Disclaimer: this is a draft and probably contains many typos and mistakes. Please report them to Manuel.Bodirsky@tu-dresden.de.

Recommendations. Chapter 1-4 are suitable for a 3rd year bachelor course. Chapter 5-8 are for a master-level course.

Notes concerning text-book literature. We use material from 12 33 34 42 57. Oligomorphic permutation groups are the topic of the short and stimulating book by Peter Cameron 33. The book on permutation groups by the same author 34 is mostly about finite permutation groups, but Section 5 covers oligomorphic permutation groups. There is even less material on infinite permutation groups in 42. The notes in 12 are on infinite permutation groups, but they neglect the topological aspect of the topic. Hodges 58 writes about model theory, but covers many fundamental things for permutation groups on the way, including topological aspects.

Prerequisites. This text is essentially self-contained. It contains only a few theorems that are just stated but not proved, for instance

- the theorem of Ryll-Nardzewski (Theorem 3.2.3); however, we have extracted all the consequences of the (proof of the) Ryll-Nardzewski theorem that we need in this text into Theorem 3.1.1 which we do prove. The full proof of Theorem 3.2.3 can be found in many text books on model theory.
- the Theorem of Birkhoff-Kakutani (Theorem 4.2.14); this is covered for example in 51 65.
CHAPTER 1

Permutation Groups

A permutation of a set $X$ is a bijection between $X$ and $X$. A permutation group $G$ on a set $X$ is a set of permutations of $X$ that

1. contains the identity permutation $\text{id}_X$,
2. contains for every permutation $u \in G$ also its inverse $u^{-1}$, and
3. contains for all $u, v \in G$ their composition $u \circ v$, defined by $(u \circ v)(x) = u(v(x))$ for all $x \in X$.

Examples.

- The set of all permutations of $X$, denoted by $\text{Sym}(X)$.
- The set of functions $\{t_a: \mathbb{Z} \to \mathbb{Z} | t_a(x) = x + a, a \in \mathbb{Z}\} \subset \text{Sym}(\mathbb{Z})$.
- The set of all order-preserving permutations of $\mathbb{Q}$.

Erlanger Programm of Felix Klein: “Understanding structure (geometry in the case of Felix Klein) by understanding its symmetry”

1.1. Relational Structures

A signature $\tau$ is a set of relation and function symbols, each equipped with an arity $k \in \mathbb{N}$. A $\tau$-structure $\mathcal{A}$ is a set $A$ (the domain of $\mathcal{A}$) together with

- a relation $R^A \subseteq A^k$ for each $k$-ary relation symbol in $\tau$, and
- a function $f^A: A^k \to A$ for each $k$-ary function symbol in $\tau$; here we allow the case $k = 0$ to model constant symbols.

Unless stated otherwise, $A, B, C, \ldots$ denote the domains of the structures $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$, respectively. We sometimes write $G(A; R_1^A, R_2^A, \ldots, f_1^A, f_2^A, \ldots)$ for the relational structure $\mathcal{A}$ with relations $R_1^A, R_2^A, \ldots$ and functions $f_1^A, f_2^A, \ldots$. When there is no danger of confusion, we use the same symbol for a function and its function symbol, and for a relation and its relation symbol. We say that a structure is infinite if its domain is infinite.

Example 1.1.1. A (simple, undirected) graph is a pair $(V, E)$ consisting of a set of vertices $V$ and a set of edges $E \subseteq \binom{V}{2}$, that is, $E$ is a set of 2-element subsets of $V$. Graphs can be modelled using relational structures $G$ using a signature that contains a single binary relation symbol $R$, putting $R^G := E$. If we insist that a structure with this signature satisfies $(x, y) \in R^G \Rightarrow (y, x) \in R^G$ and not $(x, x) \in R^G$, then we can associate to such a structure an undirected graph and obtain a bijective correspondence between undirected graphs and structures $G$ as described above.

1.1.1. Extensions and substructures. A $\tau$-structure $\mathcal{A}$ is a substructure of a $\tau$-structure $\mathcal{B}$ iff

- $A \subseteq B$,
- for each $R \in \tau$, and for all tuples $\bar{a}$ from $A$, $\bar{a} \in R^A$ iff $\bar{a} \in R^B$, and
- for each $f \in \tau$ we have that $f^A(\bar{a}) = f^B(\bar{a})$. 

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In this case, we also say that \( B \) is an extension of \( A \). Substructures \( A \) of \( B \) and extensions \( B \) of \( A \) are called proper if the domains of \( A \) and \( B \) are distinct.

Note that for every subset \( S \) of the domain of \( B \) there is a unique smallest substructure of \( B \) whose domain contains \( S \), which is called the substructure of \( B \) generated by \( S \), and which is denoted by \( B[S] \).

Example 1.1.2. A group is a structure \( G \) with a binary function symbol \( \cdot \) for multiplication, a unary function symbol \( ^{-1} \) for taking the inverse, and a constant denoted by \( e \), satisfying the sentences
\[
\begin{align*}
&\forall x,y,z. (x \cdot (y \cdot z)) = ((x \cdot y) \cdot z), \\
&\forall x. x \cdot x^{-1} = 1, \\
&\forall x. e \cdot x = x, \text{ and } \forall x. x \cdot e = x.
\end{align*}
\]
In this signature, the subgroups of \( G \) are precisely the substructures \( G \) as defined above; we also write \( H \leq G \) if \( H \) is a subgroup of \( G \). To distinguish groups from permutation groups, we might also refer to a group as an abstract group. Clearly, every permutation group \( G \) gives rise to an abstract group \( G \) where \( \circ \) takes the role of multiplication.

