New Characterizations of Homogeneous and Mixed Poisson Processes

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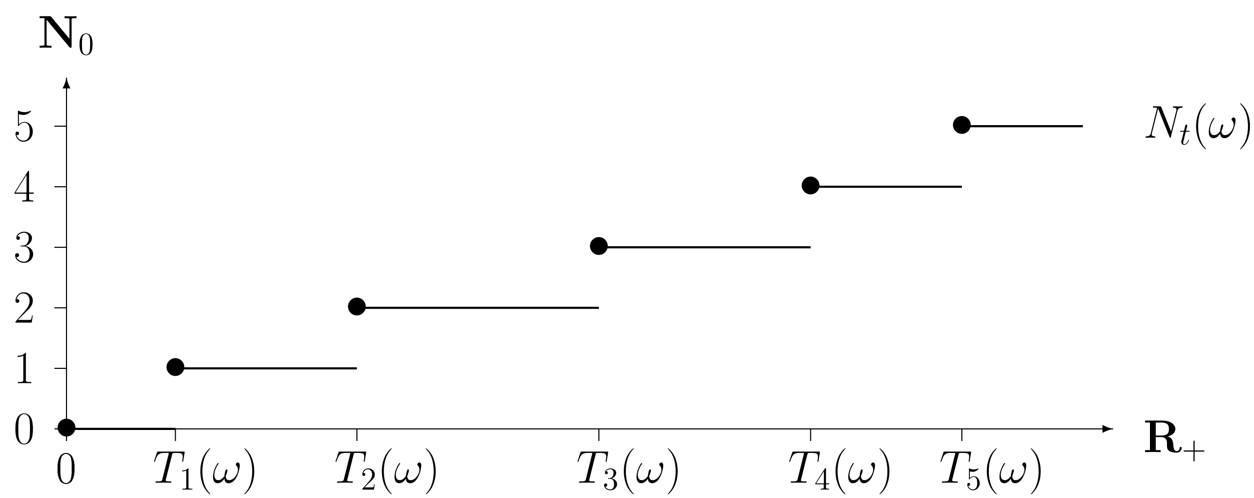
1 Claim Number Processes

A stochastic process $\{N_t\}_{t\in\mathbf{R}_+}$ is a **claim number process** if the following properties hold for all $\omega \in \Omega$:

- $N_0(\omega) = 0$,
- $N_t(\omega) \in \mathbf{N}_0$ for all $t \in (0, \infty)$,
- $N_t(\omega) = \inf_{s\in(t,\infty)} N_s(\omega)$ for all $t \in \mathbf{R}_+$,
- $\sup_{s\in[0,t)} N_s(\omega) \le N_t(\omega) \le \sup_{s\in[0,t)} N_s(\omega) + 1$ for all $t \in \mathbf{R}_+$, and
- $\sup_{t\in\mathbf{R}_+} N_t(\omega) = \infty$.



2 Homogeneous Poisson Processes

A claim number process $\{N_t\}_{t\in\mathbf{R}_+}$ is a **homogeneous Poisson process** with parameter $\alpha \in (0,\infty)$ if

$$P\left[\bigcap_{j=1}^{m}\{N_{t_j}-N_{t_{j-1}}=k_j\}\right] \;=\; \prod_{j=1}^{m} e^{-\alpha(t_j-t_{j-1})}\,\frac{(\alpha(t_j-t_{j-1}))^{k_j}}{k_j!}$$

holds for all $m\in\mathbf{N}$, $0=t_0<t_1<\cdots<t_m$ and $k_1,\dots,k_m\in\mathbf{N}_0$.

2.1 Lemma. *For a claim number process $\{N_t\}_{t\in\mathbf{R}_+}$ and $\alpha\in(0,\infty)$, the following are equivalent:*

(a) $\{N_t\}_{t\in\mathbf{R}_+}$ *is a homogeneous Poisson process with parameter α.*

(b) *The identities*

$$P\left[\bigcap_{j=1}^{m}\{N_{t_j}-N_{t_{j-1}}=k_j\}\right] \;=\; \frac{n_m!}{\prod_{j=1}^{m}k_j!}\prod_{j=1}^{m}\left(\frac{t_j-t_{j-1}}{t_m}\right)^{k_j}\cdot P[\{N_{t_m}=n_m\}]$$

and

$$P[\{N_{t_m}=n_m\}] \;=\; e^{-\alpha t_m}\,\frac{(\alpha t_m)^{n_m}}{n_m!}$$

hold for all $m\in\mathbf{N}$, $0=t_0<t_1<\cdots<t_m$, $k_1,\dots,k_m\in\mathbf{N}_0$ and $n_m=\sum_{j=1}^{m}k_j$.

