Homogenization of Symmetric Lévy Processes on $\mathbb{R}^d$

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Abstract

In this short note we study homogenization of symmetric d-dimensional Lévy processes. Homogenization of one-dimensional pure jump Markov processes has been investigated by Tanaka *et al.* in [4]; their motivation was the work by Benssousan *et al.* [1] on the homogenization of diffusion processes in $\mathbb{R}^d$, see also [2] and [10]. We investigate a similar problem for a class of symmetric pure-jump Lévy processes on $\mathbb{R}^d$ and we identify – using Mosco convergence – the limit process.

A symmetric Lévy process $(X_t)_{t\geq 0}$ is a stochastic process in $\mathbb{R}^d$ with stationary and independent increments, càdlàg paths and symmetric laws $X_t \sim -X_t$. We can characterize the (finite-dimensional distributions of the) process by its characteristic function $\mathbb{E}e^{i\langle \xi, X_t\rangle}$, $\xi \in \mathbb{R}^d$, $t > 0$, which is of the form $\exp(-t\psi(\xi))$; due to the symmetry of X_t, the characteristic exponent ψ is real-valued. It is given by the Lévy–Khintchine formula

$$\psi(\xi) = \frac{1}{2}\langle \xi, \Sigma\xi\rangle + \int_{h\neq 0} \big(1 - \cos\langle \xi, h\rangle\big)\, \nu(dh), \quad \xi \in \mathbb{R}^d. \tag{1}$$

$\Sigma \in \mathbb{R}^{d\times d}$ is the positive semidefinite *diffusion matrix* and $\nu(dh)$ is the *Lévy measure*, that is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{h\neq 0} \big(1 \wedge |h|^2\big)\, \nu(dh)$ is finite. It is clear from (1) that we have $\nu(dh) = \nu(-dh)$. Throughout this paper we assume $\Sigma \equiv 0$ and that $\nu(dh)$ has a (necessarily symmetric) density w.r.t. Lebesgue measure; in abuse of notation we write $\nu(dh) = \nu(h)\, dh$.

Let $Q = (0,1)^d$ be the open unit cube in $\mathbb{R}^d$ and $a : \mathbb{R}^d \to \mathbb{R}$ a function in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $1 < p \leq \infty$. We assume that a is *Q-periodic* in the sense that

$$a(h + ke_i) = a(h) > 0 \ \text{ for all } k \in \mathbb{Z}^d,\ i = 1, 2, \dots, d \ \text{ and a.a. } h \in Q; \tag{2}$$

as usual, e_i denotes the ith unit vector of $\mathbb{R}^d$. Moreover, we assume that

$$\int_{h\neq 0} \big(1 \wedge |h|^2\big) a(h)\nu(h)\, dh < \infty \ \text{ and } \ \nu(h) = \nu(-h) > 0,\ \ h \neq 0. \tag{3}$$

By $\bar{a}$ we denote the *mean value* of a,

$$\bar{a} := \int_Q a(h)\, dh; \tag{4}$$

moreover, we assume that $a_\delta(h) := a\left(\delta^{-1}h\right)$ satisfies

$$\int_{h\neq 0} \left(1\wedge |h|^2\right) a_\delta(h)\nu(h)\,dh < \infty \quad \text{for all} \quad \delta > 0. \tag{5}$$

For each $\delta > 0$ we consider the following quadratic form in $L^2(\mathbb{R}^d)$ which is defined for Lipschitz continuous functions with compact support $u, v \in C_0^{\mathrm{lip}}(\mathbb{R}^d)$

$$\mathcal{E}^\delta(u,v) := \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(x)-u(y)\big)\big(u(x)-u(y)\big) a_\delta(y-x)\nu(y-x)\,dy\,dx. \tag{6}$$

From the assumptions (2) and (5), we easily see that $(\mathcal{E}^\delta, C_0^{\mathrm{lip}}(\mathbb{R}^d))$ is a closable symmetric form in $L^2(\mathbb{R}^d)$ which is translation invariant, see [3]. Its closure $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ is a translation invariant regular symmetric Dirichlet form in $L^2(\mathbb{R}^d)$, and the associated stochastic process is a symmetric Lévy process. If we use (1) and some elementary Fourier analysis, we obtain the following characterization of the Dirichlet form $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ based on the characteristic exponent ψ_δ, cf. [5, Example 4.7.28] and [3, Example 1.4.1],

$$\begin{cases} \mathcal{E}^\delta(u,v) = \displaystyle\int_{\mathbb{R}^d} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\psi_\delta(\xi)\,d\xi \\ \qquad\ \mathcal{F}^\delta = \left\{ u\in L^2(\mathbb{R}^d) :\ \displaystyle\int_{\mathbb{R}^d} \left|\widehat{u}(\xi)\right|^2 \psi_\delta(\xi)\,d\xi < \infty \right\}, \end{cases}$$

