Improved Poincaré and other classic inequalities: a new approach to prove them and some generalizations

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Dresden
November 18, 2011
COWORKERS

- G. Acosta
- I. Drelichman
- A. L. Lombardi
- F. López
- M. A. Muschietti
- M. I. Prieto
- E. Russ
- P. Tchamitchian
This talk is about several classic inequalities (Korn, Poincaré, Improved Poincaré, etc.).
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Questions we are interested in:

- What are the relations between them?
- For which domains are these inequalities valid?
- When they are not valid: Are there weaker estimates still useful for variational analysis?
More generally: in some applications it is interesting to have “uniformly valid inequalities” for a sequence of domains.

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For example: Domain decomposition methods for finite elements.

A methodology to prove the inequalities in general domains using analogous inequalities in cubes.

Main tool:

Whitney decomposition into cubes!
\[ \Omega \subset \mathbb{R}^n \quad \text{bounded domain} \quad 1 \leq p \leq \infty \]

\[ W^{1,p}(\Omega) = \left\{ v \in L^p(\Omega) : \frac{\partial v}{\partial x_j} \in L^p(\Omega), \forall j = 1, \ldots, n \right\} \]

\[ W_0^{1,p}(\Omega) = \overline{C_0^\infty(\Omega)} \subset W^{1,p}(\Omega) \]

\[ H^1(\Omega) = W^{1,2}(\Omega) \quad , \quad H_0^1(\Omega) = W_0^{1,2}(\Omega) \]
\[ L^p_0(\Omega) = \{ f \in L^p(\Omega) : \int_\Omega f = 0 \} \]

\[ W^{-1,p'}(\Omega) = W^{1,p}_0(\Omega)' \]

where \( p' \) is the dual exponent of \( p \)

\[ C = C(\cdots) \text{ constant depending on } \cdots \]
\[ \mathbf{v} \in W^{1,p}(\Omega)^n \quad , \quad D\mathbf{v} = \left( \frac{\partial v_i}{\partial x_j} \right) \]
\[ \mathbf{v} \in W^{1,p}(\Omega)^n, \quad D\mathbf{v} = \left( \frac{\partial v_i}{\partial x_j} \right) \]

\[ \varepsilon(\mathbf{v}) \text{ symmetric part of } D\mathbf{v}, \text{ i. e.,} \]

\[ \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]
1 < p < \infty, there exists \( C = C(\Omega, p) \) such that

\[
\|v\|_{W^{1,p}(\Omega)^n} \leq C \left\{ \|v\|_{L^p(\Omega)^n} + \|\varepsilon(v)\|_{L^p(\Omega)^{n\times n}} \right\}
\]
\[ f \in L^p_0(\Omega), \quad 1 < p < \infty \]
$f \in L^p_0(\Omega), 1 < p < \infty$

$\exists u \in W^{1,p}_0(\Omega)^n$ such that

$$\text{div } u = f, \quad \|u\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}$$

$C = C(\Omega, p)$
Called Lions Lemma (when $p = 2$)
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$\exists \ C = C(\Omega, p)$ such that

$$\int_{\Omega} f = 0 \implies \| f \|_{L^p(\Omega)} \leq C \| \nabla f \|_{W^{-1,p}(\Omega)^n}$$
Called Lions Lemma (when $p = 2$)

$\exists \ C = C(\Omega, p)$ such that

$$\int_\Omega f = 0 \ \Rightarrow \ \| f \|_{L^p'(\Omega)} \leq C \| \nabla f \|_{W^{-1, p'}(\Omega)^n}$$

or equivalently (inf-sup condition for Stokes)

$$\inf_{f \in L^p_0(\Omega)} \sup_{v \in W^{1, p}_0(\Omega)^n} \frac{\int_\Omega f \text{div} v}{\| f \|_{L^p} \| v \|_{W^{1, p}_0(\Omega)^n}} \geq \alpha > 0$$

with $\alpha = C^{-1}$. 