3 Mixed Poisson Processes

A claim number process $\{N_t\}_{t \in \mathbf{R}_+}$ is a **mixed Poisson process** with mixing distribution $Q : \mathcal{B}(\mathbf{R}) \to [0,1]$ satisfying $Q[(0,\infty)] = 1$ if

$$P\left[\bigcap_{j=1}^{m} \{N_{t_j} - N_{t_{j-1}} = k_j\}\right] \;=\; \int_{(0,\infty)} \prod_{j=1}^{m} e^{-\alpha(t_j - t_{j-1})} \, \frac{(\alpha(t_j - t_{j-1}))^{k_j}}{k_j!} \, dQ(\alpha)$$

holds for all $m \in \mathbf{N}$, $0 = t_0 < t_1 < \cdots < t_m$ and $k_1, \ldots, k_m \in \mathbf{N}_0$.

3.1 Lemma. *For a claim number process $\{N_t\}_{t \in \mathbf{R}_+}$ and a distribution $Q : \mathcal{B}(\mathbf{R}) \to [0,1]$ satisfying $Q[(0,\infty)] = 1$, the following are equivalent:*

(a) *$\{N_t\}_{t \in \mathbf{R}_+}$ is a mixed Poisson process with mixing distribution Q.*

(b) *The identities*

$$P\left[\bigcap_{j=1}^{m} \{N_{t_j} - N_{t_{j-1}} = k_j\}\right] \;=\; \frac{n_m!}{\prod_{j=1}^{m} k_j!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{k_j} \cdot P[\{N_{t_m} = n_m\}]$$

and

$$P[\{N_{t_m} = n_m\}] \;=\; \int_{(0,\infty)} e^{-\alpha t_m} \, \frac{(\alpha t_m)^{n_m}}{n_m!} \, dQ(\alpha)$$

hold for all $m \in \mathbf{N}$, $0 = t_0 < t_1 < \cdots < t_m$, $k_1, \ldots, k_m \in \mathbf{N}_0$ and $n_m = \sum_{j=1}^{m} k_j$.

4 The Multinomial Property

A claim number process $\{N_t\}_{t \in \mathbf{R}_+}$ has the **multinomial property** if the identity

$$P\left[\bigcap_{j=1}^{m}\{N_{t_j}-N_{t_{j-1}}=k_j\}\right] = \frac{n_m!}{\prod_{j=1}^{m} k_j!} \prod_{j=1}^{m}\left(\frac{t_j-t_{j-1}}{t_m}\right)^{k_j} \cdot P[\{N_{t_m}=n_m\}]$$

holds for all $m \in \mathbf{N}$, $0=t_0<t_1<\cdots<t_m$, $k_1,\ldots,k_m \in \mathbf{N}_0$ and $n_m=\sum_{j=1}^{m} k_j$.
If $P[\{N_{t_m}=n_m\}]>0$, then the previous identity can be written as

$$P\left[\bigcap_{j=1}^{m}\{N_{t_j}-N_{t_{j-1}}=k_j\} \,\middle|\, \{N_{t_m}=n_m\}\right] = \frac{n_m!}{\prod_{j=1}^{m} k_j!} \prod_{j=1}^{m}\left(\frac{t_j-t_{j-1}}{t_m}\right)^{k_j}$$

which explains the name of the multinomial property.

Claim number processes having the multinomial property are of interest, since their finite–dimensional distributions are completely determined by their one–dimensional distributions.

The multinomial property is suitable for statistical tests.

Problem 1: Characterize all claim number processes having the multinomial property.

5 Related Properties

A claim number process $\{N_t\}_{t\in\mathbf{R}_+}$ has the **binomial property** if the identity

$$P[\{N_s = k\} \cap \{N_t - N_s = n-k\}] \; = \; \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k} \cdot P[\{N_t = n\}]$$

holds for all $0 < s < t$ and $k, n \in \mathbf{N}_0$ such that $k \leq n$.

A claim number process $\{N_t\}_{t\in\mathbf{R}_+}$ has the **Markov property** if the identity

$$P\left[\bigcap_{j=1}^{m+1}\{N_{t_j}-N_{t_{j-1}}=k_j\}\right]\cdot P[\{N_{t_m}=n_m\}]$$
$$= \; P\left[\bigcap_{j=1}^{m}\{N_{t_j}-N_{t_{j-1}}=k_j\}\right]\cdot P[\{N_{t_m}=n_m\}\cap\{N_{t_{m+1}}-N_{t_m}=k_{m+1}\}]$$

holds for all $m\in\mathbf{N}$, $0=t_0<t_1<\cdots<t_m<t_{m+1}$, $k_1,\ldots,k_m,k_{m+1}\in\mathbf{N}_0$ and $n_m=\sum_{j=1}^m k_j$. If $P[\bigcap_{j=1}^m\{N_{t_j}=n_j\}]>0$, then the previous identity can be written as

$$P\left[\{N_{t_{m+1}}=n_{m+1}\}\,\middle|\,\bigcap_{j=1}^{m}\{N_{t_j}=n_j\}\right] \;=\; P[\{N_{t_{m+1}}=n_{m+1}\}|\{N_{t_m}=n_m\}]$$

with $n_j=\sum_{i=1}^j k_i$ for all $j\in\{1,\ldots,m,m+1\}$.