$\widehat{u}(\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle}u(x)\,dx$ denotes the Fourier transform and

$$\psi_\delta(\xi) = \int_{h\neq 0} \big(1-\cos\langle\xi,h\rangle\big) a_\delta(h)\nu(h)\,dh, \quad \xi\in\mathbb{R}^d. \tag{7}$$

Condition (5) ensures that $a_\delta(h)\nu(h)$ is the density of a Lévy measure. If $\nu(h)$ is the density of a Lévy measure and if a is a bounded, nonnegative (and 1-periodic) function, then (5) clearly holds. The following example illustrates that for *unbounded* functions a the situation is different.

1 Example. a) Let $0 < \beta < 2$ and pick some δ such that $0 < \delta < 1\wedge(2-\beta)$. Define functions α_0 on $[0, 1/2]$ and α_1 on $[0,1]$ by

$$\alpha_0(x) := \begin{cases} 0, & x = 0, \\ x^{-\delta}, & 0 < x \le \frac14, \\ 4^\delta, & \frac14 \le x \le \frac12, \end{cases} \quad \text{and} \quad \alpha_1(x) := \begin{cases} \alpha_0(x), & 0\le x\le \frac12, \\ \alpha_0(1-x), & \frac12 \le x \le 1. \end{cases}$$

Denote by $a:\mathbb{R}\to\mathbb{R}$ the 1-periodic extension of α_1 to the real line. It is obvious that $a\in L^p_{\mathrm{loc}}(\mathbb{R})$ for all $1 < p < 1/\delta$. Define a further function $b = b(x)$ on $\mathbb{R}$ by $b(x) := a(x-1/2)$ for $x\in\mathbb{R}$ and set

$$\nu(h) = \frac{b(h)}{|h|^{1+\beta}}, \quad h \neq 0.$$

Clearly, $\nu(h) = \nu(-h)$; let us show that $\nu(h)$ is the density of a Lévy measure, *i.e.* $\int_{h\neq 0}\left(1\wedge h^2\right)a(h)\nu(h)\,dh < \infty$.

Since a and ν are even functions, we see

$$\int_{h\neq 0} \big(1\wedge h^2\big) a(h)\nu(h)\,dh = 2\int_0^1 h^2 a(h)\nu(h)\,dh + 2\sum_{\ell=1}^{\infty}\int_{\ell}^{\ell+1} a(h)\nu(h)\,dh.$$

For the first term we get

$$\begin{aligned}\int_0^1 h^2 a(h)\nu(h)\,dh &= \int_0^1 h^2 a(h)b(h)h^{-1-\beta}\,dh\\ &= 4^{\delta}\int_0^{1/4} h^{1-\delta-\beta}\,dh + 4^{\delta}\int_{1/4}^{1/2} h^{1-\beta}(1/2-h)^{-\delta}\,dh\\ &\quad + 4^{\delta}\int_{1/2}^{3/4} h^{1-\beta}(h-1/2)^{-\delta}\,dh + 4^{\delta}\int_{3/4}^{1} h^{1-\beta}(1-h)^{-\delta}\,dh\\ &= c(\delta) < \infty.\end{aligned}$$

The integrals under the sum appearing in the second term can be estimated using the periodicity of a and b; for all $\ell \geq 1$ we have

$$\begin{aligned}\int_{\ell}^{\ell+1} a(h)\nu(h)\,dh &= \int_0^1 a(h+\ell)b(h+\ell)(h+\ell)^{-1-\beta}\,dh\\ &= \int_0^1 a(h)b(h)(h+\ell)^{-1-\beta}\,dh\\ &\leq \ell^{-1-\beta}\int_0^1 a(h)b(h)\,dh.\end{aligned}$$