Example 1.1.3. Let \( G \) be a permutation group on a set \( A \). Let \( A \) be a structure with domain \( A \) whose signature only contains unary function symbols that denote permutations of \( A \). Note that every \( \tau \)-term must have exactly one variable, and every \( \tau \)-term \( t(x) \) defines over \( A \) a permutation \( t^A \). Then \( A \) is called a \( G \)-set if the set of all permutations obtained in this way is precisely \( G \). (We do allow that several function symbols denote the same element of \( G \).)

1.1.2. Homomorphisms, Automorphisms, etc. In the following, let \( A \) and \( B \) be \( \tau \)-structures. A homomorphism \( h \) from \( A \) to \( B \) is a mapping from \( A \) to \( B \) that preserves each function and each relation for the symbols in \( \tau \); that is,
- if \( (a_1, \ldots, a_n) \) is in \( R^A \), then \( (h(a_1), \ldots, h(a_n)) \) must be in \( R^B \);
- \( f^B(h(a_1), \ldots, h(a_k)) = h(f^A(a_1, \ldots, a_k)) \).
A homomorphism from \( A \) to \( B \) is called a strong homomorphism if it also preserves the complements of the relations from \( A \). Injective strong homomorphisms are called embeddings.

Example 1.1.4. When \( G \) and \( H \) are groups and \( h: G \to H \) is a map, then it suffices to prove that \( h \) preserves \( \circ \) in order to verify that \( h \) is a homomorphism (Why?). Moreover, note that injective homomorphisms are embeddings (this is not true for structures where the signature contains relation symbols! Find a counterexample!).

Surjective embeddings are called isomorphisms. Let \( G \) be a permutation group on a set \( A \) and \( H \) a permutation group on a set \( B \). Then a bijection \( i \) between \( A \) and \( B \) is called a (permutation group) isomorphism if there exists a bijection \( \xi \) between \( G \) and \( H \) such that for all \( g \in G \) and \( a \in A \) we have
\[
i(g(a)) \Leftrightarrow \xi(g)(i(a)).
\]

Proposition 1.1.5. \( G \) and \( H \) are isomorphic (as permutation groups) if and only if there exists a \( G \)-set \( A \) and an \( H \)-set \( B \) such that \( A \) and \( B \) are isomorphic as structures (as introduced above).

Note that if \( G \) and \( H \) are isomorphic (as permutation groups), then the corresponding abstract groups are isomorphic as well, but the converse need not be true (see Corollary 5.1.2 for a characterisation of permutation groups that are isomorphic as abstract groups).
Homomorphisms and isomorphisms from $B$ to itself are called endomorphisms and automorphisms, respectively. When $f: A \rightarrow B$ and $g: B \rightarrow C$, then $g \circ f$ denotes the composed function $x \mapsto g(f(x))$. Clearly, the composition of two homomorphisms (embeddings, automorphisms) is again a homomorphism (embedding, automorphism). Let $\text{Aut}(A)$ and $\text{End}(A)$ be the sets of automorphisms and endomorphisms, respectively, of $A$. The set $\text{Aut}(A)$ can be viewed as a group, and $\text{End}(A)$ as a monoid with respect to composition.

1.1.3. Expansions and reducts. Let $\sigma, \tau$ be signatures with $\sigma \subseteq \tau$. When $A$ is a $\sigma$-structure and $B$ is a $\tau$-structure, both with the same domain, such that $R^A = R^B$ for all relations $R \in \sigma$ and $f^A = f^B$ for all functions and constants $f \in \sigma$, then $A$ is called a reduct of $B$, and $B$ is called an expansion of $A$.

1.1.4. Disjoint Unions and Direct Products. Let $\tau$ be a relational signature. A disjoint union $A \uplus B$ of two $\tau$-structures $A$ and $B$ is the union of isomorphic copies of $A$ and $B$ with disjoint domains. That is, for all $R \in \tau$ we have $R^A \uplus R^B = R^A \cup R^B$. As disjoint unions are unique up to isomorphism, we usually speak of the disjoint union of $A$ and $B$. The disjoint union of a set of $\tau$-structures $\mathcal{C}$ is defined analogously (and the disjoint union of an empty set of structures is the $\tau$-structure with empty domain). A relational structure is called connected if it is not the disjoint union of two nonempty structures.

Let $A$ and $B$ be $\tau$-structures. Then the (direct, or categorical) product $C = A \times B$ is the $\tau$-structure with domain $A \times B$, which has for each $k$-ary $R \in \tau$ the relation that contains a tuple $((a_1, b_1), \ldots, (a_k, b_k))$ if and only if $R(a_1, \ldots, a_k)$ holds in $A$ and $R(b_1, \ldots, b_k)$ holds in $B$. For each $k$-ary $f \in \tau$ the structure $C$ has the operation that maps $((a_1, b_1), \ldots, (a_k, b_k))$ to $(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k))$. The direct product $A \times A$ is also denoted by $A^2$, and the $k$-fold product $A \times \cdots \times A$, defined analogously, by $A^k$ (it is straightforward to define this also for infinite $k$).

Exercises.

(1) Let $G$ be a permutation group, and let $A$ be a $G$-set. What are the substructures of $A$?

(2) Prove Proposition 1.1.3.

(3) Consider the structures

$$
\Gamma_1 := (\mathbb{Q}; \{(x, y) : x = y + 1\})
$$

$$
\Gamma_2 := (\mathbb{Q}; \{(x, y, u, v) : x - y = u - v \in \{1, -1\}\})
$$

$$
\Gamma_3 := (\mathbb{Q}; \{(x, y) : |x - y| = 1\})
$$

Show that

$$
\{\text{id}_{\mathbb{Q}}\} \subseteq \text{Aut}(\Gamma_1) \subseteq \text{Aut}(\Gamma_2) \subseteq \text{Aut}(\Gamma_3) \subseteq \text{Sym}(\mathbb{Q})
$$

1.1.5. Congruences. Let $A$ be a structure with a purely functional signature $\tau$. A congruence of $A$ is an equivalence relation $E$ on $A$ that is preserved by all functions of $A$ in the sense that for a $k$-ary $f \in \tau$ the function $f^A$ is a homomorphism from $(A, E)^k$ to $(A, E)$.