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Poincaré and related inequalities
\[ d(x) = \text{distance to } \partial \Omega \]
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\[ \exists C = C(\Omega, p) \text{ such that} \]

\[ \int_{\Omega} f = 0 \implies \| f \|_{L^p(\Omega)} \leq C \| d \nabla f \|_{L^p(\Omega)^n} \]
$f \in L_{0}^{p'}(\Omega) \Rightarrow \exists u \in L^{p'}(\Omega)^n$ such that
\[ f \in L_0^{p'}(\Omega) \Rightarrow \exists u \in L^{p'}(\Omega)^n \text{ such that} \]

\[ \text{div } u = f \quad , \quad \| u \|_{L^{p'}(\Omega)^n} \leq C \| f \|_{L^{p'}(\Omega)} \]

\[ u \cdot n = 0 \quad \text{en} \quad \partial \Omega \]
Several connections between these inequalities are known:
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Right Inverse of Div $\Rightarrow$ Korn
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Right Inverse of Div $\Rightarrow$ Korn

Improved Poincaré $\Rightarrow$ Korn
(Kondratiev-Oleinik)
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Right Inverse of Div $\Rightarrow$ Korn

Improved Poincaré $\Rightarrow$ Korn (Kondratiev-Oleinik)

Explicit relation between constants in 2d for $C^1$ domains (Horgan-Payne)
A SIMPLE EXAMPLE

THE DOMAIN CANNOT BE ARBITRARY

\[ \Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < x^2 \} \]
A SIMPLE EXAMPLE

THE DOMAIN CANNOT BE ARBITRARY

EXAMPLE (G. ACOSTA):

\[ \Omega = \{ (x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < x^2 \} \]
\[ f(x, y) = \frac{1}{x^2} - 3 \Rightarrow \int f = 0 \]

\[ \int f^2 = \int_0^1 \int_0^{x^2} f^2 \, dy \, dx \simeq \int_0^1 \frac{1}{x^2} \Rightarrow f \notin L^2(\Omega) \]

\[-2yx^{-3} \in L^2(\Omega) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial (-2yx^{-3})}{\partial y} \in H^{-1}(\Omega)\]
A SIMPLE EXAMPLE

THEN:

\[ \| f \|_{L^2(\Omega)} \lesssim C \| \nabla f \|_{H^{-1}(\Omega)} \]

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Poincaré and related inequalities
SAME EXAMPLE \implies

\| f \|_{L^2(\Omega)} \not\leq C \| d\nabla f \|_{L^2(\Omega)^n}
OUR MAIN RESULT

“THEOREM”

\[ \| f \|_{L^1(\Omega)} \leq C \| d \nabla f \|_{L^1(\Omega)^n} \quad \forall f \in L^1_0(\Omega) \]

\[ \Downarrow \]

“All” the inequalities valid in cubes are valid in \( \Omega \)
\((\forall p)!!\)
Why are relations between inequalities interesting?
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But, in many situations it is important to have information of the constants in terms of the geometry of the domain.
Why are relations between inequalities interesting?
We could think that, if we are not interested in strange domains, we already know that all these inequalities are valid (for example for $\Omega$ a Lipschitz domain)
But, in many situations it is important to have information of the constants in terms of the geometry of the domain
Then, we can translate information from one inequality to other
For example:
ESTIMATES IN TERMS OF GEOMETRIC PROPERTIES

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Let $\Omega$ be a convex domain with diameter $D$ and inner diameter $\rho$. 
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Let \( \Omega \) be a convex domain with diameter \( D \) and inner diameter \( \rho \). Then, it was recently proved by M. Barchiesi, F. Cagnetti, and N. Fusco that

\[
\|f\|_{L^1(\Omega)} \leq C \frac{D}{\rho} \|d \nabla f\|_{L^1(\Omega)^n} \quad \forall f \in L^1_0(\Omega)
\]
Consequently, we obtain from our results that, for $\Omega$ convex,
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$$\|f\|_{L^p(\Omega)} \leq C \frac{D}{\rho} \|\nabla f\|_{W^{-1,p}(\Omega)^n} \quad \forall f \in L^p_0(\Omega)$$
Consequently, we obtain from our results that, for $\Omega$ convex,

$$\| f \|_{L^p(\Omega)} \leq C \frac{D}{\rho} \| \nabla f \|_{W^{-1,p}(\Omega)^n} \quad \forall f \in L^p_0(\Omega)$$

$$1 < p < \infty$$

$$C = C(p, n)$$
The case of a rectangular domain shows that the dependence of the constant in terms of the eccentricity $D/\rho$ cannot be improved. Indeed (let us consider $p = 2$ for simplicity)
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\[
\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, \ -\varepsilon < y < \varepsilon\}
\]
The case of a rectangular domain shows that the dependence of the constant in terms of the eccentricity $D/\rho$ cannot be improved. Indeed (let us consider $p = 2$ for simplicity)

$$
\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, -\varepsilon < y < \varepsilon\}
$$

$$
\|f\|_{L^2(\Omega_\varepsilon)} \leq C_\varepsilon \|\nabla f\|_{H^{-1}(\Omega_\varepsilon)^n}
\downarrow

