Multiplicative Tariffs and Marginal–Sum Equations

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Multiplicative Tariff (1)

Motor car portfolio of an insurance company (exposures):

Region	**Mileage in km/year** 0-20.000	 20.000–40.000	 40.000–∞
$\vdots$			
DD	600	300	100
$\vdots$			
SB	180	75	20
$\vdots$			

Rating factors:

- Region
- Mileage

The exposure in a single cell may be too small for statistics.

Multiplicative Tariff (2)

Net premiums in percent of the overall premium level:

Region	**Mileage in km/year**			
	0-20.000	20.000–40.000	40.000–∞	Factor
⋮				
DD	0,54	0,72	0,90	0,90
⋮				
SB	0,42	0,56	0,70	0,70
⋮				
Factor	0,60	0,80	1,00	

Advantages of a multiplicative tariff:

- The number of tariff levels is smaller than the number of tariff cells.
- Separation of the overall premium level and the structure of the portfolio.

Multiplicative Tariff (3)

Notation for two rating factors $\mathcal{I}$ and $\mathcal{K}$:

- tariff levels $i \in \{1, \dots, I\}$ for rating factor $\mathcal{I}$
- tariff levels $k \in \{1, \dots, K\}$ for rating factor $\mathcal{K}$
- tariff cells $(i, k) \in \{1, \dots, I\} \times \{1, \dots, K\} =: Q$
- exposure $v_{i,k} \in (0, \infty)$ in tariff cell $(i, k) \in Q$
- claim size $S_{i,k}$ in tariff cell (i, k)
- claim size per risk in tariff cell (i, k):

$$\frac{S_{i,k}}{v_{i,k}}$$

- net premium per risk in tariff cell (i, k):

$$\mathrm{E}\left[\frac{S_{i,k}}{v_{i,k}}\right]$$

Multiplicative Tariff (4)

A collection of net premiums for the different cells is said to be a multiplicative tariff if there exist parameters $\mu, \alpha_1, \ldots, \alpha_I, \beta_1, \ldots, \beta_K \in (0, \infty)$ such that the net premiums satisfy

$$\mathrm{E}\left[\frac{S_{i,k}}{v_{i,k}}\right] = \mu\, \alpha_i\, \beta_k$$

for all $(i, k) \in Q$.

Possible scalings:

- $\max_{i=1}^{I} \alpha_i = 1 = \max_{k=1}^{K} \beta_k$
- $\sum_{i=1}^{I} \alpha_i = 1 = \sum_{k=1}^{K} \beta_k$

Notation:

- overall premium level: μ
- tariff factors: $\alpha_1, \ldots, \alpha_I$ and $\beta_1, \ldots, \beta_K$

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Collective Model (1)

Assume that

- $(\Omega, \mathcal{F}, P)$ is a probability space and that $(M, \mathcal{M})$ is a measurable space,
- N is a random variable $\Omega \to \mathbb{N}_0$, and
- $\{Y_j\}_{j\in\mathbb{N}}$ is a sequence of random variables $\Omega \to M$.

The pair

$$\langle N, \{Y_j\}_{j\in\mathbb{N}} \rangle$$

is said to be a collective model if the sequence $\{Y_j\}_{j\in\mathbb{N}}$ is independent and identically distributed and independent of N.

Interpretation:

- N is the claim number of a portfolio of risks
- Y_j is die claim variable of claim j of the portfolio, indicating the tariff cell, to which the risk causing this claim belongs, and the claim size.

Collective Model (2)

Aim: Decomposition of the collective model $\langle N, \{Y_j\}_{j \in \mathbb{N}} \rangle$ with respect to a measurable partition

$$M_1, \dots, M_m$$

of M with

$$\eta_p := \mathrm{P}[\{Y \in M_p\}]$$

and $\eta_p \in (0,1)$ for all $p \in \{1, \dots, m\}$,
where Y denotes a random variable having the same distribution as each of the random variables Y_j.

To prepare the decomposition, we start with thinning of the collective model with respect to M_p for some $p \in \{1, \dots, m\}$.