Example 1.1.6. Let $G$ be a permutation group on a set $A$, and let $A$ be a $G$-set. Recall that $A$ is a structure with a purely functional signature (all functions are unary). Then the congruences of $A$ are equivalence relations on $A$ that are preserved by all permutations in $G$. Such equivalence relations are also called congruences of $G$ (without reference to a $G$-set).

Example 1.1.7. Let $G$ be a group. Then there is a natural bijection between the congruences of $G$ and the normal subgroups of $G$: this will be treated in Section 4.6.
1.2. Automorphism Groups

Let \( G \) be a permutation group on a set \( A \). When is \( G \) the automorphism group of a structure with domain \( A \)? This has the following elegant answer. We say that \( G \) is \((\text{locally})\) closed (or closed in \( \text{Sym}(A) \)) if it contains all \( f \in \text{Sym}(A) \) with the property that for all finite \( F \subseteq X \) there exists a \( g \in G \) such that \( f(x) = g(x) \) for all \( x \in F \).

**Proposition 1.2.1.** A permutation group \( G \) on a set \( A \) is the automorphism group of a relational structure with domain \( A \) if and only if \( G \) is closed in \( \text{Sym}(A) \).

In the proof of this proposition, the following concept is useful. We write \( \text{slnv}(G) \) for the set of all relations over \( A \) that are strongly preserved by all permutations \( \alpha_G \), i.e., both \( \alpha \) and \( \alpha^{-1} \) preserve the relation. A relational structure with domain \( A \) whose relations are exactly the relations from \( \text{slnv}(G) \) is called a canonical structure for \( G \).

**Proof of Proposition 1.2.1.** For the forwards implication, suppose that \( G = \text{Aut}(A) \) and that \( f \in \text{Sym}(A) \setminus G \). Then \( f \) or \( f^{-1} \) does not preserve a relation \( R \) or function from \( A \). Suppose that \( R(a_1, \ldots, a_n) \) but not \( R(f(a_1), \ldots, f(a_n)) \). Then there is no \( g \in G \) such that \( g(x) = f(x) \) for all \( x \in \{a_1, \ldots, a_n\} \). The proof for \( f^{-1} \), and for functions not preserved by \( f \) or by \( f^{-1} \) is analogous.

For the reverse implication, let \( A \) be a canonical structure for \( G \). Clearly, every \( g \in G \) is an automorphism of \( A \). Conversely, let \( f \in \text{Aut}(A) \). By assumption, to show that \( f \) is in \( G \), it suffices to show that for any finite tuple \( (a_1, \ldots, a_n) \) of elements from \( X \) there exists an \( g \in G \) such that \( f(x) = g(x) \) for all \( x \in \{a_1, \ldots, a_n\} \). The relation \( R \) := \( \{(g(a_1), \ldots, g(a_n)) : g \in G\} \) is preserved by all operations in \( G \) and hence belongs to the relations of \( A \). Thus, \( f \) preserves \( R \). Also \( \text{id} \in G \), and therefore \( (a_1, \ldots, a_n) \in R \), and so \( R \) contains \( (f(a_1), \ldots, f(a_n)) = (g(a_1), \ldots, g(a_n)) \) for some \( g \in G \). We therefore have \( G = \text{Aut}(A) \) as desired.

The word closed suggests a topology, and indeed there is corresponding topology on \( G \), called the topology of pointwise convergence. Topological aspects will be treated properly in our chapter on topological groups, Chapter 4. However, we already give some of the basic topological definitions now, specialised to the topology of pointwise convergence on \( \text{Sym}(A) \), which is the topology we will be working with in the following sections. The complement of a closed subset of \( \text{Sym}(A) \) will be called open. A basic open set is a subset of \( \text{Sym}(A) \) of the form \( S(a, b) := \{g \in \text{Sym}(A) : g(a) = b\} \) where \( a, b \in X^n \) for some \( n \geq 1 \). The set \( S(a, b) \) is open since \( \text{Sym}(A) \setminus S(a, b) = \{f \mid f(a) \neq b\} \) is clearly closed.

**Proposition 1.2.2.** The open subsets of \( \text{Sym}(A) \) are precisely the unions of basic open sets, and they define a topology on \( G \).

**Proof.** Let \( S \subseteq \text{Sym}(A) \). Then \( S \) is open if and only if \( C := \text{Sym}(A) \setminus S \) is closed:

\[
C = \{f \mid \forall n, a \in A^n \exists g \in C : f(a) = g(a)\}
\]

\[
\Leftrightarrow S = \{f \mid \exists n, a \in A^n \forall g \in C : f(a) \neq g(a)\}
\]

\[
\Leftrightarrow S = \bigcup_{a, b \in A^n \text{ s.t. } \forall g \in C : g(a) \neq b} S(a, b)
\]

and the latter is a union of basic open sets. Hence, by de Morgan, an intersection \( I \) of two open sets is a union of intersections of basic open sets. The intersection of two basic open \( S(a, b) \) and \( S(c, d) \) equals \( S((a, c), (b, d)) \), again a basic open set. Hence, \( I \) is open. An arbitrary union of open sets is again a union of basic open sets, again by
the first part of the statement, and hence open. Finally, the empty set and Sym(A) are clearly open.

**Theorem 1.2.3** (Corollary 4.1.5 in [58]). Let $G \leq \text{Sym}(N)$ be closed. Then the following are equivalent.

1. There is an $a \in N^n$, $n \in \mathbb{N}$ such that $|G_a| = 1$.
2. $|G| \leq \omega$.
3. $|G| < 2^\omega$.

**Sketch of proof.** The implications from (1) to (2) and from (2) to (3) are trivial. For the implication from (3) to (1), suppose that $\sim(1)$. Construct a binary branching tree with the levels indexed by $\mathbb{N}$; by the assumption that $G$ is closed in Sym($\mathbb{N}$) this can be done in such a way that the infinite branches of the tree correspond to permutations in $G$; we deduce that $\sim(3)$ since there are uncountably many infinite branches of the tree.