As in the previous calculus, noting $0<\delta<1$ we see that

$$\begin{aligned}\int_0^1 a(h)b(h)\,dh &= 4^{\delta}\int_0^{1/4} h^{-\delta}dh + 4^{\delta}\int_{1/4}^{1/2} \big(1/2-h\big)^{-\delta}dh\\ &\quad + 4^{\delta}\int_{1/2}^{3/4} (h-1/2)^{-\delta}dh + 4^{\delta}\int_{3/4}^{1} (1-h)^{-\delta}dh < \infty.\end{aligned}$$

So $c := \int_0^1 a(h)b(h)\,dh < \infty$. Thus,

$$\int_{h\neq 0} \big(1\wedge h^2\big) a(h)\nu(h)\,dh \leq 2c(\delta) + c\sum_{\ell=1}^{\infty} \ell^{-1-\beta} < \infty.$$

On the other hand, we also find that

$$\begin{aligned}\int_{h\neq 0} \big(1\wedge h^2\big) a_{1/2}(h)\nu(h)\,dh &= \int_{h\neq 0} \big(1\wedge h^2\big) a(2h)b(h)|h|^{-1-\beta}\,dh\\ &\geq \int_{3/8}^{1/2} h^2 a(2h)b(h)h^{-1-\beta}\,dh\\ &= \int_{3/8}^{1/2} h^{1-\beta}(1-2h)^{-\delta}(1/2-h)^{-\delta}\,dh\\ &= 2^{\delta}\int_{3/8}^{1/2} h^{1-\beta}(1-2h)^{-2\delta}\,dh,\end{aligned}$$

and this integral blows up if $0 < \beta < 3/2$ and $1/2 \le \delta < 1 \wedge (2-\beta)$. In a similar way we can show that

$$\int_{h\neq 0} \left(1 \wedge h^2\right) a_\delta(h)\nu(h)\, dh = \infty$$

for infinitely many $\delta > 0$.

b) Let $a = a(x)$ on $\mathbb{R}$ be as in part (a)). Set $\nu(h) = |h|^{-1-\beta}$ for $h \neq 0$. Then we can show that this pair (a, ν) satisfies the conditions (2)–(5).

We will now discuss the limit of $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ as $\delta \downarrow 0$. To this end, we take a sequence of positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ such that $\delta_n \downarrow 0$ as $n \to \infty$.

2 Lemma. *Suppose that (2) and (5) hold for the function a. The measures $\{a_{\delta_n}(h)\, dh\}_{n\in\mathbb{N}}$ converge to the measure $\bar{a}\, dh$ in the vargue topology, i.e. for all compactly supported continuous functions $g \in C_0(\mathbb{R}^d)$ one has*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g(h) a_{\delta_n}(h)\, dh = \bar{a} \int_{\mathbb{R}^d} g(h)\, dh, \quad g \in C_0(\mathbb{R}^d). \tag{8}$$

Proof. We will show this lemma only in dimension $d = 1$, the case $d > 1$ adds only complications in notation. Fix $n \in \mathbb{N}$ and take any $g \in C_0(\mathbb{R})$. We have

$$\begin{aligned}\int_{\mathbb{R}} g(h) a_{\delta_n}(h)\, dh &= \int_{\mathbb{R}} g(h) a\left(\delta_n^{-1} h\right)\, dh \\ &= \delta_n \int_{\mathbb{R}} g(\delta_n h) a(h)\, dh = \delta_n \sum_{k=-\infty}^{\infty} \int_k^{k+1} g(\delta_n h) a(h)\, dh.\end{aligned}$$

Because of the periodicity of a, it follows that

$$\begin{aligned}\int_{\mathbb{R}} g(h) a_{\delta_n}(h)\, dh &= \delta_n \sum_{k=-\infty}^{\infty} \int_0^1 g(\delta_n(h+k)) a(h+k)\, dh \\ &= \delta_n \sum_{k=-\infty}^{\infty} \int_0^1 g(\delta_n(h+k)) a(h)\, dh.\end{aligned}$$

Since g has compact support,

$$g_{n,h}(\xi) := \delta_n \sum_{k=-\infty}^{\infty} g(\delta_n(h+k)) \mathbb{1}_{(\delta_n k, \delta_n(k+1)]}(\xi)$$

is, for fixed $h \in [0,1]$, a family of step functions (each with finitely many values) indexed by δ_n. It converges uniformly to $\int_{\mathbb{R}} g(\xi)\, d\xi$ as $n \to \infty$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}} g(h) a_{\delta_n}(h)\, dh &= \int_0^1 \left(\delta_n \sum_{k=-\infty}^{\infty} g(\delta_n(h+k))\right) a(h)\, dh \\ &= \int_0^1 \int_{\mathbb{R}} g_{n,h}(\xi)\, d\xi\, a(h)\, dh \\ &\xrightarrow[n\to\infty]{} \int_0^1 \left(\int_{\mathbb{R}} g(\xi)\, d\xi\right) a(h)\, dh.\end{aligned}$$

This proves (8) for $d = 1$. □

A similar argument leads to the following variant of Lemma 2.