5.1 Lemma. *For a claim number process $\{N_t\}_{t\in\mathbf{R}_+}$, the following are equivalent:*
(a) $\{N_t\}_{t\in\mathbf{R}_+}$ *has the multinomial property.*
(b) $\{N_t\}_{t\in\mathbf{R}_+}$ *has the binomial property and the Markov property.*

A claim number process $\{N_t\}_{t \in \mathbf{R}_+}$ has **independent increments** if

$$P\left[\bigcap_{j=1}^{m} \{N_{t_j} - N_{t_{j-1}} = k_j\}\right] = \prod_{j=1}^{m} P[\{N_{t_j} - N_{t_{j-1}} = k_j\}]$$

holds for all $m \in \mathbf{N}$, $0 = t_0 < t_1 < \cdots < t_m$ and $k_1, \ldots, k_m \in \mathbf{N}_0$.

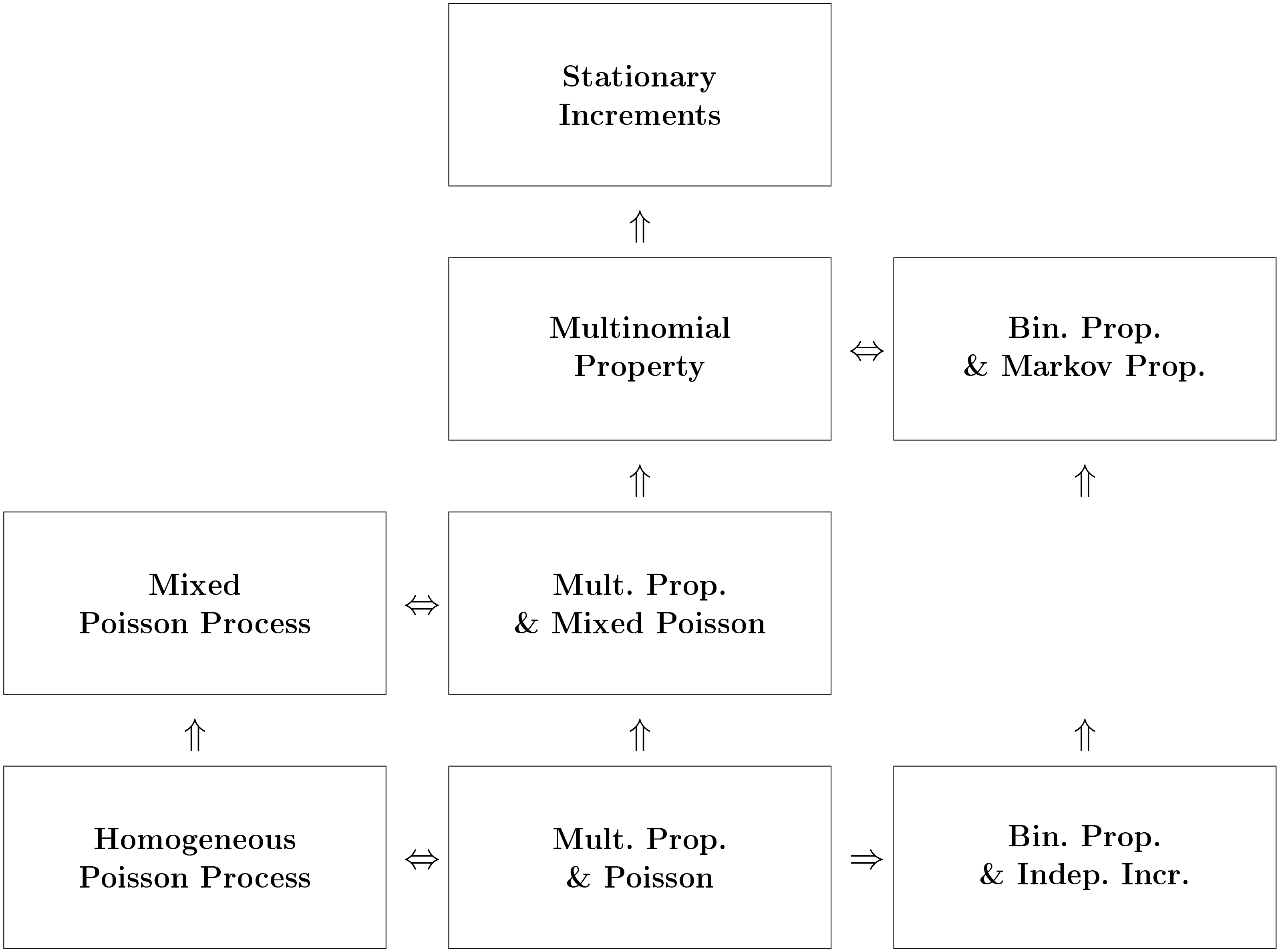
Problem 2: Characterize all claim number processes having the binomial property and independent increments.

