v^w \mathrm{e}^{-Ch(r)}\,\mathrm{d}r \geq \mathrm{e}^{-Ch(r_0)}\sqrt{\frac{2\pi}{C(h''(r_0)+2\epsilon)}} \quad \text{for large } C \gg 1.$$

The monotonicity of h on $(v, r_0]$ and $[r_0, w)$ gives

$$h(r) > h(r_0-\delta) \wedge h(r_0+\delta) > h(r_0) \quad \text{for all } |r-r_0| > \delta.$$

Consequently, as $C \to \infty$

$$\begin{aligned}\left(\int_v^{r_0-\delta} + \int_{r_0+\delta}^w\right) \mathrm{e}^{-Ch(r)}\,\mathrm{d}r &\leq \mathrm{e}^{-(C-1)[h(r_0-\delta)\wedge h(r_0+\delta)]}\left(\int_v^{r_0-\delta} + \int_{r_0+\delta}^w\right) \mathrm{e}^{-h(r)}\,\mathrm{d}r \\ &\leq \mathrm{e}^{-(C-1)[h(r_0-\delta)\wedge h(r_0+\delta)]}\int_v^w \mathrm{e}^{-h(r)}\,\mathrm{d}r \\ &= \mathsf{o}\big(\mathrm{e}^{-Ch(r_0)}C^{-1/2}\big)\,.\end{aligned}$$

Using the first inequality in (4.1) a calculation similar to (4.2) gives, for sufficiently large values of $C \gg 1$,

$$\begin{aligned}\int_v^w \mathrm{e}^{-Ch(r)}\,\mathrm{d}r &= \left(\int_v^{r_0-\delta} + \int_{r_0+\delta}^w\right) \mathrm{e}^{-Ch(r)}\,\mathrm{d}r + \int_{r_0-\delta}^{r_0+\delta} \mathrm{e}^{-Ch(r)}\,\mathrm{d}r \\ &\leq \mathrm{e}^{-Ch(r_0)}\sqrt{\frac{2\pi}{C(h''(r_0)-2\epsilon)}}.\end{aligned}$$

Combining both estimates we have shown for all $\epsilon \in (0, h''(r_0)/2)$ and $C \gg 1$

$$\mathrm{e}^{-Ch(r_0)}\sqrt{\frac{2\pi}{C(h''(r_0)+2\epsilon)}} \leq \int_v^w \mathrm{e}^{-Ch(r)}\,\mathrm{d}r \leq \mathrm{e}^{-Ch(r_0)}\sqrt{\frac{2\pi}{C(h''(r_0)-2\epsilon)}}.$$

This immediately implies the claim. □

Lemma 4.4. *Let $\phi : (0,\infty) \to (-1,\infty)$ be a continuous function such that $\phi(0+) = 0$ and $\limsup_{s\to\infty} s^{-\theta}(1+\phi(s)) < \infty$ for some $\theta \in \mathbb{R}$. For all constants $a \in \mathbb{R}$ and $b, c, d > 0$ the following asymptotics obtains*

$$\int_0^\infty s^a \mathrm{e}^{-Bs^b - cs^{-d}}\big(1+\phi(s)\big)\,\mathrm{d}s \;\sim\; I(B) \quad \textit{as } B \to \infty$$

where the value $I(B)$ is given by

$$I(B) := \sqrt{\frac{2\pi}{b+d}}\,(bB)^{-\frac{2(a+1)+d}{2(b+d)}}\,(cd)^{\frac{2(a+1)-b}{2(b+d)}}\exp\left[-(b+d)\left(b^{-1}c\right)^{\frac{b}{b+d}}\left(d^{-1}B\right)^{\frac{d}{b+d}}\right].$$

Proof. First, we prove that

$$\int_0^\infty s^a \mathrm{e}^{-Bs^b - cs^{-d}}\,\mathrm{d}s \;\sim\; I(B) \quad \text{as } B \to \infty. \tag{4.3}$$

If $a \neq -1$, changing variables according to $r = (c^{-1}B)^{(a+1)/(b+d)}\, s^{a+1}$ gives

$$\int_0^\infty s^a \mathrm{e}^{-Bs^b - cs^{-d}}\,\mathrm{d}s = \frac{1}{|a+1|}\left(cB^{-1}\right)^{\frac{a+1}{b+d}}\int_0^\infty \exp\left[-c^{\frac{b}{b+d}}B^{\frac{d}{b+d}}\left(r^{\frac{b}{a+1}} + r^{-\frac{d}{a+1}}\right)\right]\mathrm{d}r.$$

Lemma 4.3 with $v = 0$, $w = \infty$, $h(r) = r^{b/(a+1)} + r^{-d/(a+1)}$, $r_0 = (b^{-1}d)^{(a+1)/(b+d)}$ and $C = c^{b/(b+d)}B^{d/(b+d)}$ yields (4.3).