### 1.2.1. \textbf{Aut-sInv.} Recall that the automorphism group of a relational structure $A$, i.e., the set of all automorphisms of $A$, is denoted by $\text{Aut}(A)$. In the following it will be convenient to define the operator $\text{Aut}$ also on sets $R$ of relations over the same domain $A$, in which case $\text{Aut}(R)$ denotes the set of all permutations $\rho$ of $A$ such that $\rho$ and its inverse $\rho^{-1}$ preserve all relations form $R$.

For $P \subseteq \text{Sym}(A)$, and sets $R$ of relations over the domain $A$, we present a description of the closure operator $P \mapsto \text{Aut}(\text{sInv}(P))$; the closure operator $R \mapsto \text{sInv}(\text{Aut}(R))$ will be described in Section 3.1.

**Definition 1.2.4.** For $P \subseteq \text{Sym}(A)$, we define

- $\langle P \rangle$, the permutation group generated by $P$, to be the smallest permutation group on $A$ that contains $P$.
- $\overline{P}$, the closure of $P$ in $\text{Sym}(A)$, to be the smallest closed subset of $\text{Sym}(A)$ that contains $P$.

**Example 1.2.5.** Let $P$ be the set of permutations $f$ of $\mathbb{N}$ that have finite support, that is, the set $\{ i \in \mathbb{N} \mid f(i) \neq i \}$ is finite. Then $P \subseteq \overline{P} = \text{Sym}(\mathbb{N})$.

**Proposition 1.2.6.** Let $P \subseteq \text{Sym}(A)$ be arbitrary. Then $\text{Aut}(\text{sInv}(P)) = \overline{\langle P \rangle}$ equals the smallest permutation group that contains $P$ and is closed in $\text{Sym}(A)$.

**Proof.** Let $P'$ be the smallest permutation group that contains $P$ and is closed in $\text{Sym}(A)$. Since $P \subseteq P'$ and $P'$ is a permutation group, we must have $\langle P \rangle \subseteq P'$, and therefore also $\overline{P} \subseteq P'$ since $P'$ is closed in $\text{Sym}(A)$. To show the converse inclusion $P' \subseteq \overline{P}$, it suffices to verify that $\overline{P}$ is a closed subgroup of $\text{Sym}(A)$. Since $\overline{P}$ is clearly closed in $\text{Sym}(A)$ we only have to show that $\overline{P}$ contains compositions and inverses. We do the verification for closure under compositions on finite subsets $F$ of $A$. Indeed, when $f, g \in \overline{P}$, then there are $f', g' \in \langle P \rangle$ such that $f(x) = f'(x)$ for all $x \in F$ and $g(x) = g'(x)$ for all $x \in f(F)$. We therefore have $f(g(x)) = f'(g'(x))$ for all $x \in F$, and hence $f \circ g \in \overline{P}$, as desired.

We now show that $\overline{\langle P \rangle} \subseteq \text{Aut}(\text{sInv}(P))$. Let $p \in \overline{\langle P \rangle}$ be arbitrary, and let $R$ be from $\text{sInv}(P)$. We have to show that $p$ and $p^{-1}$ preserve $R$. Let $t \in R$; we have that $p(t) = q_1 \circ \cdots \circ q_k(t)$ for some permutations $q_1, \ldots, q_k \in P \cup P^{-1}$. Since $q_1, \ldots, q_k$ preserve $R$, we have that $q(t) \in R$. The argument for $p^{-1}$ is analogous.

Finally, we show $\text{Aut}(\text{sInv}(P)) \subseteq \overline{\langle P \rangle}$. Let $p$ be from $\text{Aut}(\text{sInv}(P))$. It suffices to show that for every finite subset $\{a_1, \ldots, a_n\}$ of $A$ there is a $q \in \langle P \rangle$ such that $p(a_i) = q(a_i)$ for all $i \leq n$. Consider the relation $\{(q(a_1), \ldots, q(a_n)) \mid q \in \langle P \rangle \}$. It is preserved by all permutations in $P$. Therefore, $p$ preserves this relation, and so there exists $q \in \langle P \rangle$ as required.
Exercises.

(4) Let $G$ be the permutation group on $\mathbb{Z}$ that is generated by the transpositions $\tau_i := (i, -i)$, for $i \in \mathbb{Z}$. What is the cardinality of $G$, and what is the cardinality of $G^*$?

(5) The finitary alternating group $A$ on $\mathbb{N}$ is the set of all permutations of $\mathbb{N}$ that can be written as a composition of an even number of transpositions. Determine $A$.

(6) Let $P = \{f, g\} \subseteq \text{Sym}(\mathbb{Z})$ where $f$ is a transposition and $g$ is $x \mapsto x + 1$. Determine the cardinalities of $\langle P \rangle$, $P$, and $\langle P \rangle$.

1.3. Group Actions

We now consider abstract groups, that is, algebraic structures $G$ over a set $G$ of group elements, with a function symbol for multiplication of group elements, a function for the inverse of a group element, and the constant for the identity. The link to permutation groups is given by the concept of an action of such a group on a set, which is described below.

**Definition 1.3.1.** Let $G$ be a group and $X$ a set. An action of $G$ on $X$ is a homomorphism $\phi$ from $G$ into $\text{Sym}(X)$. An action $\phi$ is called faithful if $\phi$ is injective.

Clearly, to every action of $G$ on $X$ we can associate a permutation group as considered before, namely the image of the action in $\text{Sym}(X)$. Conversely, to every permutation group $G$ on a set $X$ we can associate an abstract group $G$ whose domain is $G$ (the permutations), where composition and inverse are defined in the obvious way, and which acts on $X$ faithfully by $\phi(g)(x) := g(x)$.