\|x\|_{L^2} \leq C_\varepsilon \|\nabla x\|_{H^{-1}} = C_\varepsilon \|\nabla y\|_{H^{-1}} \leq C_\varepsilon \|y\|_{L^2}
$$
\[ \| X \|_{L^2(\Omega_\varepsilon)} \sim \varepsilon^{\frac{1}{2}}, \quad \| Y \|_{L^2(\Omega_\varepsilon)} \sim \varepsilon^{\frac{3}{2}} \]

\[ \Downarrow \]

\[ C_\varepsilon \geq \frac{C}{\varepsilon} \sim \frac{D}{\rho} \]
In many cases it is also possible to go in the opposite way:
In many cases it is also possible to go in the opposite way: For example, if $\Omega \subset \mathbb{R}^2$ has diameter $D$ and is star-shaped with respect to a ball of radius $\rho$, we could prove that

$$\|f\|_{L^2(\Omega)} \leq C \frac{D}{\rho} \left( \log \frac{D}{\rho} \right) \|\nabla f\|_{H^{-1}(\Omega)^n} \quad \forall f \in L^2_0(\Omega)$$
ESTIMATES IN TERMS OF GEOMETRIC PROPERTIES

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$$\| f \|_{L^2(\Omega)} \leq C \frac{D}{\rho} \left( \log \frac{D}{\rho} \right) \| \nabla f \|_{H^{-1}(\Omega)^n} \quad \forall f \in L^2_0(\Omega)$$

And from this estimate it can be deduced that

$$\| f \|_{L^2(\Omega)} \leq C \frac{D}{\rho} \left( \log \frac{D}{\rho} \right) \| d\nabla f \|_{L^2(\Omega)^n} \quad \forall f \in L^2_0(\Omega)$$
Proof of

\[ \|f\|_{L^1(\Omega)} \leq C_1 \|d\nabla f\|_{L^1(\Omega)^n} \quad \forall f \in L^1_0(\Omega) \]
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\[ \|f\|_{L^1(\Omega)} \leq C_1 \|d\nabla f\|_{L^1(\Omega)^n} \quad \forall f \in L^1_0(\Omega) \]

\[ \Downarrow \]

\[ \forall f \in L^p_0(\Omega), 1 < p < \infty \]

\[ \exists u \in W^{1,p}_0(\Omega)^n \quad \text{such that} \]

\[ \text{div } u = f \quad , \quad \|u\|_{W^{1,p}(\Omega)^n} \leq C \|f\|_{L^p(\Omega)} \]
Proof of

\[ \| f \|_{L^1(\Omega)} \leq C_1 \| d\nabla f \|_{L^1(\Omega)^n} \quad \forall f \in L_0^1(\Omega) \]

\[ \Downarrow \]

\[ \forall f \in L_0^p(\Omega), \, 1 < p < \infty \]

\[ \exists u \in W_0^{1,p}(\Omega)^n \quad \text{such that} \]

\[ \text{div } u = f \quad , \quad \| u \|_{W^{1,p}(\Omega)^n} \leq C \| f \|_{L^p(\Omega)} \]

\[ C = C(n, p, C_1) \]
THREE STEPS

Improved Poincaré for $p = 1 \Rightarrow$ Improved Poincaré for $1 < p < \infty$
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- Improved Poincaré for $p'$ allows us to reduce the problem to “local problems” in cubes
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- Improved Poincaré for $p = 1$ $\Rightarrow$ Improved Poincaré for $1 < p < \infty$
- Improved Poincaré for $p'$ allows us to reduce the problem to “local problems” in cubes
- Solve for the divergence in cubes and sum
STEP 2

Assume improved Poincaré for $p'$
Assume improved Poincaré for $p'$
Take a Whitney decomposition of $\Omega$, i.e.,

$$\Omega = \bigcup_{j} Q_j,$$

$s_j \approx \text{dist}(Q_j, \partial \Omega) =: d_j$.

For $f \in L^p(\Omega)$, there exists a decomposition $f = \sum_j f_j$. 