If $a = -1$, we change variables according to $r = \log s + (b+d)^{-1} \log(c^{-1}B)$, and use Lemma 4.3 with $v = -\infty$, $w = \infty$, $h(r) = \mathrm{e}^{br} + \mathrm{e}^{-dr}$, $r_0 = (b+d)^{-1}\log(b^{-1}d)$ and $C = c^{b/(b+d)}B^{d/(b+d)}$ to obtain (4.3).

We still have to check that

$$\lim_{B\to\infty} B^{\frac{2(a+1)+d}{2(b+d)}} \exp\left[(b+d)\left(b^{-1}c\right)^{\frac{b}{b+d}}\left(d^{-1}B\right)^{\frac{d}{b+d}}\right] \int_0^\infty s^a \mathrm{e}^{-Bs^b - cs^{-d}} \phi(s)\,\mathrm{d}s = 0.$$

To this end, we fix $n \in \mathbb{N}$ and observe that

$$\begin{aligned} I_1(n,B) := \int_0^{1/n} s^a \mathrm{e}^{-Bs^b-cs^{-d}} |\phi(s)|\,\mathrm{d}s &\le \left\|\phi \mathbb{1}_{(0,1/n)}\right\|_\infty \int_0^{1/n} s^a \mathrm{e}^{-Bs^b-cs^{-d}}\,\mathrm{d}s \\ &\le \left\|\phi \mathbb{1}_{(0,1/n)}\right\|_\infty \int_0^\infty s^a \mathrm{e}^{-Bs^b-cs^{-d}}\,\mathrm{d}s. \end{aligned}$$

Moreover, set

$$I_2(n,B) := \int_{1/n}^\infty s^a \mathrm{e}^{-Bs^b-cs^{-d}} |\phi(s)|\,\mathrm{d}s.$$

By our assumption, there exists a constant $C(n) > 0$ depending on n such that $1 + \phi(s) \le C(n)s^\theta$ for all $s \ge 1/n$. Thus,

$$|\phi(s)| \le 1 + (1+\phi(s)) \le (ns)^{\theta\vee 0} + C(n)s^\theta \le C(n,\theta)s^{\theta\vee 0}, \quad s \ge 1/n,$$

where $C(n,\theta) := n^{\theta\vee 0} + C(n)n^{(-\theta)\vee 0}$. Using the dominated convergence theorem we deduce

$$\begin{aligned} &B^{\frac{2(a+1)+d}{2(b+d)}} \exp\left[(b+d)\left(b^{-1}c\right)^{\frac{b}{b+d}}\left(d^{-1}B\right)^{\frac{d}{b+d}}\right] I_2(n,B) \\ &\le C(n,\theta)\int_{1/n}^\infty B^{\frac{2(a+1)+d}{2(b+d)}} s^{a+(\theta\vee 0)} \exp\left[(b+d)\left(b^{-1}c\right)^{\frac{b}{b+d}}\left(d^{-1}B\right)^{\frac{d}{b+d}} - Bs^b - cs^{-d}\right]\mathrm{d}s \\ &\xrightarrow[B\to\infty]{} C(n,\theta)\cdot 0 = 0. \end{aligned}$$