We give an alternative characterisation of action which in many texts is taken to be the official definition.

**Proposition 1.3.2.** Let $G$ be a group and $X$ a set. The $\phi: G \to \text{Sym}(X)$ is an action of $G$ on $X$ if and only if the map $\cdot : G \times X \to X$ defined by $g \cdot x := \phi(g)(x)$ satisfies

- $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in S$, and
- $e \cdot x = x$ for every $x \in X$.

The action $\phi$ is faithful if and only if for any two distinct $g, h \in G$ there exists an $x \in X$ such that $g \cdot x \neq h \cdot x$.

**Proof.** The proof is just moving symbols. \qed

When $x \in X$, the orbit of $x$ with respect to an action of $G$ on $X$ is the set $\{g \cdot x \mid g \in G\}$. Hence, an orbit of $k$-tuples in the corresponding permutation group on $X$ is an orbit of the action of $G$ on $X^k$ that is defined componentwise, that is, $g$ maps $(x_1, \ldots, x_k)$ to $(gx_1, \ldots, gx_k)$. In this way we can also use other terminology introduced for permutation groups (such as a transitivity, congruences, primitivity, etc.) for group actions.

Cayley’s theorem states that every group has a representation as a permutation group.

**Theorem 1.3.3 (Cayley’s theorem).** Let $G$ be any group. Then $G$ has a faithful action on $G$.

**Proof.** The action $\xi$ on $G$ is by left translation: for $g \in G$, we define $\xi(g)$ by $\xi(g)(h) := gh$ for all $h \in G$. It is straightforward to verify that this map is an injective group homomorphism. \qed
We close this section with some important examples of group actions.

Example 1.3.4 (Action by left translation). A left coset of a subgroup \(H\) of \(G\) is a set of the form \(\{gh \mid h \in H\}\) for \(g \in G\), also written \(gH\). Clearly, the set of all left cosets of \(H\) partitions \(G\), and is denoted by \(G/H\). The cardinality of \(G/H\) is called the index of \(H\) in \(G\). We define an action \(\xi\) of \(G\) on \(G/H\) by setting \(\xi(f)(gH) := (fg)H\). This action is also called the action of \(G\) on \(G/H\) by left translation. It is transitive since for any \(g_1H, g_2H \in G/H\) the map \(\xi(g_2g_1^{-1})\) takes \(g_1H\) to \(g_2H\). Note that this example generalises the construction in the proof of Cayley’s theorem since we may take \(H = \{1\}\).

Example 1.3.5 (Action by conjugation). The map \(\xi : G \to G\) given by \(\xi(g)(h) := ghg^{-1}\) defines a group action:

\[
\xi(g_1g_2)(h) = (g_1g_2)h(g_1g_2)^{-1} = g_1(g_2hg_2^{-1})g_2^{-1} = \xi(g_1)(\xi(g_2)h)
\]

called the action of \(G\) on \(G\) by conjugation. The orbits of this action are called the conjugacy classes, and the stabiliser of \(h \in G\) with respect to this action is the centraliser \(C_G(h)\) of \(h\):

\[
C_G(h) = \{g \in G \mid gh = hg\}.
\]

Example 1.3.6 (The componentwise action on \(A^n\)). If \(G\) is a permutation group on a set \(X\), then the componentwise action of \(G\) on \(X^n\) is given by

\[
\xi(g)(x_1, \ldots, x_n) := (g(x_1), \ldots, g(x_n)).
\]

Example 1.3.7 (The setwise action on \(\binom{\ell}{k}\)). If \(G\) is a permutation group on a set \(X\), then the setwise action of \(G\) on \(\binom{\ell}{k}\) is given by

\[
\xi(g)(\{x_1, \ldots, x_n\}) := \{g(x_1), \ldots, g(x_n)\}.
\]

In the following we review the classical theory how permutation groups can be build from simpler permutation groups forming various forms of products. The direct product of a sequence of groups \((G_i)_{i \in I}\) is the product of this sequence as defined in general in Section 1.1.4, note that the product is again a group. Products appear in several ways when studying permutation groups; the first is when we want to describe the relation between a permutation group and its ‘transitive constituents’, described in the following.

1.3.1. The intransitive action of the direct product. When \(G\) acts on a set \(X\) and \(S \subset X\) is an orbit with respect to this action, then \(G\) naturally acts transitively on \(S\) by restriction; we call the corresponding group \(H\) the group induced by \(S\), or a transitive constituent.

Proposition 1.3.8 (see §33). Let \(G\) be a group acting on a set \(X\), and let \((G_i)_{i \in I}\) be the groups induced by the orbits of \(G\) on \(X\). Then \(G\) is isomorphic to a subgroup of \(\prod_{i \in I} G_i\), and there are surjective homomorphisms from \(G\) to \(G_i\), for each \(i\).

Definition 1.3.9. Let \(G_1\) and \(G_2\) be groups acting on disjoint sets \(X\) and \(Y\), respectively. Then the action of \(G_1 \times G_2\) on \(X \cup Y\) defined by \((g_1, g_2) \cdot z = g_1z\) if \(z \in X\), and \(g_2z\) if \(y \in Y\), is called the natural intransitive action of \(G_1 \times G_2\) on \(X \cup Y\).

\(^{1}\)We say that an action \(\xi : G \to \text{Sym}(S)\) is transitive if the permutation group \(\xi(G) \leq \text{Sym}(S)\) is transitive.
When $G_1$, and $G_2$ are the automorphism groups of $\omega$-categorical relational structures $A$ and $B$ with disjoint domains $A$ and $B$, respectively, then the image of the natural intransitive action on $A \cup B$ (as a homomorphism from $G_1 \times G_2$ to $\text{Sym}(A \cup B)$) can also be described as the automorphism group of a relational structure $C$, we can take for $C$ the disjoint union of $A$ and $B$, expanded by a unary predicate that contains exactly the elements of $A$. Since reducts of $\omega$-categorical structures are again $\omega$-categorical, this shows in particular that the disjoint union of two $\omega$-categorical structures is again $\omega$-categorical.