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Thinning (1)

Thinning for fixed $p \in \{1, \dots, m\}$:

- Let
$$N_p := \sum_{j=1}^{N} \chi_{\{Y_j \in M_p\}}$$
denote the number of claims (for which the claim variable assumes a value) in M_p.
- Let $\nu_{p;0} := 0$ and for $j \in \mathbb{N}$ let
$$\nu_{p;j} := \inf\Big\{ l \in \mathbb{N} \;\Big|\; \nu_{p;j-1} < l \text{ und } Y_l \in M_p \Big\}$$
denote the number of the j–th claim in M_p.
- For $j \in \mathbb{N}$, let
$$Y_{p;j} := \sum_{l=1}^{\infty} \chi_{\{\nu_{p;j}=l\}} Y_l$$
denote claim variable of the j–th claim in M_p.

Thinning (2)

Theorem. For every $p \in \{1, \ldots, m\}$, the pair

$$\langle N_p, \{Y_{p;j}\}_{j\in\mathbb{N}} \rangle$$

is a collective model.

The independence of N_p and $\{Y_{p;j}\}_{j\in\mathbb{N}}$ is quite remarkable since each of these random variables has been constructed from the same sequence $\{Y_j\}_{j\in\mathbb{N}}$.

General application in statistics:
Post–stratifying does not destroy the property of sampling variables being independent and identically distributed.

It remains to investigate the dependencies between the thinned collective models.

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Decomposition (1)

Theorem. The families

$$\{Y_{1;j}\}_{j\in\mathbb{N}}, \dots, \{Y_{m;j}\}_{j\in\mathbb{N}}$$

are independent of each other.

Since $\langle N, \{Y_j\}_{j\in\mathbb{N}}\rangle$ is a collective model, each of

$$\langle N, \{\chi_{\{X_j\in M_p\}}\}_{j\in\mathbb{N}}\rangle$$

with $p \in \{1, \dots, m\}$ is a collective model as well.
In particular, Wald's identities

$$\begin{aligned} E[N_p] &= \eta_p\, E[N] \\ \mathrm{var}[N_p] &= \eta_p\, E[N] + \eta_p^2\,(\mathrm{var}[N] - E[N]) \\ \mathrm{cov}[N_p, N_q] &= \eta_p\,\eta_q\,(\mathrm{var}[N] - E[N]) \end{aligned}$$

hold for all $p, q \in \{1, \dots, m\}$ with $p \neq q$.

Decomposition (2)

Theorem. The claim numbers satisfy

$$N = \sum_{p=1}^{m} N_p$$

and the identity

$$\mathrm{P}\left[\bigcap_{p=1}^{m} \{N_p = n_p\} \;\middle|\; \{N = n\}\right] = \frac{n!}{\prod_{p=1}^{m} n_p!} \prod_{p=1}^{m} \eta_p^{n_p}$$

holds for every $n \in \mathbb{N}_0$ and every family $\{n_p\}_{p \in \{1,\dots,m\}} \subseteq \mathbb{N}_0$ satisfying $\sum_{p=1}^{m} n_p = n$.

Theorem. The following are equivalent:

- The family $\{N_p\}_{p \in \{1,\dots,m\}}$ is independent.
- N has a Poisson distribution.

In this case, every claim number N_p has a Poisson distribution as well.

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Decomposition with Respect to Tariff Cells (1)

We consider the collective model $\langle N, \{Y_j\}_{j\in\mathbb{N}}\rangle$ with

$$M := (\{1,\dots,I\}\times\{1,\dots,K\})\times\mathbb{R} = Q\times\mathbb{R}$$

and $\mathcal{M} := 2^{\{1,\dots,I\}\times\{1,\dots,K\}}\otimes\mathcal{B}(\mathbb{R}) = 2^Q\otimes\mathcal{B}(\mathbb{R})$. Then the individual claim variables can be represented in the form

$$Y_j = \begin{pmatrix} T_j \\ X_j \end{pmatrix}$$

where

- T_j denotes the tariff cell to which the risk causing claim j belongs, and
- X_j denotes the claim size of claim j.

We assume that, for every claim $j\in\mathbb{N}$, the random variables T_j and X_j are independent with $P[\{X_j\geq 0\}] = 1$.