A claim number process $\{N_t\}_{t \in \mathbf{R}_+}$ has **stationary increments** if

$$P\left[\bigcap_{j=1}^{m} \{N_{t_j+h} - N_{t_{j-1}+h} = k_j\}\right] = P\left[\bigcap_{j=1}^{m} \{N_{t_j} - N_{t_{j-1}} = k_j\}\right]$$

holds for all $m \in \mathbf{N}$, $0 = t_0 < t_1 < \cdots < t_m$, $h \in (0, \infty)$ and $k_1, \ldots, k_m \in \mathbf{N}_0$.

5.2 Lemma. *Every claim number process having the multinomial property has stationary increments.*


Stationary Increments
Multinomial Property
Bin. Prop. & Markov Prop.
Mixed Poisson Process
Mult. Prop. & Mixed Poisson
Homogeneous Poisson Process
Mult. Prop. & Poisson
Bin. Prop. & Indep. Incr.

6 Homogeneous Poisson Processes

6.1 Lemma. *If* $\{N_t\}_{t \in \mathbf{R}_+}$ *is a claim number process having the binomial property, then*

$$P[\{N_t = n\}] \;>\; 0$$

and

$$P[\{N_t - N_s = n\}] \;>\; 0$$

holds for all $s, t \in (0, \infty)$ *such that* $s < t$ *and for all* $n \in \mathbf{N}_0$*.*

6.2 Theorem. *For a claim number process* $\{N_t\}_{t \in \mathbf{R}_+}$*, the following are equivalent:*

(a) $\{N_t\}_{t \in \mathbf{R}_+}$ *is a homogeneous Poisson process.*

(b) $\{N_t\}_{t \in \mathbf{R}_+}$ *has the binomial property and independent increments.*

(c) $\{N_t\}_{t \in \mathbf{R}_+}$ *has the multinomial property and there exists some* $\alpha \in (0, \infty)$ *such that*

$$P[\{N_t = n\}] \;=\; e^{-\alpha t} \frac{(\alpha\, t)^n}{n!}$$

holds for all $t \in (0, \infty)$ *and all* $n \in \mathbf{N}_0$*.*

Proof. (c) $\Longleftrightarrow$ (a) $\Longrightarrow$ (b): Straightforward.

(b) $\Longrightarrow$ (a): By assumption, we have

$$P[\{N_s = k\}] \cdot P[\{N_t - N_s = l\}] \;=\; \binom{k+l}{k} \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^l \cdot P[\{N_t = k+l\}] \qquad (*)$$

and Lemma 6.1 yields $P[\{N_t = n\}] > 0$ and $P[\{N_t - N_s = n\}] > 0$.

Using $(*)$, we obtain

$$P[\{N_s = n+1\}] \cdot P[\{N_t - N_s = 0\}] \;=\; \left(\frac{s}{t}\right)^{n+1} \cdot P[\{N_t = n+1\}]$$

$$P[\{N_s = n\}] \cdot P[\{N_t - N_s = 0\}] \;=\; \left(\frac{s}{t}\right)^{n} \cdot P[\{N_t = n\}]$$

and hence

$$\frac{n+1}{s} \, \frac{P[\{N_s = n+1\}]}{P[\{N_s = n\}]} \;=\; \frac{n+1}{t} \, \frac{P[\{N_t = n+1\}]}{P[\{N_t = n\}]}$$

Thus, the right hand side is independent of t.

Using $(*)$ again, we obtain

$$P[\{N_s = n\}] \cdot P[\{N_t - N_s = 1\}] \;=\; (n+1)\left(\frac{s}{t}\right)^n \frac{t-s}{t} \cdot P[\{N_t = n+1\}]$$

$$P[\{N_s = n\}] \cdot P[\{N_t - N_s = 0\}] \;=\; \left(\frac{s}{t}\right)^n \cdot P[\{N_t = n\}]$$

and hence

$$\frac{1}{t-s}\,\frac{P[\{N_t - N_s = 1\}]}{P[\{N_t - N_s = 0\}]} \;=\; \frac{n+1}{t}\,\frac{P[\{N_t = n+1\}]}{P[\{N_t = n\}]}$$

Thus, the right hand side is independent of n.