3 Corollary. *Suppose that (2) and (5) hold. The family $\{a_{\delta_n}\}_{n\in\mathbb{N}}$ converges to the constant $\bar a := \int_Q a(h)\,dh$ weakly in $L^p_{\text{loc}}(\mathbb{R}^d)$, $1 < p < \infty$, i.e. for any compact set K of $\mathbb{R}^d$,*

$$\lim_{n\to\infty} \int_K g(x)a_{\delta_n}(x)\,dx = \bar a \int_K g(x)\,dx, \quad g \in L^q(K), \tag{9}$$

where p and q are conjugate $1/p + 1/q = 1$.

Proof. Let $K \subset \mathbb{R}^d$ be a compact set. We may regard $g\mathbb{1}_K$ as an element of $L^q(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^q(\mathbb{R}^d)$, there is for each $\epsilon > 0$, some $\phi = \phi_\epsilon \in C_0^\infty(\mathbb{R}^d)$ such that $\|\phi - g\mathbb{1}_K\|_{L^q(\mathbb{R}^d)} < \epsilon$ and $\operatorname{supp}\phi \subset K_1 := \{x + y : x \in K,\ |y| \le 1\} =: K_1$. So,

$$\begin{aligned}
&\left|\int_K g(x)a_{\delta_n}(x)\,dx - \bar a \int_K g(x)\,dx\right| \\
&= \left|\int_{\mathbb{R}^d} g(x)\mathbb{1}_K(x)a_{\delta_n}(x)\,dx - \bar a \int_{\mathbb{R}^d} g(x)\mathbb{1}_K(x)\,dx\right| \\
&\le \int_{K_1} \big|g(x)\mathbb{1}_K(x) - \phi(x)\big|a_{\delta_n}(x)\,dx + \left|\int_{\mathbb{R}^d} \phi(x)\big(a_{\delta_n}(x) - \bar a\big)\,dx\right| \\
&\quad + \bar a \int_{K_1} \big|\phi(x) - g(x)\mathbb{1}_K(x)\big|\,dx \\
&\le \|g\mathbb{1}_K - \phi\|_{L^q(\mathbb{R}^d)} \cdot \left(\int_{K_1} a_{\delta_n}(x)^p\,dx\right)^{1/p} + \left|\int_{\mathbb{R}^d} \phi(x)\big(a_{\delta_n}(x) - \bar a\big)\,dx\right| \\
&\quad + \bar a \cdot \operatorname{vol}(K_1)^{1/p} \cdot \|g\mathbb{1}_K - \phi\|_{L^q(\mathbb{R}^d)}.
\end{aligned}$$

By the previous lemma, the second term on the right hand side tends to 0 as $n \to \infty$. The expression $\|g\mathbb{1}_K - \phi\|_{L^q(\mathbb{R}^d)}$ can be made arbitrarily small if we choose ϕ accordingly. This means that it is enough to show that $\int_{K_1} a_\delta(x)^p\,dx$ is bounded for $0 < \delta < 1$. Again, we consider the one-dimensional case, the arguments for $d > 1$ just have heavier notation.

Without loss of generality we may assume that $K_1 = [-N, N]$ for some $N \in \mathbb{N}$. Take $k := \lfloor N/\delta \rfloor + 1 \in \mathbb{N}$, the smallest integer which is bigger or equal N/δ. We have

$$I = \int_{-N}^{N} a_\delta(x)^p\,dx = \delta \int_{-N/\delta}^{N/\delta} a(x)^p\,dx \le \delta \int_{-k}^{k-1} a(x)^p\,dx = \delta \sum_{\ell=-k}^{k-1} \int_\ell^{\ell+1} a(x)^p\,dx,$$

and, because of the periodicity of a,

$$I \le \delta \sum_{\ell=-k}^{k-1} \int_0^1 a(x+\ell)^p\,dx = 2k\delta \int_0^1 a(x)^p\,dx \le 2(N+1) \int_0^1 a(x)^p\,dx.$$