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Poincaré and related inequalities
Assume improved Poincaré for $p'$

Take a Whitney decomposition of $\Omega$, i. e.,

$$\Omega = \bigcup_j Q_j , \quad Q_j^0 \cap Q_i^0 = \emptyset$$

$$\text{diam } Q_j \sim \text{dist}(Q_j, \partial \Omega) =: d_j$$
Assume improved Poincaré for $p'$

Take a Whitney decomposition of $\Omega$, i.e.,

$$\Omega = \bigcup_j Q_j, \quad Q_j^0 \cap Q_i^0 = \emptyset$$

$$\text{diam } Q_j \sim \text{dist } (Q_j, \partial \Omega) =: d_j$$

For $f \in L^p_0(\Omega)$, there exists a decomposition

$$f = \sum_j f_j$$
such that
such that

$$f_j \in L_0^p(\tilde{Q}_j)$$

and

$$\tilde{Q}_j := \frac{9}{8} Q_j$$
such that

\[ f_j \in L^p_0(\tilde{Q}_j) \quad , \quad \tilde{Q}_j := \frac{9}{8} Q_j \]

and

\[ \|f\|_{L^p(\Omega)} \sim \sum_j \|f_j\|_{L^p(\tilde{Q}_j)} \]
Proof of the existence of this decomposition:
Proof of the existence of this decomposition:

Take a partition of unity associated with the Whitney decomposition

\[ \sum_j \phi_j = 1 \quad , \quad \text{sop } \phi_j \subset \tilde{Q}_j = \frac{9}{8}Q_j \]

\[ \| \phi_j \|_{L^\infty} \leq 1 \quad , \quad \| \nabla \phi_j \|_{L^\infty} \leq C/d_j \]
Assuming the improved Poincaré for $p'$, we obtain by duality:
DECOMPOSITION OF FUNCTIONS

Assuming the improved Poincaré for \( p' \), we obtain by duality:

For \( f \in L^p_0(\Omega) \) there exists \( u \in L^p(\Omega)^n \) such that

\[
\operatorname{div} u = f \quad \text{in} \quad \Omega \quad \quad u \cdot n = 0 \quad \text{on} \quad \partial \Omega
\]
Assuming the improved Poincaré for $p'$, we obtain by duality:

For $f \in L^p_0(\Omega)$ there exists $u \in L^p(\Omega)^n$ such that

$$\text{div } u = f \quad \text{in } \Omega \quad u \cdot n = 0 \quad \text{on } \partial \Omega$$

$$\|u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$
Then, we define
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\[ f_j = \text{div} (\phi_j u) \]
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and so
Then, we define

\[ f_j = \text{div} \left( \phi_j u \right) \]

and so

\[
\begin{align*}
  f &= \text{div} u = \text{div} u \sum_j \phi_j = \sum_j \text{div} (\phi_j u) = \sum_j f_j \\
  \text{sop} \phi_j &\subset \tilde{Q}_j \Rightarrow \text{sop} f_j \subset \tilde{Q}_j , \quad \int f_j = 0
\end{align*}
\]
From finite superposition:

\[ |f(x)|^p \leq C \sum_{j} |f_j(x)|^p \]
From finite superposition:

\[ |f(x)|^p \leq C \sum_j |f_j(x)|^p \]

and therefore

\[ \|f\|_{L^p(\Omega)}^p \leq C \sum_j \|f_j\|_{L^p(\tilde{Q}_j)}^p \]
To prove the other estimate we use

\[ \| \phi_j \|_{L^\infty} \leq 1, \quad \| \nabla \phi_j \|_{L^\infty} \leq \frac{C}{d_j} \]
To prove the other estimate we use

\[ \| \phi_j \|_{L^\infty} \leq 1 \quad , \quad \| \nabla \phi_j \|_{L^\infty} \leq C/d_j \]

Then,

\[ f_j = \text{div} (\phi_j u) = \text{div} u \phi_j + \nabla \phi_j \cdot u = f \phi_j + \nabla \phi_j \cdot u \]
\[ \| f_j \|_{L^p(\tilde{Q}_j)}^p \leq C \left\{ \| f \|_{L^p(\tilde{Q}_j)}^p + \| u \|_{L^p(\tilde{Q}_j)}^p \right\} \]
\[ \| f_j \|_{L^p(Q_j)}^p \leq C \left\{ \| f \|_{L^p(Q_j)}^p + \| \frac{u}{d} \|_{L^p(Q_j)}^p \right\} \]