Decomposition with Respect to Tariff Cells (2)

Since $\langle N, \{Y_j\}_{j\in\mathbb{N}}\rangle$ is a collective model,

$$\langle N, \{X_j\}_{j\in\mathbb{N}}\rangle$$

is a collective model as well.

Therefore, the claim size of the portfolio

$$S := \sum_{j=1}^{N} X_j$$

satisfies Wald's identity

$$\mathrm{E}[S] = \mathrm{E}[N]\,\mathrm{E}[X]$$

where X denotes any random variable having the same distribution as each of the individual claim sizes X_j.

Decomposition with Respect to Tariff Cells (3)

Decomposition of the collective model $\langle N, \{Y_j\}_{j\in\mathbb{N}}\rangle$ with respect to the sets

$$M_{i,k} := \{(i,k)\} \times \mathbb{R}$$

with $\eta_{i,k} := \mathrm{P}[\{Y \in M_{i,k}\}] = \mathrm{P}[\{T = (i,k)\}]$ and $\eta_{i,k} \in (0,1)$ yields the collective models

$$\langle N_{i,k}, \{Y_{i,k;j}\}_{j\in\mathbb{N}}\rangle$$

for the tariff cells. Moreover, the individual claim sizes of tariff cell (i,k) satisfy

$$Y_{i,k;j} = \begin{pmatrix} (i,k) \\ X_{i,k;j} \end{pmatrix}$$

and the claim size of tariff cell (i,k) satisfies

$$S_{i,k} := \sum_{j=1}^{N_{i,k}} X_{i,k;j}$$

Decomposition with Respect to Tariff Cells (4)

Theorem. The following identities hold for every tariff cell $(i,k) \in Q$:

- $\mathrm{P}[\{X_{i,k} \in B\}] = \mathrm{P}[\{X \in B\}]$ for all $B \in \mathcal{B}(\mathbb{R})$.
- $\mathrm{E}[N_{i,k}] = \eta_{i,k}\,\mathrm{E}[N]$.
- $\mathrm{E}[S_{i,k}] = \mathrm{E}[N_{i,k}]\,\mathrm{E}[X_{i,k}] = \eta_{i,k}\,\mathrm{E}[N]\,\mathrm{E}[X] = \eta_{i,k}\,\mathrm{E}[S]$.

Theorem.

- $N = \sum_{(i,k)\in Q} N_{i,k}$ and $S = \sum_{(i,k)\in Q} S_{i,k}$.
- The identity

$$\mathrm{P}\left[\bigcap_{(i,k)\in Q} \{N_{i,k} = n_{i,k}\} \,\middle|\, \{N = n\}\right] = \frac{n!}{\prod_{(i,k)\in Q} n_{i,k}!} \prod_{(i,k)\in Q} \eta_{i,k}^{n_{i,k}}$$

holds for all $n \in \mathbb{N}_0$ and every family $\{n_{i,k}\}_{(i,k)\in Q} \subseteq \mathbb{N}_0$ satisfying $\sum_{(i,k)\in Q} n_{i,k} = n$ (conditional multinomial distribution).

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Assumptions (1)

Given: Known measures of exposure

$$v_{i,k} \in \mathbb{R}_+$$

for the tariff cells $(i, k) \in Q$.

Assumption: There exist parameters $\alpha_1, \dots, \alpha_I \in (0, \infty)$ and $\beta_1, \dots, \beta_K \in (0, \infty)$ such that

$$\eta_{i,k} = \alpha_i \, \beta_k \, v_{i,k}$$

holds for all $(i, k) \in Q$ (threefold proportionality).