Therefore,

$$\alpha \;:=\; \frac{n+1}{t}\,\frac{P[\{N_t = n+1\}]}{P[\{N_t = n\}]}$$

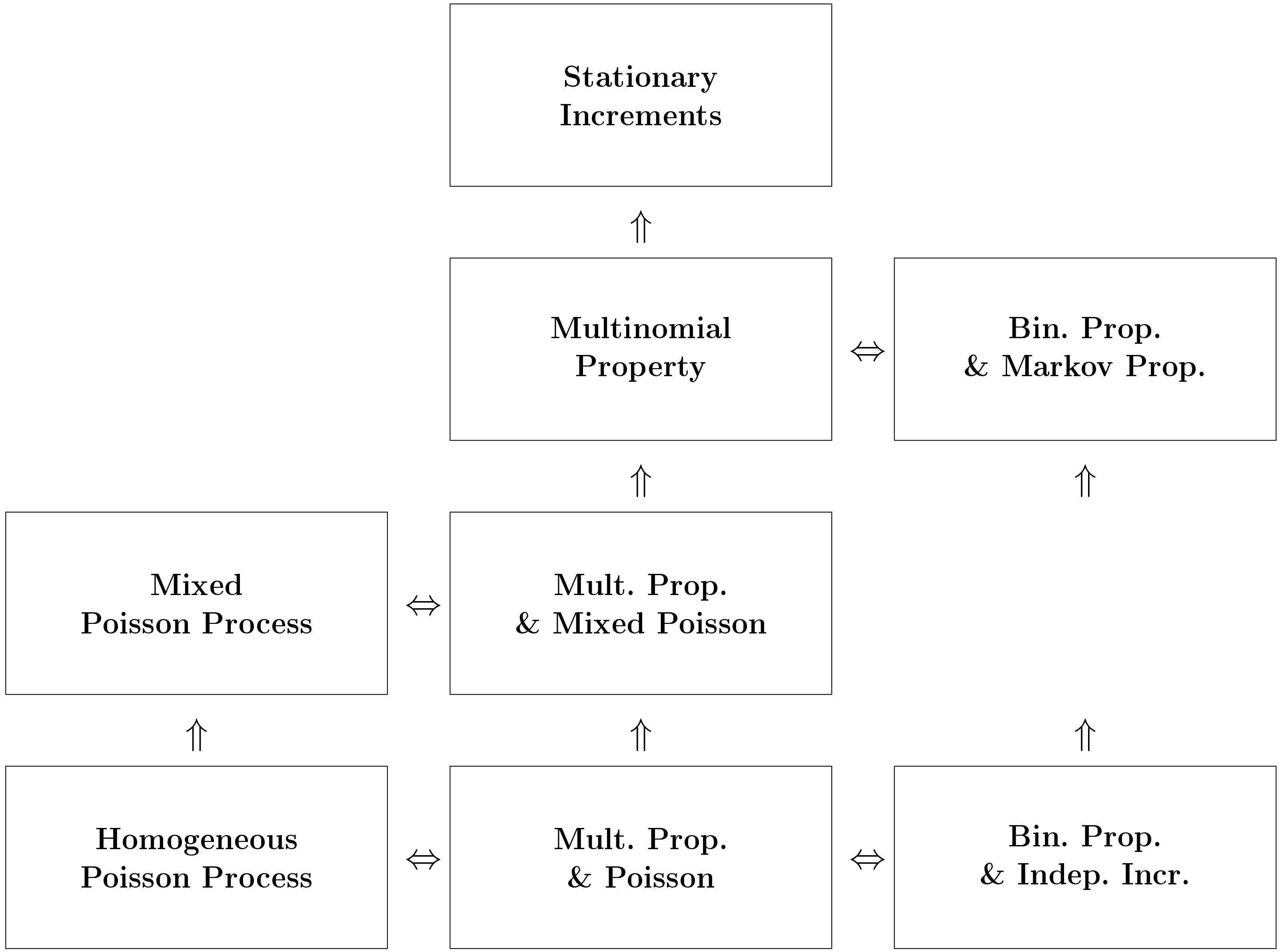
is independent of t and n.

Since

$$\frac{P[\{N_t = n+1\}]}{P[\{N_t = n\}]} = \frac{\alpha\, t}{n+1}$$

N_t has the Poisson distribution with parameter αt.

Using $(*)$ again, it now follows by straightforward calculation that $N_t - N_s$ has the Poisson distribution with parameter $\alpha(t-s)$. □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7 Mixed Poisson Processes

7.1 Lemma. *If* $\{N_t\}_{t \in \mathbf{R}_+}$ *is a claim number process having the binomial property, then there exists a probability measure* $Q : \mathcal{B}(\mathbf{R}) \to [0,1]$ *with* $Q[(0,\infty)] = 1$ *such that*

$$P[\{N_t = n\}] \;=\; \int_{(0,\infty)} e^{-\alpha t} \frac{(\alpha\, t)^n}{n!} \, dQ(\alpha)$$

holds for all $t \in (0,\infty)$ *and all* $n \in \mathbf{N}_0$.