□

4 Corollary. *Assume that (2)–(5) hold and let $\{\delta_n\}_{n\in\mathbb{N}}$ be a monotonically decreasing sequence of positive numbers such that $\delta_n \to 0$ as $n \to \infty$. For any compact set $K \subset \mathbb{R}^d \times \mathbb{R}^d$, let $g_n \in L^q(K)$ be a sequence of functions which converges in L^q to some $g \in L^q(K)$. Then the following limit exists*

$$\lim_{n\to\infty} \iint_K g_n(x,y)a_{\delta_n}(x-y)\,dx\,dy = \bar a \iint_K g(x,y)\,dx\,dy. \tag{10}$$

Proof. Note that

$$
\begin{aligned}
&\Big|\iint\limits_K g_n(x,y)a_{\delta_n}(x-y)\,dx\,dy - \bar a\iint\limits_K g(x,y)\,dx\,dy\Big| \\
&\le \Big|\iint\limits_K \big(g_n(x,y)-g(x,y)\big)a_{\delta_n}(x-y)\,dx\,dy\Big| + \Big|\iint\limits_K g(x,y)\big(a_{\delta_n}(x-y)-\bar a\big)\,dx\,dy\Big| \\
&\le \Big[\iint\limits_K |g_n(x,y)-g(x,y)|^q\,dx\,dy\Big]^{\frac1q}\Big[\iint\limits_K a_{\delta_n}(x-y)^p\,dx\,dy\Big]^{\frac1p} + \Big|\int\limits_{\mathbb{R}^d} H(z)\big(a_{\delta_n}(z)-\bar a\big)\,dz\Big|
\end{aligned}
$$

where we use

$$H(z) := \int_{\mathbb{R}^d} \mathbb{1}_K(y+z,y)g(y+z,y)\,dy, \quad z\in\mathbb{R}^d.$$

Since K is a compact set of $\mathbb{R}^d\times\mathbb{R}^d$, we have (i) $\sup_{n\in\mathbb{N}}\iint_K a_{\delta_n}(x-y)^p\,dx\,dy<\infty$ as in the proof of Corollary 3, and (ii) the function H has compact support, hence $H\in L^q(\mathbb{R}^d)$. Therefore, the first term on the right hand side converges to 0 as $n\to\infty$, while the second term tends to 0 because of Corollary 3. □

Recall that a sequence of quadratic forms $\{(\mathcal{E}^n,\mathcal{F}^n)\}_{n\in\mathbb{N}}$ defined on $L^2(\mathbb{R}^d)$ is called *Mosco-convergent*, if the following two conditions are satisfied

(M1) For all $u\in L^2(\mathbb{R}^d)$ and all sequences $(u_n)_{n\in\mathbb{N}}\subset L^2(\mathbb{R}^d)$ such that $u_n\rightharpoonup u$ (weak convergence in L^2) we have $\liminf\limits_{n\to\infty}\mathcal{E}^n(u_n,u_n)\ge\mathcal{E}(u,u)$.

(M2) For every $u\in\mathcal{F}$ there exist elements $u_n\in\mathcal{F}^n$, $n\in\mathbb{N}$, such that $u_n\to u$ (strong convergence in L^2) and $\limsup\limits_{n\to\infty}\mathcal{E}^n(u_n,u_n)\le\mathcal{E}(u,u)$.

Note that **(M1)** entails that we have $\limsup_{n\to\infty}\mathcal{E}^n(u_n,u_n)=\mathcal{E}(u,u)$ in **(M2)**.

We can now state the main result of our paper.

5 Theorem. *Assume that* (2)–(5) *hold for the functions a and ν, and let ν be locally bounded as a function defined on $\mathbb{R}^d\setminus\{0\}$. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be a monotonically decreasing sequence of positive numbers such that $\delta_n\to0$ as $n\to\infty$. For each $n\in\mathbb{N}$ we consider the Dirichlet forms $(\mathcal{E}^n,\mathcal{F}^n):=(\mathcal{E}^{\delta_n},\mathcal{F}^{\delta_n})$ defined in* (6)*. The Dirichlet forms $(\mathcal{E}^n,\mathcal{F}^n)$ converge to $(\mathcal{E},\mathcal{F})$ in the sense of Mosco. The limit $(\mathcal{E},\mathcal{F})$ is the closure of $(\mathcal{E},C_0^{\mathrm{lip}}(\mathbb{R}^d))$ which is given by*

$$\mathcal{E}(u,v) := \bar a\iint_{\mathbb{R}\times\mathbb{R}}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\nu(y-x)\,dy\,dx, \quad u,v\in C_0^{\mathrm{lip}}(\mathbb{R}^d).$$