The assumption implies

$$\sum\nolimits_{(i,k)\in Q} \alpha_i \, \beta_k \, v_{i,k} = 1$$

Assumptions (2)

Because of the assumption $\eta_{i,k} = \alpha_i \, \beta_k \, v_{i,k}$,

- the expected claim number in tariff cell (i, k) satisfies

$$\mathrm{E}[N_{i,k}] = \eta_{i,k} \, \mathrm{E}[N] = \alpha_i \, \beta_k \, v_{i,k} \, \mathrm{E}[N]$$

- the expected claim size in tariff cell (i, k) satisfies

$$\mathrm{E}[S_{i,k}] = \eta_{i,k} \, \mathrm{E}[S] = \alpha_i \, \beta_k \, v_{i,k} \, \mathrm{E}[S]$$

- the net premium per risk in tariff cell (i, k) satisfies

$$\mathrm{E}[S_{i,k}/v_{i,k}] = \alpha_i \, \beta_k \, \mathrm{E}[S]$$

In particular, the assumption yields a multiplicative tariff.

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Marginal–Sum Estimation: Claim Sizes per Cell (1)

The claim sizes of the tariff cells and the total claim size of the portfolio are related by the identities

$$\sum\nolimits_{(i,k)\in Q} S_{i,k} = S$$

and

$$\mathrm{E}[S_{i,k}] = \alpha_i \, \beta_k \, v_{i,k} \, \mathrm{E}[S]$$

Marginal–Sum Estimation: Claim Sizes per Cell (2)

Summation over the equations $E[S_{i,k}] = \alpha_i \, \beta_k \, v_{i,k} \, E[S]$ yields

$$\begin{aligned} \sum\nolimits_{k=1}^{K} E[S_{i,k}] &= \sum\nolimits_{k=1}^{K} \alpha_i \, \beta_k \, v_{i,k} \, E[S] \qquad & i \in \{1, \dots, I\} \\ \sum\nolimits_{i=1}^{I} E[S_{i,k}] &= \sum\nolimits_{i=1}^{I} \alpha_i \, \beta_k \, v_{i,k} \, E[S] \qquad & k \in \{1, \dots, K\} \end{aligned}$$

Estimators $\widehat{\alpha}_1, \dots, \widehat{\alpha}_I$ and $\widehat{\beta}_1, \dots, \widehat{\beta}_K$ of the parameters $\alpha_1, \dots, \alpha_I$ and $\beta_1, \dots, \beta_K$ are said to be Marginal–Sum estimators if they satisfy the Marginal–Sum equations

$$\begin{aligned} \sum\nolimits_{k=1}^{K} \widehat{\alpha}_i \, \widehat{\beta}_k \, v_{i,k} &= \sum\nolimits_{k=1}^{K} \frac{S_{i,k}}{S} \qquad & i \in \{1, \dots, I\} \\ \sum\nolimits_{i=1}^{I} \widehat{\alpha}_i \, \widehat{\beta}_k \, v_{i,k} &= \sum\nolimits_{i=1}^{I} \frac{S_{i,k}}{S} \qquad & k \in \{1, \dots, K\} \end{aligned}$$

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Marginal–Sum Estimation: Claim Numbers (1)

The claim numbers of the tariff cells and the total claim number of the portfolio are related by the identities

$$\sum_{(i,k)\in Q} N_{i,k} = N$$

and

$$E[N_{i,k}] = \alpha_i \, \beta_k \, v_{i,k} \, E[N]$$

Note that

$$E[S] = E[N] \, E[X]$$

which because of $E[X_{i,k}] = E[X]$ implies

$$E[S_{i,k}] = E[N_{i,k}] \, E[X]$$

Marginal–Sum Estimation: Claim Numbers (2)

Summation over the equations $\mathrm{E}[N_{i,k}] = \alpha_i \, \beta_k \, v_{i,k} \, \mathrm{E}[N]$ yields

$$\begin{aligned} \sum\nolimits_{k=1}^{K} \mathrm{E}[N_{i,k}] &= \sum\nolimits_{k=1}^{K} \alpha_i \, \beta_k \, v_{i,k} \, \mathrm{E}[N] \qquad i \in \{1, \dots, I\} \\ \sum\nolimits_{i=1}^{I} \mathrm{E}[N_{i,k}] &= \sum\nolimits_{i=1}^{I} \alpha_i \, \beta_k \, v_{i,k} \, \mathrm{E}[N] \qquad k \in \{1, \dots, K\} \end{aligned}$$

and hence the Marginal–Sum equations

$$\begin{aligned} \sum\nolimits_{k=1}^{K} \widehat{\alpha}_i \, \widehat{\beta}_k \, v_{i,k} &= \sum\nolimits_{k=1}^{K} \frac{N_{i,k}}{N} \qquad i \in \{1, \dots, I\} \\ \sum\nolimits_{i=1}^{I} \widehat{\alpha}_i \, \widehat{\beta}_k \, v_{i,k} &= \sum\nolimits_{i=1}^{I} \frac{N_{i,k}}{N} \qquad k \in \{1, \dots, K\} \end{aligned}$$

(now with $N_{i,k}/N$ in the place of $S_{i,k}/S$).