and therefore, using

\[ \| \frac{u}{d} \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)} \]
DECOMPOSITION OF FUNCTIONS

\[ \| f_j \|_{L^p(Q_j)}^p \leq C \left\{ \| f \|_{L^p(Q_j)}^p + \| u \|_{L^p(Q_j)}^p \right\} \]

and therefore, using

\[ \| u \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)} \]

we obtain,
\[ \| f_j \|_{L^p(\tilde{Q}_j)}^p \leq C \left\{ \| f \|_{L^p(\tilde{Q}_j)}^p + \| \frac{u}{d} \|_{L^p(\tilde{Q}_j)}^p \right\} \]

and therefore, using

\[ \| \frac{u}{d} \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)} \]

we obtain,

\[ \sum_j \| f_j \|_{L^p(\tilde{Q}_j)}^p \leq C \| f \|_{L^p(\Omega)}^p \]
Then, to solve $\text{div } u = f$
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solve \( \text{div} \, u_j = f_j \)
Then, to solve \( \text{div} \mathbf{u} = f \)

solve \( \text{div} \mathbf{u}_j = f_j \)

\[ \mathbf{u}_j \in W_0^{1,p}(\tilde{Q}_j) \quad , \quad \| \mathbf{u}_j \|_{W^{1,p}(\tilde{Q}_j)} \leq C \| f_j \|_{L^p(\tilde{Q}_j)} \]
Then, to solve \( \text{div} \, u = f \)

solve \( \text{div} \, u_j = f_j \)

\[
    u_j \in W^{1,p}_0(\tilde{Q}_j), \quad \|u_j\|_{W^{1,p}(\tilde{Q}_j)} \leq C \|f_j\|_{L^p(\tilde{Q}_j)}
\]

Observe that \( C = C(n, p) \)
Then, to solve \( \text{div} \mathbf{u} = f \)

solve \( \text{div} \mathbf{u}_j = f_j \)

\[
\mathbf{u}_j \in W^{1,p}_0(\tilde{Q}_j), \quad \| \mathbf{u}_j \|_{W^{1,p}(\tilde{Q}_j)} \leq C \| f_j \|_{L^p(\tilde{Q}_j)}
\]

Observe that \( C = C(n, p) \)

and
Then, to solve \( \text{div} \, u = f \)

solve \( \text{div} \, u_j = f_j \)

\[
\begin{align*}
  u_j &\in W^{1,p}_0(\tilde{Q}_j), & \|u_j\|_{W^{1,p}(\tilde{Q}_j)} &\leq C\|f_j\|_{L^p(\tilde{Q}_j)}
\end{align*}
\]

Observe that \( C = C(n, p) \)

and

\[
  u = \sum_j u_j
\]

is the required solution!
To conclude the proof of our theorem we need:
To conclude the proof of our theorem we need:

\[ \| f - f_\Omega \|_{L^1(\Omega)} \leq C \| d\nabla f \|_{L^1(\Omega)} \]

\[ \Downarrow \]

\[ \| f - f_\Omega \|_{L^p(\Omega)} \leq C \| d\nabla f \|_{L^p(\Omega)} \]
OR MORE GENERALLY

\[ \| f - f_\Omega \|_{L^1(\Omega)} \leq C \| w \nabla f \|_{L^1(\Omega)} \]

\[ \Downarrow \]

\[ \| f - f_\Omega \|_{L^p(\Omega)} \leq C \| w \nabla f \|_{L^p(\Omega)} \]

\[ \forall p < \infty \]
1) \[ \| f - f_\Omega \|_{L^p(\Omega)} \leq C \| w \nabla f \|_{L^p(\Omega)} \]
Poincaré in $L^1$ implies Poincaré in $L^p$, $p < \infty$

1) \[ \| f - f_\Omega \|_{L^p(\Omega)} \leq C \| w \nabla f \|_{L^p(\Omega)} \]

2) \( \forall E \subset \Omega \) such that \( |E| \geq \frac{1}{2} |\Omega| \)
\( f|_E = 0 \Rightarrow \| f \|_{L^p(\Omega)} \leq C \| w \nabla f \|_{L^p(\Omega)} \)

with \( C \) independent of \( E \) and \( f \).
Case $p = 1$:
Case $p = 1$:

$$f = f_+ - f_-$$

Suppose $|\{f_+ = 0\}| \geq \frac{1}{2} |\Omega|$

$$\Rightarrow \quad \|f_+\|_{L^1(\Omega)} \leq C \|w \nabla f_+\|_{L^1(\Omega)} \leq C \|w \nabla f\|_{L^1(\Omega)}$$
But
Poincaré in $L^1$ implies Poincaré in $L^p$, $p < \infty$

