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Hu, Yaozhong: *Analysis on Gaussian Spaces*. World Scientific, Hackensack (NJ) 2017. xi + 470 pp., £129.00, US-$ 147.25 (RRP). ISBN 978-981-3142-17-6.

In infinite dimensions there is no non-trivial measure which is locally finite and invariant under translation, that is: There is no Lebesgue measure in Banach space. On the other hand, it is well known that the concept of Gaussian measures extends from the finite-dimensional to the infinite-dimensional setting. The present monograph provides an introduction to Gaussian measures in finite dimensional spaces and to abstract Wiener space, Malliavin's calculus (stochastic calculus of variations) and Gaussian measures on Banach spaces. In the second half of the book the power of the so far developed method is illustrated by several applications.

This monograph differs from many existing and usually more specialized books in at least two respects: First of all, despite the advanced character of the material, it is mainly self-contained (only a few topics are deferred to the appendix, and—if feasible—complete proofs are given), and secondly by the choice of the material: It is indeed a pleasant surprise to find material on classical Gaussian analysis, Malliavin calculus and the stochastic analysis of Gaussian processes not only next to each other but also combined in a consistent treatment. The mainly analytical and from-scratch presentation of the material does give novices and colleagues from various fields the possibility to get a foothold in these exciting topics.

The first half of the book (Chapters 2–6, pp. 7–217, Chapter 1 is just a short guide to the monograph) are a concise introduction to Gaussian measures in finite- and infinite-dimensional spaces. The starting point is a self-contained proof of the powerful but still little-observed classical Garsia–Rodemich–Rumsey inequalities, including a new extension to the case of multiparameter processes, and its application to the path regularity of Gaussian random processes and fields. Chapter 3 discusses the basics of Gaussian measure in Euclidean spaces, with an elegant proof of the Brunn–Minkowski inequality, the spectral gap and log-Sobolev inequality and hypercontractivity (with a clever new semigroup proof) – to mention but a few topics. In this chapter also Hermite polynomials and functions are introduced, preparing the field for iterated stochastic integrals and chaos decompositions. Gaussian measures on Banach spaces are the topic of Chapter 4. The author begins with a discussion of Gaussian vectors and probability in Banach space (Kahane–Khintchine inequalities, Itô-Nisio lemma) and then moves on to introduce abstract as well as canonical Wiener space. The last two sections are devoted to right-tail and left-tail (aka: small ball probability) exponential estimates for Gaussian random variables in a Banach space. The next two chapters are devoted to nonlinear functionals on abstract Wiener space: Chapter 5 gives an introduction to Fock space, chaos expansions and their relation to multiple Wiener–Itô integrals. Very useful is the rarely-found discussion on multiple Stratronovich integrals. As applications, we encounter exponential right-tail estimates of a homogeneous Wiener chaos and a chaos expansion of the exit time of a Wiener functional. This line of thought continues in Chapter 6 with a further analysis of Wiener functionals. This is essentially an introduction to Malliavin's calculus, following the classical approach of P.A. Meyer in the presentation of S. Watanabe [*Lectures on Stochastic Differential Equations and Malliavin Calculus*. Tata Institute of Fundamental Research & Springer, Bombay 1984. Zbl 0546.60054], see also Ikeda and Watanabe [*Stochastic Differential Equations and Diffusion Processes*. North–Holland & Kodansha, Amsterdam & Tokyo 1989, 2nd edn. Zbl 0684.60040]. This is rounded-off by a careful discussion of Girsanov's theorem in the classical setting as well as in infinite dimension. The last sections of this chapter give an introduction to Wick products – thus making the relation to Hida's white noise analysis – and to the noncommutative composition of Wiener functionals; the latter is then applied to functionals of a Brownian motion stopped at an anticipative exit time.

The second half of the book (Chapters 7–11, pp. 219–426) contains applications of the material developed in the first part. The opening chapter (Chapter 7) continues the exposition on Malliavin's calculus and the chaos decomposition. First, necessary and sufficient criteria are derived, guaranteeing that the second quantization of (the complexification of) a linear map between two abstract Wiener spaces is hypercontractive. Then the theory of L^p-style Sobolev spaces on an abstract Wiener space is developed by discussing P.A. Meyer's inequalities, the related multiplier theorem and Meyer's version of the Littlewood–Paley–Stein theory. The chapter closes with interpolation inequalities for Sobolev spaces on an abstract Wiener space and Grothendieck's inequality. In Chapter 8 C. Stein's characterization of Gaussian measure (aka Stein's method) is discussed in connection with general Wiener functionals. The main thrust of the exposition is on the existence and the convergence (to the normal density) of densities of certain Wiener functionals; this includes also a discussion of the smoothness of the density and – if applicable – the representation of the derivatives. Chapter 9 has a discussion of Brownian local time and self-intersection local time; this is fairly standard material. The last two chapters of the monograph are devoted to the study of

Brownian-driven SDEs. First the existence and uniqueness of solutions (in an L^p-setting and for the formally Malliavin-differentiated solutions) is discussed. Then, understanding the solution as a functional in abstract Wiener space, Hörmander's theorem and the smoothness of the solution in Wiener–Sobolev spaces is treated. In a one-dimensional setting the Itô–Wiener chaos expansion of the solution is presented. Finally, conditions for hypercontractivity and the spectral and log-Sobolev inequalities for the solutions are presented from a semigroup point of view. In the last chapter the author studies Wong–Zakai type approximation schemes for Brownian SDEs, in particular the rate of convergence in L^p- and Wiener–Sobolev spaces.

Overall, this book is an interesting and welcome addition to the existing literature; mathematically, it is sound and carefully written. Although the proofs are very detailed, the literary style is somewhat 'abrupt' and there is a general lack of introducing and explaining texts. There are quite a number of typos which make the book less enjoyable to read as it could be. The ordering and presentation of the bibliography make me doubt whether there was any serious proof-reading. A three-and-a-half page Index for a 400-plus page monograph is meager and important keywords (e.g. Stein's method etc.) are indeed missing. Unfortunately, a list of notation also remains a desideratum.

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