Proof. For all $n \in \mathbf{N}_0$, define a map $\Pi_n : \mathbf{R}_+ \to [0,1]$ by letting

$$\Pi_n(t) \;:=\; P[\{N_t = n\}]$$

The binomial property yields

$$\begin{aligned}
\Pi_k(s) &= P[\{N_s = k\}] \\
&= \sum_{n=k}^{\infty} P[\{N_s = k\} \cap \{N_t = n\}] \\
&= \sum_{n=k}^{\infty} P[\{N_s = k\} \cap \{N_t - N_s = n-k\}] \\
&= \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^{n-k} P[\{N_t = n\}] \\
&= \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^{n-k} \Pi_n(t)
\end{aligned}$$

In particular, we have

$$\Pi_0(s) \;=\; \sum_{n=0}^{\infty} \left(\frac{t-s}{t}\right)^n \Pi_n(t)$$

The power series Π_0 is absolutely convergent on $[0, 2t]$ and differentiation yields

$$\begin{aligned}\Pi_0^{(k)}(s) &= \sum_{n=k}^{\infty} k! \binom{n}{k} \left(\frac{t-s}{t}\right)^{n-k} \left(-\frac{1}{t}\right)^k \Pi_n(t) \\ &= k! \left(-\frac{1}{s}\right)^k \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^{n-k} \Pi_n(t) \\ &= k! \left(-\frac{1}{s}\right)^k \Pi_k(s)\end{aligned}$$

and hence

$$\Pi_k(s) = (-1)^k \frac{s^k}{k!} \Pi_0^{(k)}(s)$$

It follows that the inequality

$$(-1)^k \Pi_0^{(k)}(s) \geq 0$$

holds for all $s \in (0, \infty)$ (which means that Π_0 is completely monotone on $(0, \infty)$).

Since the paths $\{N_t(\omega)\}_{t\in\mathbf{R}_+}$ of a claim number process are increasing and right continuous with $N_0(\omega) = 0$, we also have

$$\begin{aligned}
\lim_{s\to 0} \Pi_0(s) &= \lim_{s\to 0} P[\{N_s = 0\}] \\
&= \sup_{s\in(0,\infty)} P[\{N_s = 0\}] \\
&= P\left[\bigcup_{s\in(0,\infty)} \{N_s = 0\}\right] \\
&= P\left[\left\{\inf_{s\in(0,\infty)} N_s = 0\right\}\right] \\
&= P[\{N_0 = 0\}] \\
&= 1
\end{aligned}$$

Now the theorem of Bernstein and Widder yields the existence of a probability measure $Q : \mathcal{B}(\mathbf{R}) \to [0, 1]$ with $Q[\mathbf{R}_+] = 1$ such that

$$\Pi_0(s) = \int_{\mathbf{R}_+} e^{-\alpha s}\, dQ(\alpha)$$

holds for all $s \in \mathbf{R}_+$.

Differentiation of

$$\Pi_0(s) \; = \; \int_{\mathbf{R}_+} e^{-\alpha s} \, dQ(\alpha)$$

yields

$$\Pi_0^{(k)}(s) \; = \; \int_{\mathbf{R}_+} (-\alpha)^k \, e^{-\alpha s} \, dQ(\alpha)$$

and hence

$$\begin{aligned} P[\{N_t = k\}] \; &= \; \Pi_k(s) \\ &= \; (-1)^k \, \frac{s^k}{k!} \, \Pi_0^{(k)}(s) \\ &= \; (-1)^k \, \frac{s^k}{k!} \int_{\mathbf{R}_+} (-\alpha)^k \, e^{-\alpha s} \, dQ(\alpha) \\ &= \; \int_{\mathbf{R}_+} e^{-\alpha s} \, \frac{(\alpha s)^k}{k!} \, dQ(\alpha) \end{aligned}$$

for all $s \in (0, \infty)$ and all $k \in \mathbf{N}_0$.

Finally, since the paths of a claim number process increase to infinity,

$$
\begin{aligned}
0 &= P\left[\left\{\sup_{s\in(0,\infty)} N_s = 0\right\}\right] \\
&= P\left[\bigcap_{s\in(0,\infty)} \{N_s = 0\}\right] \\
&= \inf_{s\in(0,\infty)} P[\{N_s = 0\}] \\
&= \inf_{s\in(0,\infty)} \Pi_0(s) \\
&= \inf_{s\in(0,\infty)} \int_{\mathbf{R}_+} e^{-\alpha s}\, dQ(\alpha) \\
&\geq Q[\{0\}]
\end{aligned}
$$

and hence $Q[\{0\}] = 0$. Therefore, we have

$$
\Pi_k(s) = \int_{(0,\infty)} e^{-\alpha s} \frac{(\alpha s)^k}{k!}\, dQ(\alpha)
$$

for all $s \in (0, \infty)$ and all $k \in \mathbf{N}_0$. $\square$

7.2 Theorem. *For a claim number process* $\{N_t\}_{t \in \mathbf{R}_+}$, *the following are equivalent:*
(a) $\{N_t\}_{t \in \mathbf{R}_+}$ *is a mixed Poisson process.*
(b) $\{N_t\}_{t \in \mathbf{R}_+}$ *has the binomial property and the Markov property.*
(c) $\{N_t\}_{t \in \mathbf{R}_+}$ *has the multinomial property.*

Proof. (a) $\Longrightarrow$ (c) $\Longleftrightarrow$ (b): Straightforward.