Proof. We will check the conditions **(M1)** and **(M2)** of Mosco convergence. For **(M1)** we take any $u\in L^2(\mathbb{R}^d)$ and any sequence $\{u_n\}\subset L^2(\mathbb{R}^d)$ such that $u_n\rightharpoonup u$ as $n\to\infty$. Without loss, we may assume that $\liminf_{n\to\infty}\mathcal{E}^n(u_n,u_n)<\infty$.

We will use the Friedrichs mollifier. This is a family of convolution operators

$$J_\epsilon[u](x) = \int_{\mathbb{R}^d} u(x-y)\rho_\epsilon(y)\,dy, \quad x\in\mathbb{R}^d,\ \epsilon>0,$$

given by the kernels $\{\rho_\epsilon\}_{\epsilon>0}$ for a C^∞-kernel $\rho : \mathbb{R}^d \to [0,\infty)$ satisfying

$$0 \le \rho(x) = \rho(-x), \quad \int_{\mathbb{R}^d} \rho(x)\,dx = 1, \quad \operatorname{supp}[\rho] = \big\{x \in \mathbb{R}^d : \ |x| \le 1\big\}$$
$$\text{and} \quad \rho_\epsilon(x) := \rho(x/\epsilon), \quad \text{for } \epsilon > 0 \text{ and } x \in \mathbb{R}^d.$$

We then have

$$\begin{aligned}
&\mathcal{E}^n(u_n,u_n)\\
&\quad = \iint_{x\neq y} \big(u_n(x)-u_n(y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,dy\,dx\\
&\quad = \int_{\mathbb{R}^d}\left(\iint_{x\neq y} \big(u_n(x)-u_n(y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,dy\,dx\right)\rho_\epsilon(z)\,dz\\
&\quad = \int_{\mathbb{R}^d}\left(\iint_{x\neq y} \big(u_n(x-z)-u_n(y-z)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,dy\,dx\right)\rho_\epsilon(z)\,dz,
\end{aligned}$$

and using the Fubini theorem and Jensen's inequality yields, for any compact set K so that $K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$,

$$\begin{aligned}
&\mathcal{E}^n(u_n,u_n)\\
&\quad = \iint_{x\neq y}\left(\int_{\mathbb{R}^d} \big(u_n(x-z)-u_n(y-z)\big)^2 \rho_\epsilon(z)\,dz\right) a_{\delta_n}(y-x)\nu(y-x)\,dy\,dx\\
&\quad \ge \iint_{x\neq y}\left(\int_{\mathbb{R}^d} \big(u_n(x-z)-u_n(y-z)\big) \rho_\epsilon(z)\,dz\right)^2 a_{\delta_n}(y-x)\nu(y-x)\,dy\,dx\\
&\quad \ge \iint_{K} \big(J_\epsilon[u_n](x)-J_\epsilon[u_n](y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,dy\,dx.
\end{aligned}$$

Note that $\sup_{n\in\mathbb{N}} \|u_n\|_{L^2} < \infty$ because of the weak convergence $u_n \rightharpoonup u$. Again by weak convergence, $u_n \rightharpoonup u$, and we conclude that $u_{n,\epsilon} = J_\epsilon[u_n]$ converges pointwise to $u_\epsilon := J_\epsilon[u]$. Using the local boundedness of ν on $\mathbb{R}^d\setminus\{0\}$ and the fact that K is a compact set satisfying $K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$, we see that $\big(u_{n,\epsilon}(x) - u_{n,\epsilon}(y)\big)^2\nu(y-x)$ converges in $L^q(K)$ to

$$\big(u_\epsilon(x)-u_\epsilon(y)\big)^2\nu(y-x) \quad \text{as } n\to\infty.$$

From (10) we get

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{E}^n(u_n,u_n) &\ge \liminf_{n\to\infty} \mathcal{E}^n(u_{n,\epsilon},u_{n,\epsilon})\\
&\ge \liminf_{n\to\infty} \iint_K \big(u_{n,\epsilon}(x)-u_{n,\epsilon}(y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,dy\,dx\\
&= \bar{a}\iint_K \big(u_\epsilon(x)-u_\epsilon(y)\big)^2\nu(y-x)\,dy\,dx.
\end{aligned}$$