But

$$\int_{\Omega} f = 0 \Rightarrow \int_{\Omega} f_+ = \int_{\Omega} f_-$$
But

\[
\int_{\Omega} f = 0 \Rightarrow \int_{\Omega} f^+ = \int_{\Omega} f^-
\]

and therefore,

\[
\|f\|_{L^1(\Omega)} = \int_{\Omega} f^+ + f^- = 2 \int_{\Omega} f^+ \leq 2C \|w \nabla f^+\|_{L^1(\Omega)}
\]
Poincaré in $L^1$ implies Poincaré in $L^p$, $p < \infty$

If

$$\int_\Omega f_+^p = \int_\Omega f_-^p$$
Poincaré in $L^1$ implies Poincaré in $L^p$, $p < \infty$

If

$$\int_{\Omega} f_+^p = \int_{\Omega} f_-^p$$

the same argument than for $p = 1$ applies!
If
\[ \int_\Omega f_+^p = \int_\Omega f_-^p \]
the same argument than for \( p = 1 \) applies!

But, from Bolzano's theorem
\[ \exists \lambda \in \mathbb{R} \]
such that
\[ \int_\Omega (f - \lambda)_+^p = \int_\Omega (f - \lambda)_-^p \]
Now,

\[ f|_E = 0 \Rightarrow |f|^p|_E = 0 \]
Now,

$$f|_{E} = 0 \Rightarrow |f|^p|_{E} = 0$$

Apply 1–Poincaré to $|f|^p$
Poincaré in $L^1$ implies Poincaré in $L^p$, $p < \infty$

$$\int_{\Omega} |f|^p \leq C_p \int_{\Omega} |f|^{p-1} |\nabla f|_w$$

$$\leq \left( \int_{\Omega} |f|^p \right)^{1/p'} \left( \int_{\Omega} |\nabla f|^{p_w} \right)^{1/p}$$
Poincaré in $L^1$ implies Poincaré in $L^p$, $p < \infty$

$$\int_{\Omega} |f|^p \leq C_p \int_{\Omega} |f|^{p-1} |\nabla f| w$$

$$\leq \left( \int_{\Omega} |f|^p \right)^{1/p'} \left( \int_{\Omega} |\nabla f|^p w^p \right)^{1/p}$$

therefore,

$$\|f\|_{L^p(\Omega)} \leq C \|w \nabla f\|_{L^p(\Omega)}$$
Now, to complete our arguments we have to prove the improved Poincaré in $L^1$. 
Proof of the improved Poincaré in $L^1$

Now, to complete our arguments we have to prove the improved Poincaré in $L^1$.

We will show that it can be proved for a very general class of domains by elementary calculus arguments.
$$f(y) - \bar{f} = - \int_{\Omega} G(x, y) \cdot \nabla f(x) \, dx$$
\[ f(y) - \bar{f} = - \int_{\Omega} G(x, y) \cdot \nabla f(x) \, dx \]

Assume \( \Omega \) star-shaped with respect to a ball \( B(0, \delta) \)
\[ f(y) - \bar{f} = - \int_{\Omega} G(x, y) \cdot \nabla f(x) \, dx \]

Assume \( \Omega \) star-shaped with respect to a ball \( B(0, \delta) \)

\[ \omega \in C_0^\infty(B_{\delta/2}) \quad \int_{B_{\delta/2}} \omega = 1 \quad \bar{f} = \int_{\Omega} f \omega \]

\[ y \in \Omega, \quad z \in B_{\delta/2} \]

\[ f(y) - f(z) = \int_0^1 (y - z) \cdot \nabla f(y + t(z - y)) \, dt \]
Multiplying by $\omega(z)$ and integrating:
Multiplying by \( \omega(z) \) and integrating:

\[
f(y) - \bar{f} = \int_{\Omega} \int_{0}^{1} (y - z) \cdot \nabla f(y + t(z - y)) \omega(z) dtdz
\]
Multiplying by $\omega(z)$ and integrating:

$$f(y) - \bar{f} = \int_{\Omega} \int_{0}^{1} (y - z) \cdot \nabla f(y + t(z - y)) \omega(z) dtdz$$