1.3.2. The product action. When $G_1$ is a group acting on a set $X$, and $G_2$ a group acting on a set $Y$, there is another important natural action of $G := G_1 \times G_2$ besides the intransitive natural action of $G$, which is called the product action of $G$. In this action, $G$ acts on $X \times Y$ by $(g_1, g_2) \cdot (x, y) = (g_1x, g_2y)$. If the actions of $G_1$ and $G_2$ are transitive, then the product action is clearly transitive, too. We claim that when the actions of $G_1$ and $G_2$ are oligomorphic, then the product action is also oligomorphic. Let $F_1(n)$ and $F_2(n)$ then the number of orbits of the componentwise action of $G_1$ on $X^n$ and $Y^n$, respectively. Then the number of orbits of the componentwise action of $G$ on $X \times Y$ is $F_1(n)F_2(n)$, and in particular finite, which proves the claim.

When $G_1$ and $G_2$ are the automorphism groups of $\omega$-categorical structures $A$ and $B$, then the image of the product action of $G$ in $\text{Sym}(A \times B)$ is the automorphism group of the following structure, which we call the full product structure of two relational structures $A$ and $B$, and denote by $A \boxtimes B$. Let $\sigma$ be the signature of $A$, and $\tau$ be the signature of $B$; we assume that $\sigma$ and $\tau$ are disjoint, otherwise we rename the relations so that the assumption is satisfied. For each $k$-ary $R \in \sigma$, the structure $A \boxtimes B$ contains the relation $\{((a_1, b_1), \ldots, (a_k, b_k)) \mid (a_1, \ldots, a_k) \in R^A, (b_1, \ldots, b_k) \in R^B\}$, and for each $k$-ary $R \in \tau$, it contains the relation $\{((a_1, b_1), \ldots, (a_k, b_k)) \mid (b_1, \ldots, b_k) \in R^A, (a_1, \ldots, a_k) \in R^B\}$. Finally, we also add the relations $P_1 = \{((a_1, b_1), (a_2, b_2)) \mid a_1 = a_2\}$ and $P_2 = \{((a_1, b_1), (a_2, b_2)) \mid b_1 = b_2\}$ to $A \boxtimes B$.

**Proposition 1.3.10.** The automorphism group of $C := A \boxtimes B$ is $G_1 \times G_2$ in its product action on $A \times B$.

**Proof.** Let $h$ be the product action of $G = G_1 \times G_2$ on $A \times B$, viewed as a homomorphism from $G$ to $\text{Sym}(A \times B)$. Let $(g_1, g_2)$ be an element of $G$. Then $h((g_1, g_2))$ is the permutation $(x, y) \mapsto (g_1x, g_2y)$ of $A \times B$, and this map preserves $C$: when $((a_1, b_1), \ldots, (a_k, b_k)) \in R^A$, for $R \in \sigma$, then $(a_1, \ldots, a_k) \in R^A$, and so $(g_1a_1, \ldots, g_1a_k) \in R^A$. Therefore, $((g_1a_1, g_2b_1), \ldots, (g_1a_k, g_2b_k)) \in R^B$. The proof for the relation symbols $R \in \tau$ is analogous.

We now show that conversely, every automorphism $g$ of $C$ is in the image of $h$. Let $a_0 \in A, b_0 \in B$. Let $g_1$ be the permutation of $A$ that maps $a \in A$ to the point $a'$ such that $h((a, b_0)) = (a', b_0)$. Similarly, let $g_2$ be the permutation of $B$ that maps $b \in B$ to the point $b'$ such that $h((a_0, b)) = (a', b')$. Since $g$ preserves $P_1, P_2$, the definition of $g_1$ and $g_2$ does not depend on the choice of $a_0$ and $b_0$. Moreover, $g_1$ is from $G_1$, since $g$ preserves the relations for the symbols from $\sigma$. Similarly, $g_2$ is from $G_2$. Then $g' := h((g_1, g_2))$ equals $g$, since $g'(a, b) = (g_1a, g_2b) = g(a, b)$. Hence, $g$ is a permutation of $A \times B$ that lies in the image of $h$. □

Note that Proposition 1.3.10 becomes false in general when we omit the relations $P_1$ and $P_2$ in $A \boxtimes B$. Consider for example the structure without structure $B$ (that is, $B$ has empty signature). Then the automorphism group of $B \boxtimes B$ is imprimitive, but without the relations $P_1$ and $P_2$, the structure is isomorphic to $B$ and hence primitive.
Also note that when $A$ and $B$ are ordered structures, we could omit $P_1$ and $P_2$ in the definition of the full product without sacrificing Proposition 1.3.10, since $P_1(x,y)$ is definable from the order $<$ of $A$ by the formula $\neg (x < y) \land \neg (y < x)$, and similarly $P_2$ is definable from the order of $B$.

Finally we remark that $(A \boxtimes B) \boxtimes C$ and $A \boxtimes (B \boxtimes C)$ have the same automorphism group (on the domain $A \times B \times C$). We explicitly define the $d$-fold full product as follows.

**Definition 1.3.11** (Full product of $d$ structures). Let $B_1, \ldots, B_d$ be structures with disjoint relational signatures $\tau_1, \ldots, \tau_d$. We denote by $\boxtimes_{i=1}^d B_i$ the structure with domain $B := B_1 \times \cdots \times B_d$ that contains for every $i \leq d$, and every $m$-ary $R \in (\tau_i \cup \{=\})$ an $m$-ary relation defined by

$$\{(x^1_i, \ldots, x^d_i), (x^1_m, \ldots, x^m_m) \mid (x^1_i, \ldots, x^i_i) \in R^B_i\}.$$ If $B := B_1 = \cdots = B_d$, then we first rename $R \in \tau_i$ into $R_i$ so that the factors have pairwise disjoint signatures, and then write $B^d$ for $\boxtimes_{i=1}^d B_i$.