Combining these calculations gives

$$\begin{aligned} B^{\frac{2(a+1)+d}{2(b+d)}} &\exp\left[(b+d)\left(b^{-1}c\right)^{\frac{b}{b+d}}\left(d^{-1}B\right)^{\frac{d}{b+d}}\right] \cdot \left|\int_0^\infty s^a \mathrm{e}^{-Bs^b-cs^{-d}}\phi(s)\,\mathrm{d}s\right| \\ &\le B^{\frac{2(a+1)+d}{2(b+d)}} \exp\left[(b+d)\left(b^{-1}c\right)^{\frac{b}{b+d}}\left(d^{-1}B\right)^{\frac{d}{b+d}}\right] \cdot \big(I_1(n,B) + I_2(n,B)\big) \\ &\le B^{\frac{2(a+1)+d}{2(b+d)}} \exp\left[(b+d)\left(b^{-1}c\right)^{\frac{b}{b+d}}\left(d^{-1}B\right)^{\frac{d}{b+d}}\right] \\ &\qquad \times \left(\left\|\phi\mathbb{1}_{(0,1/n)}\right\|_\infty \int_0^\infty s^a \mathrm{e}^{-Bs^b-cs^{-d}}\,\mathrm{d}s + I_2(n,B)\right) \\ &\xrightarrow[B\to\infty]{} \left\|\phi\mathbb{1}_{(0,1/n)}\right\|_\infty \sqrt{\frac{2\pi}{b+d}}\, b^{-\frac{2(a+1)+d}{2(b+d)n}} (cd)^{\frac{2(a+1)-b}{2(b+d)}} + 0 \xrightarrow[n\to\infty]{} 0 + 0 = 0. \end{aligned}$$

This completes the proof. □

References

[1] B. Baeumer, M.M. Meerschaert: Stochastic solutions for fractional Cauchy problems. *Fract. Calc. Appl. Anal.* **4** (2001) 481–500.

[2] A. Bendikov: Asymptotic formulas for symmetric stable semigroups. *Expo. Math.* **12** (1994) 381–384.

[3] R.M. Blumenthal, R.K. Getoor: Some theorems on stable processes. *Trans. Amer. Math. Soc.* **95** (1960) 263–273.
[4] S. Bochner: Diffusion equation and stochastic processes. *Proc. Natl. Acad. Sci. U.S.A.* **35** (1949) 368–370.
[5] Z.-Q. Chen, P. Kim, T. Kumagai, J. Wang: Heat kernel estimates for time fractional equations. arXiv:1708.05863v2.
[6] N.G. de Bruijn: *Asymptotic Methods in Analysis.* North–Holland, Amsterdam 1958.
[7] A. Grigor'yan, T. Kumagai: On the dichotomy in the heat kernel two sided estimates. *Proc. Sympos. Pure Math.* **77** (2008) 199–210.
[8] N. Jacob: *Pseudo Differential Operators and Markov Processes. Vol. 1–3.* Imperial College Press, London 2001–2005.
[9] M. Magdziarz: Path properties of subdiffusion–a martingale approach. *Stoch. Models* **26** (2010) 256–271.
[10] M. Magdziarz, R.L. Schilling: Asymptotic properties of Brownian motion delayed by inverse subordinators. *Proc. Amer. Math. Soc.* **143** (2015) 4485–4501.
[11] M. Magdziarz, A. Weron, K. Weron: Fractional Fokker–Planck dynamics: Stochastic representation and computer simulation. *Phys. Rev. E* **75** (2007) 016708.
[12] M.M. Meerschaert, E. Nane, Y. Xiao: Large deviations for local time fractional Brownian motion and applications. *J. Math. Anal. Appl.* **346** (2008) 432–445.
[13] M.M. Meerschaert, H.P. Scheffler: Limit theorems for continuous time random walks with infinite mean waiting times. *J. Appl. Probab.* **41** (2004) 623–638.
[14] R. Metzler, J. Klafter: The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339** (2000) 1–77.
[15] A. Piryatinska, A.I. Saichev, W.A. Woyczynski: Models of anomalous diffusion: The subdiffusive case. *Phys. A* **349** (2005) 375–420.
[16] G. Pólya: On the zeros of an integral function represented by Fourier's integral. *Messenger Math.* **52** (1923) 185–188.
[17] S.G. Samko, A.A. Kilbas, O.I. Marichev: *Fractional Integrals and Derivatives.* Gordon and Breach, 1993.
[18] K. Sato: *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge 1999.
[19] R.L. Schilling, R. Song, Z. Vondraček: *Bernstein Functions. Theory and Applications.* De Gruyter, Berlin 2012 (2nd edn).
[20] V.V. Uchaikin, V.M. Zolotarev: *Chance and Stability. Stable Distributions and their Applications.* VSP, Utrecht 1999.

(C.-S. Deng) School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
E-mail address: dengcs@whu.edu.cn

(R.L. Schilling) TU Dresden, Fakultät Mathematik, Institut für Mathematische Stochastik, 01062 Dresden, Germany
E-mail address: rene.schilling@tu-dresden.de