Marginal–Sum Estimation: Claim Numbers (3)

In the case where $v_{i,k} > 0$ holds for all (i,k),
Marginal–Sum estimation based on claim numbers can be justified by the Maximum–Likelihood principle:

Define $\mathrm{p} : \mathbb{N}_0 \to \mathbb{R}_+$ by letting

$$\mathrm{p}(n) := \mathrm{P}[\{N = n\}]$$

Then every family $\{n_{i,k}\}_{(i,k)\in Q} \subseteq \mathbb{N}_0$ with $n := \sum_{(i,k)\in Q} n_{i,k}$ satisfies

$$\begin{aligned}\mathrm{P}\left[\bigcap_{(i,k)\in Q} \{N_{i,k} = n_{i,k}\}\right] &= \frac{n!}{\prod\limits_{(i,k)\in Q} n_{i,k}!} \prod_{(i,k)\in Q} \eta_{i,k}^{n_{i,k}} \cdot \mathrm{p}(n) \\ &= \frac{n!}{\prod\limits_{(i,k)\in Q} n_{i,k}!} \prod_{(i,k)\in Q} (\alpha_i \beta_k v_{i,k})^{n_{i,k}} \cdot \mathrm{p}(n)\end{aligned}$$

Marginal–Sum Estimation: Claim Numbers (4)

Likelihood Function:

$$L\Big(\widehat{\alpha}_1,\dots,\widehat{\alpha}_I,\widehat{\beta}_1,\dots,\widehat{\beta}_K\;\Big|\;\{N_{i,k}\}_{(i,k)\in Q}\Big) = \frac{N!}{\prod_{(i,k)\in Q} N_{i,k}!}\prod_{(i,k)\in Q}(\widehat{\alpha}_i\widehat{\beta}_k v_{i,k})^{N_{i,k}}\cdot \mathrm{p}(N)$$

Log–Likelihood Function:

$$(\log\circ L)\Big(\widehat{\alpha}_1,\dots,\widehat{\alpha}_I,\widehat{\beta}_1,\dots,\widehat{\beta}_K\;\Big|\;\{N_{i,k}\}_{(i,k)\in Q}\Big) = \sum_{(i,k)\in Q} N_{i,k}\log(\widehat{\alpha}_i\widehat{\beta}_k v_{i,k}) + C$$

Marginal–Sum Estimation: Claim Numbers (5)

Lagrange–Approach: Minimize

$$\ell\Big(\widehat{\alpha}_1,\dots,\widehat{\alpha}_I,\widehat{\beta}_1,\dots,\widehat{\beta}_K,\lambda\;\Big|\;\{N_{i,k}\}_{(i,k)\in Q}\Big) = \sum_{(i,k)\in Q} N_{i,k}\log(\widehat{\alpha}_i\widehat{\beta}_k v_{i,k}) + \lambda\left(1-\sum_{(i,k)\in Q}\widehat{\alpha}_i\widehat{\beta}_k v_{i,k}\right)$$

Partial differentiation yields

$$\begin{aligned}
\sum\nolimits_{k=1}^{K}\frac{N_{i,k}}{\widehat{\alpha}_i} &= \lambda\sum\nolimits_{k=1}^{K}\widehat{\beta}_k v_{i,k} \qquad i\in\{1,\dots,I\}\\
\sum\nolimits_{i=1}^{I}\frac{N_{i,k}}{\widehat{\beta}_k} &= \lambda\sum\nolimits_{i=1}^{I}\widehat{\alpha}_i v_{i,k} \qquad k\in\{1,\dots,K\}\\
1 &= \sum\nolimits_{(i,k)\in Q}\widehat{\alpha}_i\widehat{\beta}_k v_{i,k}
\end{aligned}$$