Since $K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$ is an arbitrary compact set, we can approximate $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$ by such sets. Using monotone convergence and the fact that the left hand side is independent of K, L, we arrive at

$$\begin{aligned}
\sup_{0<\epsilon<1} \mathcal{E}(u_\epsilon,u_\epsilon) &= \sup_{0<\epsilon<1} \sup_{\substack{K:\ \text{compact}\\ K\subset\mathbb{R}^d\times\mathbb{R}^d\setminus\{(x,x):x\in\mathbb{R}^d\}}} \bar{a}\iint_K \big(u_\epsilon(x)-u_\epsilon(y)\big)^2\nu(y-x)\,dy\,dx\\
&\le \liminf_{n\to\infty} \mathcal{E}^n(u_n,u_n) < \infty. \qquad (11)
\end{aligned}$$

Theorem 2.4 in [8] now shows that $u_\epsilon \in \mathcal{F} \cap C^\infty(\mathbb{R}^d)$ for each $\epsilon \in (0,1)$. Since J_ϵ is an L^2-contraction operator for each $\epsilon > 0$, we see that the family $\{u_\epsilon\}_{\epsilon>0}$, $u_\epsilon = J_\epsilon[u]$, is bounded w.r.t. $\mathcal{E}_1(\bullet,\bullet) := \mathcal{E}(\bullet,\bullet) + (\bullet,\bullet)_{L^2}$ by (11). The Banach–Alaoglu theorem guarantees that there is an $\mathcal{E}_1$-weakly convergent subsequence $u_{\epsilon(n)}$, $\epsilon(n) \downarrow 0$, and a function v so that $u_{\epsilon(n)}$ converges $\mathcal{E}_1$-weakly to $v \in \mathcal{F}$. Using the Banach–Saks theorem shows that the Cesàro means $\frac{1}{n}\sum_{k=1}^n u_{\epsilon(n_k)}$ of a further subsequence converge $\mathcal{E}_1$-strongly, hence in $L^2(\mathbb{R}^d)$, to v. As u_ϵ converges to u in $L^2(\mathbb{R}^d)$, we can identify the limit as $u = v$. In particular, $u \in \mathcal{F}$ and

$$\liminf_{n\to\infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u,u).$$

In order to see **(M2)**, we use the regularity of the Dirichlet form $(\mathcal{E},\mathcal{F})$; therefore, it is enough to consider $u \in C_0^{\mathrm{lip}}(\mathbb{R}^d)$. Set $u_n = u \in C_0^{\mathrm{lip}}(\mathbb{R}^d)$ for each n. We have

$$\mathcal{E}^n(u_n,u_n) = \mathcal{E}^n(u,u) = \iint_{x\neq y} \big(u(x)-u(y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,dy\,dx,$$

and we conclude with (8) that $\lim_{n\to\infty} \mathcal{E}^n(u_n,u_n) = \mathcal{E}(u,u)$. □

6 Remark. Suppose that the function a on $\mathbb{R}$ satisfies (2)–(5), and ν is given by $\nu(x) = |x|^{-1-\alpha}$, $x \in \mathbb{R}\setminus\{0\}$, for some $0 < \alpha < 2$. Then the following quadratic form defines a translation invariant regular symmetric Dirichlet form on $L^2(\mathbb{R})$:

$$\tilde{\mathcal{E}}(u,v) := \iint_{x\neq y} \big(u(x)-u(y)\big)\big(u(x)-u(y)\big)\frac{a(x-y)}{|x-y|^{1+\alpha}}\,dx\,dy, \quad u,v \in C_0^{\mathrm{lip}}(\mathbb{R}).$$