(c) $\Longrightarrow$ (a): By Lemma 7.1, the multinomial property yields the existence of a probability measure $Q : \mathcal{B}(\mathbf{R}) \to [0,1]$ with $Q[(0,\infty)] = 1$ such that

$$P[\{N_t = n\}] \;=\; \int_{(0,\infty)} e^{-\alpha t} \, \frac{(\alpha \, t)^n}{n!} \, dQ(\alpha)$$

holds for all $t \in (0,\infty)$ and all $n \in \mathbf{N}_0$.
Using the multinomial property again, it follows that the claim number process is a mixed Poisson process. □

7.3 Corollary. *For a claim number process* $\{N_t\}_{t \in \mathbf{R}_+}$, *the following are equivalent:*
(a) $\{N_t\}_{t \in \mathbf{R}_+}$ *is a mixed Poisson process with independent increments.*
(b) $\{N_t\}_{t \in \mathbf{R}_+}$ *is a homogeneous Poisson process.*

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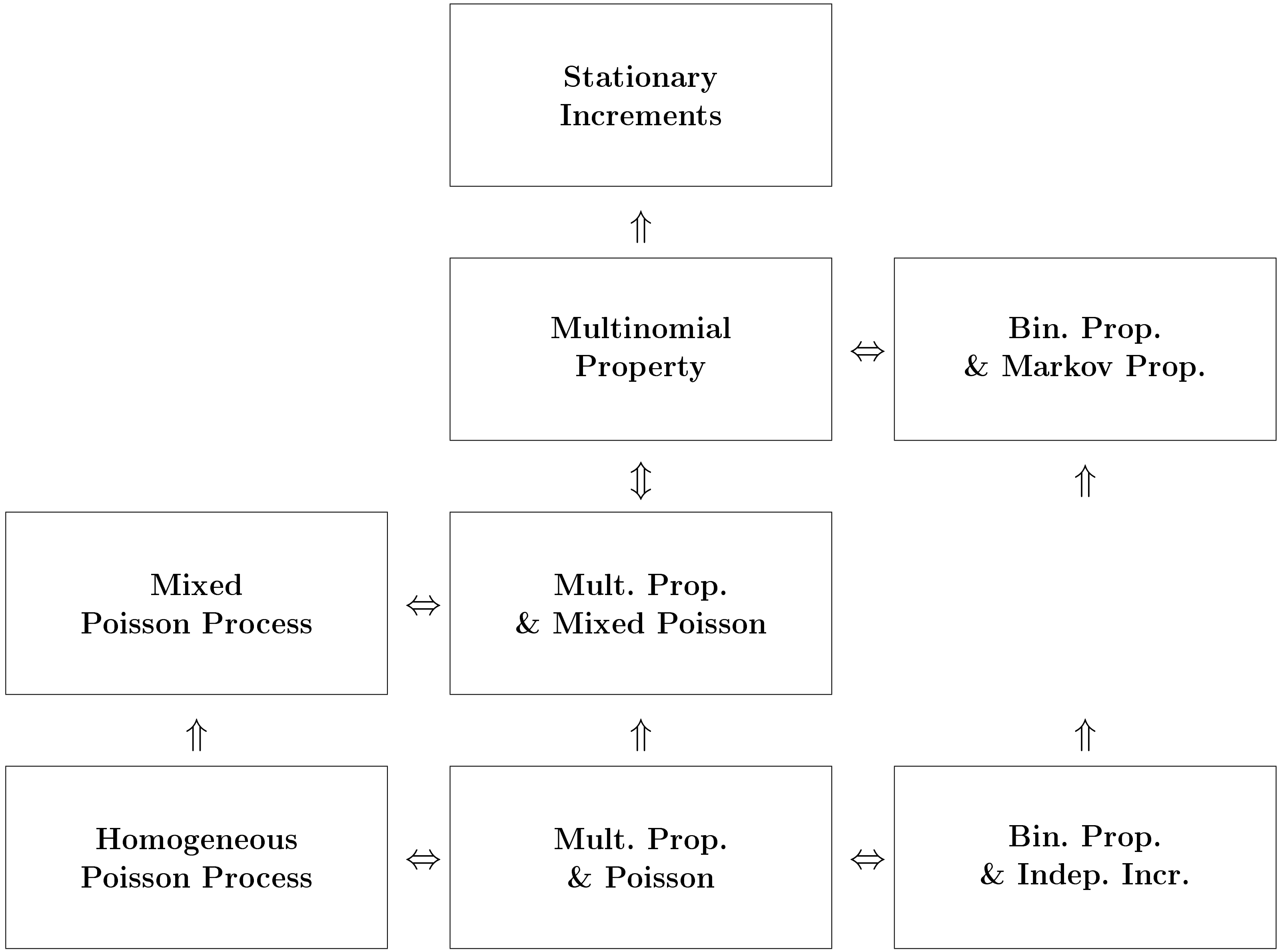