Marginal–Sum Estimation: Claim Numbers (6)

and hence

$$\begin{aligned}\sum_{k=1}^{K} N_{i,k} &= \lambda \sum_{k=1}^{K} \widehat{\alpha}_i \widehat{\beta}_k v_{i,k} && i \in \{1,\dots,I\}\\ \sum_{i=1}^{I} N_{i,k} &= \lambda \sum_{i=1}^{I} \widehat{\alpha}_i \widehat{\beta}_k v_{i,k} && k \in \{1,\dots,K\}\\ 1 &= \sum_{(i,k)\in Q} \widehat{\alpha}_i \widehat{\beta}_k v_{i,k}\end{aligned}$$

Since

$$N = \sum_{(i,k)\in Q} N_{i,k} = \lambda \sum_{(i,k)\in Q} \widehat{\alpha}_i \widehat{\beta}_k v_{i,k} = \lambda$$

this yields the Marginal–Sum equations

$$\begin{aligned}\sum_{k=1}^{K} \widehat{\alpha}_i \, \widehat{\beta}_k \, v_{i,k} &= \sum_{k=1}^{K} \frac{N_{i,k}}{N} && i \in \{1,\dots,I\}\\ \sum_{i=1}^{I} \widehat{\alpha}_i \, \widehat{\beta}_k \, v_{i,k} &= \sum_{i=1}^{I} \frac{N_{i,k}}{N} && k \in \{1,\dots,K\}\end{aligned}$$

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Marginal–Sum Problem (1)

Given: Matrices $\mathbf{V}, \mathbf{S} \in \mathbb{R}_+^{\{1,\dots,I\}\times\{1,\dots,K\}}$ with the property, that every row and every column contains at least one element $\neq 0$:

Wanted: Solution(s) $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in (0,\infty) \times (0,\infty)^I \times (0,\infty)^K$ to the Marginal–Sum equations (MS equations)

$$\begin{aligned} \sum\nolimits_{k=1}^{K} \mu\, \alpha_i\, \beta_k\, v_{i,k} &= \sum\nolimits_{k=1}^{K} s_{i,k} \qquad i \in \{1,\dots,I\} \\ \sum\nolimits_{i=1}^{I} \mu \alpha_i\, \beta_k\, v_{i,k} &= \sum\nolimits_{i=1}^{I} s_{i,k} \qquad k \in \{1,\dots,K\} \end{aligned}$$

Remark: If $(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a solution, then μ satisfies

$$\mu = \frac{\sum_{i=1}^{I} \sum_{k=1}^{K} s_{i,k}}{\sum_{i=1}^{I} \sum_{k=1}^{K} \alpha_i\, v_{i,k} \beta_k} = \frac{\mathbf{1}'\mathbf{S}\mathbf{1}}{\boldsymbol{\alpha}'\mathbf{V}\boldsymbol{\beta}}$$

Therefore, only $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ will be referred to as a solution.

Marginal–Sum Problem (2)

The MS equations have a radially unique solution if

- they have a solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ and
- every solution $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ has the form $(c\boldsymbol{\alpha}^*, d\boldsymbol{\beta}^*)$ for some $c, d \in (0, \infty)$.

Example. Assume that $v_{i,k} = v > 0$ holds for all (i, k). Then the MS equations become

$$\begin{aligned} (\mu\, v\, \mathbf{1}'\boldsymbol{\beta})\, \boldsymbol{\alpha} &= \mathbf{S1} \\ (\mu\, v\, \mathbf{1}'\boldsymbol{\alpha})\, \boldsymbol{\beta} &= \mathbf{S}'\mathbf{1} \end{aligned}$$

and hence have the (radially unique) solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ with

$$\begin{aligned} \boldsymbol{\alpha}^* &= \mathbf{S1} \\ \boldsymbol{\beta}^* &= \mathbf{S}'\mathbf{1} \end{aligned}$$

and $\mu^* = (v\, \mathbf{1}'\mathbf{S1})^{-1}$.