**1.3.3. Semidirect products.** Semidirect products can be seen either as a way to construct new groups from simpler ones (Section 1.3.3.2), or, equivalently, as a tool to decompose a given group into simpler constituents (Section 1.3.3.3). We first introduce some fundamental concepts for (abstract) groups.

1.3.3.1. **Normal subgroups.** A subgroup $N$ of $G$ with domain $N$ is called normal if $gN = Ng$ for all elements $g$ of $G$. In this section we give alternative descriptions of closed, and of open normal subgroups. Recall the following equivalent characterisations of normality of subgroups.

**Proposition 1.3.12.** Let $G$ be a group, and $N$ be a subgroup of $G$. Then the following are equivalent.

1. $N$ is normal.
2. $G/N$ has the congruence $E = \{(a, b) \mid ab^{-1}N \in N\}$.
3. There is a homomorphism $h$ from $G$ to some group such that $N = h^{-1}(0)$.
4. For every $g \in G$ and every $v \in N$ we have $gvg^{-1} \in N$.

**Proof.** (1) $\Rightarrow$ (2): to verify that $E$ is a congruence, we have to show that for all $(a_1, b_1), (a_2, b_2) \in E$, $(a_1a_2)(b_1b_2)^{-1} = a_1(a_2b_2^{-1})b_1^{-1} \in a_1Nb_1^{-1} = Na_1b_1^{-1} \subseteq NN = N$.

(2) $\Rightarrow$ (3): $g \mapsto gN$ is a group homomorphism from $G$ to $G/N$.

(3) $\Rightarrow$ (4): For $g \in G$ and $v \in h^{-1}(0)$, we must show that $gvg^{-1} \in h^{-1}(0)$.

Indeed, $h(gvg^{-1}) = h(g)h(v)h(g)^{-1} = h(g)0h(g)^{-1} = 0$.

(4) $\Rightarrow$ (1): assume that $gNg^{-1} \subseteq N$ for all $g \in G$. Let $a \in G$ be arbitrary. Applying the assumption for $g = a$ we find that $aN \subseteq Na$. Applying the assumption for $g = a^{-1}$ we find that $a^{-1}N(a^{-1})^{-1} = a^{-1}Na \subseteq N$, and hence $Na \subseteq aN$. We conclude that $aN = Na$.

**Exercises.**

7. (Exercise 1 page 9 of [33]) For $H$ a subgroup of $G$, the kernel of the action of $G$ on the coset space of $H$ is $\bigcap_{g \in G} g^{-1}Hg$. Moreover, this group is the largest normal subgroup of $G$ which is contained in $H$.

1.3.3.2. **The outer semidirect product.** Let $H$ and $N$ be groups and let $\theta : H \to \operatorname{Aut}(N)$ be a homomorphism.

**Definition 1.3.13.** The semidirect product of $N$ by $H$ with respect to $\theta$, denoted by $N \rtimes H$ (or $H \ltimes N$), is the group $G$ with the elements $N \times H$ and group multiplication defined by

$$(u, x)(v, y) := (u\theta(x)(v), xy).$$
indeed associative:

\[ \text{for all } (u, x), (v, y) \in G. \]

The definition contains some claims that we still have to verify. Multiplication is indeed associative:

\[
((u, x)(v, y))(w, z) = (u\theta(x)(v), xy)(w, z) = (u\theta(x)v\theta(y)(w), (xy)z) = (u\theta(x)v\theta(y)(w), x(yz)) \quad \text{(since } \theta \text{ is a homomorphism)}
\]

\[
(u\theta(x)v\theta(y)(w), x(yz)) = (u, x)(v\theta(y)(w), yz) = (u, x)((v, y)(w, z)).
\]

In the following, we write \( x(v) \) instead of \( \theta(x)(v) \) for better readability. Clearly, \((1, 1)\) is a neutral element, and the inverse of \((u, x)\) is \((x^{-1}(u^{-1}), x^{-1})\):

\[
(u, x)(x^{-1}(u^{-1}), x^{-1}) = (ux(x^{-1}(u^{-1})), xx^{-1}) = (uu^{-1}, 1) = (1, 1)
\]

Note that \( H^* := \{(1, x) \mid x \in H\} \) is a subgroup of \( G \) that is isomorphic to \( H \), and that \( N^* := \{(u, 1) \mid u \in N\} \) is a subgroup of \( G \) isomorphic to \( N \). The next proposition collects some further important properties of semidirect products.

**Proposition 1.3.14.** Let \( G = N \rtimes H \). Then

- \( N^* \triangleleft G \),

- \( G = N^*H^* \), and

- \( N^* \cap H^* = \{(1, 1)\} \).

**Proof.** \((u, x) \in N^* \) and \((v, y) \in G \) we have

\[
(u, x)(v, 1)(x^{-1}(u^{-1}), x^{-1}) = (ux(v), x(x^{-1}(u^{-1}), x^{-1})
\]

\[
= (ux(v)x(x^{-1}(u^{-1})), xx^{-1}) = (ux(v)u^{-1}, 1) \in N
\]

To see that \( G = N^*H^* \) it suffices to observe that \((u, x)\) can be written as \((u, 1)(1, x)\), and obviously \( N^* \cap H^* = \{(1, 1)\}\).

Finally, we mention that the action of \( H^* \) on \( N^* \) by conjugation in \( G \) reflects the original action of \( H \) on \( N \), that is,

\[
(1, x)(u, 1)(1, x)^{-1} = (x(u), x(x^{-1}(1), x^{-1}) = (x(u)x(x^{-1}(1)), xx^{-1}) = (x(u), 1).
\]

Usually, \( H^* \) and \( N^* \) are identified with \( G \) and \( H \), so we consider \( G \) and \( H \) as subgroups of the semidirect product \( N \rtimes H \).