Let $\tilde{X} = (\tilde{X}(t))_{t\geq 0}$ be the symmetric Lévy process on $\mathbb{R}$ associated with the Dirichlet form $(\tilde{\mathcal{E}},\tilde{\mathcal{F}})$ on $L^2(\mathbb{R})$. For any $n \in \mathbb{N}$, set

$$X^{(n)}(t) := \epsilon_n \tilde{X}(\epsilon_n^{-\alpha} t), \quad t > 0.$$

Then $X^{(n)} = (X^{(n)}(t))_{t\geq 0}$ is also a symmetric Lévy process and we denote for each $n \in \mathbb{N}$ by $(\mathcal{E}^{(n)},\mathcal{F}^{(n)})$ the corresponding Dirichlet form. The semigroup $\{T_t^{(n)}\}_{t>0}$ generated by $(\mathcal{E}^{(n)},\mathcal{F}^{(n)})$ is given by

$$\begin{aligned} T_t^{(n)} f(x) &= \mathbb{E}\left[f(X^{(n)}(t)) \mid X^{(n)}(0) = x\right] \\ &= \mathbb{E}_{x/\epsilon_n}\Big[f(\epsilon_n \tilde{X}(\epsilon_n^{-\alpha} t))\Big] = \Big(\tilde{T}_{\epsilon_n^{-\alpha} t} f(\epsilon_n \cdot)\Big)(\epsilon_n^{-1} x), \quad x \in \mathbb{R}. \end{aligned}$$

Since the Dirichlet form $(\mathcal{E}^{(n)},\mathcal{F}^{(n)})$ can be obtained by

$$\mathcal{E}^{(n)}(u,v) = \lim_{t\downarrow 0} \frac{1}{t}\big(u - T_t^{(n)} u, v\big)_{L^2},$$

it follows for $t > 0$ that

$$\begin{aligned} \frac{1}{t}\big(u - T_t^{(n)}u, v\big)_{L^2} &= \frac{1}{t}\int_{\mathbb{R}} \big[u(x) - T_t^{(n)} u(x)\big] v(x)\,dx \\ &= \frac{1}{t}\int_{\mathbb{R}} \big[u(\epsilon_n \cdot \epsilon_n^{-1} x) - \big(\tilde{T}_{\epsilon_n^{-\alpha} t} u(\epsilon_n \bullet)\big)(\epsilon_n^{-1} x)\big] v(x)\,dx \\ &= \frac{1}{\epsilon_n^\alpha}\cdot\frac{1}{s}\int_{\mathbb{R}} \big[u(\epsilon_n \xi) - \big(\tilde{T}_s u(\epsilon_n \bullet)\big)(\xi)\big] v(\epsilon_n \xi)\epsilon_n\,d\xi \end{aligned}$$

where we use the notation $\xi = \epsilon_n^{-1}x$ and $s = \epsilon_n^{-\alpha}t$. Letting $s \to 0$, hence $t \to 0$, yields

$$
\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\big(u - T_t^{(n)}u, v\big)_{L^2}\\
&\quad= \epsilon_n^{1-\alpha}\cdot\tilde{\mathcal{E}}\big(u(\epsilon_n\bullet), v(\epsilon_n\bullet)\big)\\
&\quad= \epsilon_n^{1-\alpha}\int_{\mathbb{R}}\big(u(\epsilon_n x) - u(\epsilon_n y)\big)\big(v(\epsilon_n x) - v(\epsilon_n y)\big)\frac{a(x-y)}{|x-y|^{1+\alpha}}\,dx\,dy\\
&\quad= \epsilon_n^{1-\alpha}\int_{\mathbb{R}}\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\frac{a\big(\epsilon_n^{-1}(x-y)\big)}{|x-y|^{1+\alpha}}\epsilon_n^{1+\alpha}\frac{dx}{\epsilon_n}\frac{dy}{\epsilon_n}\\
&\quad= \int_{\mathbb{R}}\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\frac{a(\epsilon_n^{-1}(x-y))}{|x-y|^{1+\alpha}}\,dx\,dy\\
&\quad= \mathcal{E}^{(n)}(u,v).
\end{aligned}
$$

Since Mosco convergence entails the convergence of the semigroups, hence the finite-dimensional distributions (fdd) of the processes, we may combine the above calculation with Theorem 5 to get the following result: *The processes $X^{(n)}$ associated with $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ – these are obtained by scaling $t \mapsto \epsilon_n^{-\alpha}t$ and $x \mapsto \epsilon_n x$ from the process $\tilde{X}$ given by $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ – converge, in the sense of fdd, to the process X associated with $(\mathcal{E}, \mathcal{F})$.* This is the Dirichlet form apporach to the problem discussed in [4] (see also [10, 7, 6] for related works and [9] for Mosco convergence of Dirichlet forms).

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