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Equivalent Fixed–Point Problems (1)

Define $G : (0,\infty)^I \to (0,\infty)^K$ and $H : (0,\infty)^K \to (0,\infty)^I$ by letting

$$G_k(\boldsymbol{\alpha}) := \frac{\mathbf{1}'\mathbf{S}\mathbf{e}_k}{\boldsymbol{\alpha}'\mathbf{V}\mathbf{e}_k} \qquad \text{und} \qquad H_i(\boldsymbol{\beta}) := \frac{\mathbf{e}_i'\mathbf{S}\mathbf{1}}{\mathbf{e}_i'\mathbf{V}\boldsymbol{\beta}}$$

Then:

- G and H are continuous, decreasing, and homogeneous of degree -1.
- $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is a solution to the MS equations iff there is some $\mu \in (0,\infty)$ such that

$$\mu\,\boldsymbol{\alpha} = H(\boldsymbol{\beta}) \qquad \text{and} \qquad \mu\,\boldsymbol{\beta} = G(\boldsymbol{\alpha})$$

- If $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is a solution to the MS equations, then

$$\mu = \frac{\mathbf{1}'\mathbf{S}\mathbf{1}}{\boldsymbol{\alpha}'\mathbf{V}\boldsymbol{\beta}}$$

Equivalent Fixed–Point Problem (2)

Define $\Phi : (0,\infty)^I \to (0,\infty)^I$ and $\Psi : (0,\infty)^K \to (0,\infty)^K$ by letting

$$\Phi := H \circ G \qquad \text{and} \qquad \Psi := G \circ H$$

Then Φ and Ψ are continuous, increasing, and homogeneous of degree 1 and $G \circ \Phi = \Psi \circ G$ and $\Phi \circ H = H \circ \Psi$. Thus: Every strictly positive multiple of a fixed point is a fixed point as well.

Theorem.

- If (α, β) is a solution to the MS equations, then α is a fixed point of Φ and β is a fixed point of Ψ.
- If α is a fixed point of Φ, then $G(\alpha)$ is a fixed point of Ψ and $(\alpha, G(\alpha))$ is a solution to the MS equations.
- If β is a fixed point of Ψ, then $H(\beta)$ is a fixed point of Φ and $(H(\beta), \beta)$ is a solution to the MS equations.

Equivalent Fixed–Point Problem (3)

Proof.

- Assume that (α, β) is a solution to the MS equations and let $\mu := \mathbf{1}'\mathbf{S}\mathbf{1}/\alpha'\mathbf{V}\beta$. Then

  $$\mu\,\boldsymbol{\alpha} = H(\boldsymbol{\beta}) = H(\mu^{-1}G(\boldsymbol{\alpha})) = \mu\,H(G(\boldsymbol{\alpha})) = \mu\,\Phi(\boldsymbol{\alpha})$$

  In particular, α is a fixed point of Φ.

- Assume that α is a fixed point of Φ. Then

  $$\Psi(G(\boldsymbol{\alpha})) = G(\Phi(\boldsymbol{\alpha})) = G(\boldsymbol{\alpha})$$

  In particular,

  $$\boldsymbol{\beta} := G(\boldsymbol{\alpha})$$

  is a fixed point of Ψ and satisfies

  $$\boldsymbol{\alpha} = \Phi(\boldsymbol{\alpha}) = H(G(\boldsymbol{\alpha})) = H(\boldsymbol{\beta})$$

  Therefore, (α, β) is a solution to the MS equations.

Equivalent Fixed–Point Problem (4)

Theorem. The following are equivalent:

- The MS equations have a radially unique solution.
- Φ has a radially unique fixed point.
- Ψ has a radially unique fixed point.

Existence Theorem. Assume that $v_{i,k} > 0$ holds for all (i, k). Then Φ has a fixed point.

