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M.B. Marcus and J. Rosen: Markov Processes, Gaussian Processes, and Local Times. Cambridge University Press, Cambridge studies in advanced mathematics vol. **100**, Cambridge 2006, x + 620 pp., £52.00, US-$ 95.00 ISBN 0-521-86300-7.

It is easy to tell what this book *does not want to be*: a general introduction to Markov processes, Gaussian processes and local times in one volume. Nevertheless the reader will learn a good deal of this material as it is needed to understand the true topic of the volume, the study of various isomorphism theorems and their application to local times. *Isomorphism theorems* connect mean-zero Gaussian random fields with local times for a class of Markov processes; the first of these results is due to E.B. Dynkin [*J. Funct. Anal.* **55** (1984), 344376, Zbl. 0533.60061], further theorems are due to N. Eisenbaum (and co-authors) [*Sém. Probab.* **29** (1995), 266289, Springer LNM 1613, Zbl. 0849.60075; *Ann. Probab.* **28** (2000), 17811796, Zbl. 1044.60064]. The blueprint of all isomorphism theorems are the Ray-Knight theorems which relate Brownian local times with squares of independent Brownian motions. The trouble here is that Brownian motion is both the archetype of a strongly symmetric Markov process and all Gaussian processes—and it is a major step to see that it is the Markov character which is important for the local times while the Gaussian nature is 'associated' to the local times. This connection is achieved via the covariance function of the Gaussian process which is just the 0-potential density of the Markov process. Alternatively, e.g. if there is no finite 0-potential density, one could associate the whole zoo of α-potential densities, $\alpha > 0$, of the Markov process. This point of view opens a whole new world: one can study local times of strongly symmetric Markov processes through Gaussian processes and vice versa. The authors of the present monograph started this line of research in a series of seminal papers which appeared in 1992, [*Ann. Probab.* **20** (1992) 1603-1684 and 1685-1713, Zbl. 0762.60068, 0762.60069]. Already there they gave necessary and sufficient criteria for the regularity, boundedness, variational properties etc. of the local times of a symmetric Markov process in terms of the corresponding properties of the associated Gaussian process. One of the most striking examples is probably the fact that the local time L_t^x of a strongly symmetric Markov process admits a jointly continuous version if, and only if, the associated Gaussian field is continuous.

The early proofs of the isomorphism theorems are relatively cumbersome—they rely heavily on rather intricate combinatorial methods—and the first statements are not as explicit as the Ray-Knight theorems for Brownian motion. The latter changed later on, mostly because of the authors' own and N. Eisenbaum's work, but the results and their applications remained scattered in the research literature. The primary motivation of the authors was therefore, to present the material in a unified way with much simpler proofs, many of them appearing for the first time in print. Nevertheless it takes some background knowledge in Markov and Gaussian processes, in particular potential theory and sample path properties, to understand the matter and this presents a formidable hurdle to the novice and non-specialist. Not many (readily accessible...) books on probabilistic potential theory are around and still fewer books are devoted to sample path properties of Gaussian processes; only X. Fernique's *Fonctions aléatoires Gaussiennes, vecteurs aléatoires Gaussiens* [CRM Monographs, Montréal 1997, Zbl. 0902.60030] and, to a lesser extent, Ledoux and Talagrand's *Probability on Banach spaces* [Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Ser., 1991, Zbl. 0748.60004] spring to mind. Being aware of this, the authors devoted a considerable part of their monograph to (the potential theory and local times of) Markov processes and (the study of sample paths of) Gaussian random fields. Without prior exposure to stochastic processes, I am sure it will take some long winter evenings to cope with this part, but this will be time well spent: the choice of the material and the exposition are worth it.

The books starts with a gentle reminder of Brownian motion and the Ray-Knight theorems. This is a congenial opening since it contains all the topics to come, however tightly knotted together by Brownian motion. In the subsequent chapters the authors carefully undo the knot, separate and analyse the three threads only to knit them together in the end. The first thread are Markov processes, their potential theory and their local times. Two chapters and approximately 120 pages deal with mostly symmetric Markov processes having transition and resolvent densities. Some emphasis is put on special classes of Markov processes, Feller and Lévy processes. The next three chapters, Chapters 5–7, are devoted to Gaussian processes, in particular the study of limit laws and the boundedness, continuity and exact moduli of continuity of the trajectories. Although much of the material has been around since the seventies, it appears for the first time in a monograph, now with modern and streamlined proofs. After these extensive preparations—they occupy more than 360 of the book's 620 pages—the authors can begin with the main topic: the proof of the Dynkin and Eisenbaum isomorphism theorems. In Chapter 8 they give several derivations of these results and show why they should be seen as generalizations of the Ray-Knight theorems. In the remaining four chapters several applications of the isomorphism theorems are given. The philosophy is easily described: using the isomorphism theorems one can derive necessary and sufficient criteria for properties of the local times in terms of the associated Gaussian processes; their properties, however, were reviewed before so that the all threads come together again. Chapters 9 and 10 contain the material on joint continuity and variational properties of local times and Chapter 11 investigates the most visited sites of symmetric stable Lévy processes. The last two chapters deal with the scope of the theory: in Chapter 12 it is shown that the direct counterpart of the Ray-Knight theory on h-transforms will only hold for diffusions. The last chapter takes up the problem to find a characterization of the associated Gaussian processes.

This is a masterly written text which should be accessible to advanced graduate students and non-specialists. For the researcher interested in Gaussian processes or local times it will become an indispensable standard resource.

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