and changing variables

$$x = y + t(z - y)$$
\[
f(y) - \bar{f} = \int_{\Omega} \int_{0}^{1} \frac{(y - x)}{t^{n+1}} \cdot \nabla f(x) \omega \left( y + \frac{x - y}{t} \right) \, dt \, dx
\]
that is,

\[ f(y) - \bar{f} = - \int_{\Omega} G(x, y) \cdot \nabla f(x) \, dx \]
$$G(x, y) = \int_0^1 \frac{(x - y)}{t} \omega \left( y + \frac{x - y}{t} \right) \frac{1}{t^n} dt$$
\[ G(x, y) = \int_0^1 \frac{(x - y)}{t} \omega \left( y + \frac{x - y}{t} \right) \frac{1}{t^n} dt \]

PROPERTIES OF \( G(x, y) \)
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The representation formula can be generalized to John domains (Acosta-D.-Muschietti)
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\[ G(x, y) = \int_0^1 \left( \frac{x - \gamma}{t} + \gamma'(t, y) \right) \omega \left( \frac{x - \gamma(t, y)}{t} \right) \frac{dt}{t^n} \]
The representation formula can be generalized to John domains (Acosta-D.-Muschietti)

\[ G(x, y) = \int_0^1 \left( \frac{x - \gamma}{t} + \dot{\gamma}(t, y) \right) \omega \left( \frac{x - \gamma(t, y)}{t} \right) \frac{dt}{t^n} \]

where \( \gamma(t, y) \) are “John curves”

\[ \gamma(0, y) = y, \quad \gamma(1, y) = 0 \]

\[ |\dot{\gamma}(t, y)| \leq K, \quad d(\gamma(t, y)) \geq \delta t \]
And the same properties for $G(x, y)$ holds.

$$|G(x, y)| \leq \frac{C_1}{|x - y|^{n-1}}$$

$$|x - y| > C_2 d(x) \Rightarrow G(x, y) = 0$$

with constants depending on $\delta, K y \omega$
\[
\int_{\Omega} |f(y) - \bar{f}| \, dy \leq \int_{\Omega} \int_{\Omega} |G(x, y)||\nabla f(x)| \, dx \, dy
\]
\[
\int_{\Omega} |f(y) - \bar{f}|\,dy \leq \int_{\Omega} \int_{\Omega} |G(x, y)||\nabla f(x)|\,dxdy
\]

\[
= \int_{\Omega} \int_{\Omega} |G(x, y)| \cdot |\nabla f(x)|\,dydx
\]
PROOF OF IMPROVED POINCARÉ IN $L^1$
(Drelichman-D.)

\[ \int_{\Omega} |f(y) - \bar{f}| \, dy \leq \int_{\Omega} \int_{\Omega} |G(x, y)||\nabla f(x)| \, dx \, dy \]

\[ = \int_{\Omega} \int_{\Omega} |G(x, y)| \cdot |\nabla f(x)| \, dy \, dx \]

\[ \leq C \int_{\Omega} \int_{|x-y| \leq Cd(x)} \frac{dy}{x-y^{n-1}} |\nabla f(x)| \, dx \]

\[ \leq C \| d\nabla f \|_{L^1(\Omega)} \]
If $\Omega$ satisfies the weaker improved Poincaré inequality

$$\|f\|_{L^1(\Omega)} \leq C_1 \|d^{1-\beta} \nabla f\|_{L^1(\Omega)}$$

for some $\beta < 1$ then,

$$\forall f \in L^p_0(\Omega) \exists u \in W^{1,p}_0(\Omega)$$

such that

$$\text{div}\ u = f,$$

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In this case $\beta = \alpha$
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In this case \( \beta = \alpha \)
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\[
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\]

\[
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In this case $\beta = \alpha$

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$$\forall f \in L^p_0(\Omega) \quad \exists u \in W^{1,p}_0(\Omega)^n$$

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$$\text{div } u = f, \quad \| d^{1-\alpha} \nabla u \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)}$$

The case $p = 2$ can be used to prove existence and uniqueness for the Stokes equations.
We have a general methodology to prove inequalities once they are known in cubes.
We have a general methodology to prove inequalities once they are known in cubes.

The decomposition of functions of zero integral is interesting in itself because it might have other applications (we are working in the application of this decomposition to weighted a priori estimates).
THANK YOU FOR YOUR ATTENTION!

HAPPY BIRTHDAY MARTIN!!