Sketch of the proof: Define $\overline{G} : \mathbb{R}^I_+ \setminus \{\mathbf{0}\} \to (0, \infty)^K$ by letting

$$\overline{G}_k(\alpha) := \frac{\mathbf{1}'\mathbf{S}\mathbf{e}_k}{\alpha'\mathbf{V}\mathbf{e}_k}$$

and define

$$\overline{\Phi} := H \circ \overline{G}$$

Equivalent Fixed–Point Problem (5)

- $\overline{G}$ and $\overline{\Phi}$ are continuous extensions of G and Φ.
- Let

  $$\Delta^I := \left\{\boldsymbol{\alpha} \in \mathbb{R}^I_+ \;\middle|\; \mathbf{1}'\boldsymbol{\alpha} = 1\right\}$$

  and define $\widetilde{\Phi} : \Delta^I \to \Delta^I$ by letting

  $$\widetilde{\Phi}(\boldsymbol{\alpha}) := \frac{1}{1 + \mathbf{1}'\overline{\Phi}(\boldsymbol{\alpha})} \Big(\boldsymbol{\alpha} + \overline{\Phi}(\boldsymbol{\alpha})\Big)$$

  Then Δ^I is nonempty, convex and compact, and $\widetilde{\Phi}$ is continuous.
- Brouwer's Fixed Point Theorem: $\widetilde{\Phi}$ has a fixed point $\boldsymbol{\alpha}$.
- $\boldsymbol{\alpha}$ is also a fixed point of $\overline{\Phi}$ (use definition of $\overline{\Phi}$).
- Since $\boldsymbol{\alpha} = \overline{\Phi}(\boldsymbol{\alpha}) \in (0, \infty)^I$, $\boldsymbol{\alpha}$ is also a fixed point of Φ.

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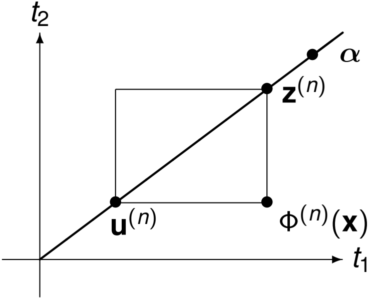
Radial Uniqueness and Iteration (1)

Theorem. Assume that $v_{i,k} > 0$ holds for all (i, k).
If $\boldsymbol{\alpha}$ is a fixed point of Φ, then, for every $\mathbf{x} \in (0, \infty)^I$, there exists some $\xi \in (0, \infty)$ such that

$$\lim_{n\to\infty} \Phi^n(\mathbf{x}) = \xi\,\boldsymbol{\alpha}$$

Sketch of the proof:



Radial Uniqueness and Iteration (2)

Corollary. Assume that $v_{i,k} > 0$ holds for all (i,k).
Then Φ has a radially unique fixed point.

Contents

Example (1)

Example. Let

$$\mathbf{V} := \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \qquad \text{und} \qquad \mathbf{S} := \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

Then we have $v_{1,2} = 0$ (!) and

$$\begin{aligned} 2\mu\alpha_1\beta_1 \quad\quad\quad\quad\quad\quad &= 1 \\ 3\mu\alpha_2\beta_1 + \quad \mu\alpha_2\beta_2 &= 4 \\ 2\mu\beta_1\alpha_1 + 3\mu\beta_1\alpha_2 &= 1 \\ \mu\beta_2\alpha_2 &= 4 \end{aligned}$$

Therefore, the MS equations have no solution.

Example (2)

We have

$$\Phi(\boldsymbol{\alpha}) = \begin{pmatrix} \dfrac{1}{2/(2\alpha_1+3\alpha_2)} \\ \dfrac{4}{3/(2\alpha_1+3\alpha_2)+4/\alpha_2} \end{pmatrix}$$

Iteration with the initial value $\mathbf{x} := (1/2)\,\mathbf{1}$ yields the following scaled iterates $\mathbf{x}^{(n)} := \Phi^{(n)}(\mathbf{x})/\mathbf{1}'\Phi^{(n)}(\mathbf{x})$:

n	$x_1^{(n)}$	$x_2^{(n)}$
0	0,5	0,5
1	0,742	0,258
10	0,952	0,048
100	0,995	0,005
1000	0,999	0,001
10000	1,000	0,000

Contents

Literature

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DSVM = Dresdner Schriften zur Versicherungsmathematik.