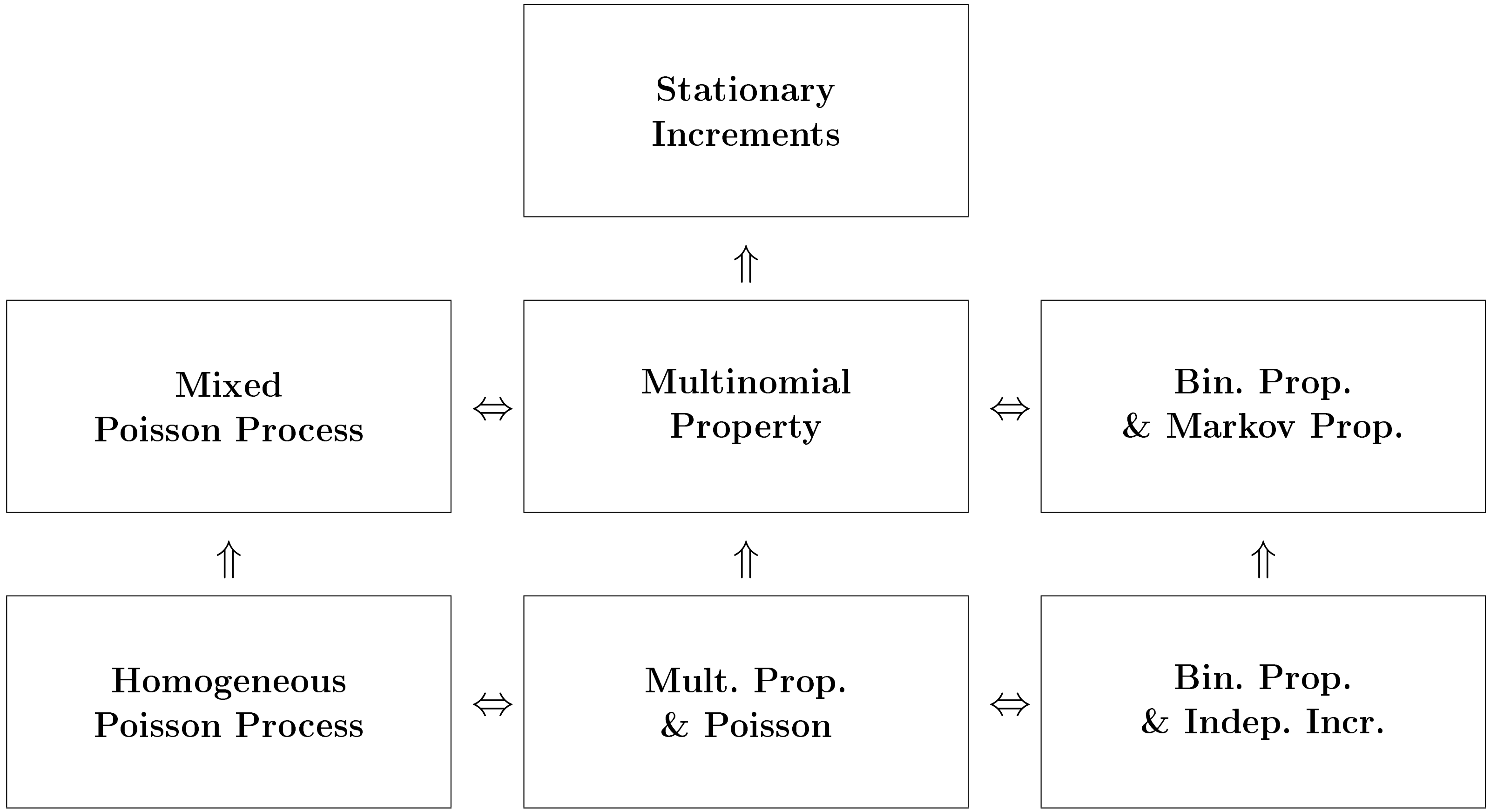
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References

Berg, C., Christensen, J. P. R., and Ressel, P. [1984]: *Harmonic Analysis on Semigroups.* Springer, Berlin – Heidelberg – New York.

Hofmann, M. [1955]: *Über zusammengesetzte Poisson–Prozesse und ihre Anwendungen in der Unfallversicherung.* Schweiz. Verein. Versicherungsmathematiker **55**, 499–575.

Lundberg, O. [1940]: *On Random Processes and Their Application to Sickness and Accident Statistics.* Amsqvist & Wicksells, Uppsala.

Mattner, L. [1993]: *Bernstein's theorem, inversion formula of Post and Widder, and the uniqueness theorem for Laplace transforms.* Expos. Math. **11**, 137–140.

Schmidt, K. D. [1996]: *Lectures on Risk Theory.* Teubner, Stuttgart.

Schmidt, K. D., and Zocher, M. []: *Claim number processes having the multinomial property.* Submitted for publication.