**1.3.3. Inner direct product.** This section provides a characterisation of those groups \( G \) that can be obtained as a semidirect product of two subgroups of \( G \). A sequence of groups \( N, G, H \) together with maps \( \alpha: A \to B \) and \( \beta: B \to C \) is called a short exact sequence, and written

\[
1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1
\]

if \( \alpha \) is injective, \( \beta \) is surjective, and the kernel of \( \beta \) equals the image of \( \alpha \). Hence, \( N \) can be considered as a normal subgroup of \( G \) and \( H \) is isomorphic to \( G/N \).

**Proposition 1.3.15.** Let \( G \) be a group, \( N \triangleleft G \), and \( H \leq G \). Then the following are equivalent.
(1) \( H \) is a complement for \( N \) in \( G \), i.e., \( G = NH := \{nh \mid n \in N, h \in H\} \) and \( N \cap H = \{1_G\} \).

(2) For every \( g \in G \) there exists a unique \( n \in N \) and \( h \in H \) such that \( g = nh \).

(3) There is a homomorphism \( \mu: G \rightarrow H \) that fixes \( H \) pointwise and whose kernel is \( N \).

(4) The restriction of the factor map \( \sigma: G \rightarrow G/N \) to \( H \) is an isomorphism between \( H \) and \( G/N \).

(5) There exists a short exact sequence

\[
1 \rightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \rightarrow 1
\]

that splits, i.e., there is a homomorphism \( \rho: H \rightarrow G \) such that \( \beta \circ \rho = \text{id}_H \).

(6) \( G \) is isomorphic to the semidirect product \( H \rtimes H \) with respect to the action of \( H \) on \( N \) by conjugation in \( G \).

**Proof.** (1) implies (2): Suppose that \( n_1, n_2 \in N \) and \( h_1, h_2 \) are such that \( n_1h_1 = n_2h_2 \). Then in particular, \( n_1H = n_2H \), which is the case if and only if \( n_2^{-1}n_1 \in N \cap H = \{1\} \), and hence \( n_1 = n_2 \). Similarly, we deduce that \( h_1 = h_2 \).

(2) implies (3). Let \( \mu: G \rightarrow H \) be the function that maps \( g \in G \) to the unique \( h \in H \) such that \( g = nh \) for some \( n \in N \). Then \( \mu \) is a homomorphism: if \( g_1 = n_1h_1 \) and \( g_2 = n_2h_2 \)

\[
\mu(g_1g_2) = \mu(n_1h_1n_2h_2)
= \mu(n_1h_1n_2^{-1}h_1^{-1}h_2)
= h_1h_2 \quad \text{(since } h_1n_2h_2^{-1} \in N) \\
= \mu(n_1h_1)\mu(n_2h_2)
= \mu(g_1)\mu(g_2).
\]

Then \( \mu^{-1}(1) = N \) and for any \( h \in H \) we have \( \mu(h) = h \).

(3) implies (4): The restriction of \( \sigma \) to \( H \) is a homomorphism from \( H \) to \( G/N \).
It is injective since for \( u \in H \) we have \( \sigma(u) = 1_{G/N} \) if and only if \( u \in N \) if and only if \( \mu(u) = 1_H \) if and only if \( u = 1_H \). It is surjective since for every \( [g]_N \in G/N \) we have that \( \mu(g) = \mu([g]) = \mu([g])_N = \sigma(\mu(g)) \).

(4) implies (5): let \( \tau: H \rightarrow G/N \) be the restriction of the factor map \( \sigma \) which is an isomorphism by (4). Then \( \beta := \tau^{-1}\sigma: G \rightarrow H \) is a surjective homomorphism whose kernel is \( N \). Choose \( \rho: H \rightarrow G \) to be the inclusion map we obtain \( \beta \circ \rho = \tau^{-1}\sigma \circ \text{id}_H \).

(5) implies (6): we may assume that \( \alpha \) and \( \rho \) are the inclusion maps. Define \( \theta: H \rightarrow \text{Aut}(N) \) as \( n \mapsto hnh^{-1} \). We claim that \( N \rtimes H \) with respect to \( \theta \) is isomorphic to \( G \), the isomorphism \( \xi \) being \( (n, h) \mapsto nh \). We verify that \( \xi \) is a homomorphism:

\[
\xi((n_1, h_1)(n_2, h_2)) = \xi(n_1h_1n_2^{-1}h_1^{-1}, h_1h_2)
= n_1h_1n_2h_2
= n_1h_1n_2
= \xi(n_1, h_1)\xi(n_2, h_2)
\]

The homomorphism \( \xi \) is injective: if \( \xi(n_1, h_1) = n_1h_1 = 1 \) then

\[
\beta(n_1h_1) = \beta(n_1)\beta(h_1) = 1 \cdot \beta(h_1) = 1,
\]

and hence \( h_1 = 1 \) since \( \beta \) is injective. Since \( n_1h_1 = 1 \) this implies that \( n_1 = 1 \), too. To show that \( \xi \) is surjective, TODO.

(6) implies (1): this is Proposition \[1.3.14\]
If the equivalent conditions in Proposition 1.3.15 apply, then $G$ is called a \textit{split extension} of $N$ by $H$. Here is an example of a short exact sequence that does not split.

\textbf{Example 1.3.16.} Let $G := \mathbb{Z}_6$. Then $h: G \rightarrow \mathbb{Z}_2$ given by $h(g) := g \mod 2$ is a surjective homomorphism, and there is an isomorphism $i$ between $\mathbb{Z}_3$ and the kernel $N$ of $h$. We then have the short exact sequence

$$1 \longrightarrow N \overset{i}{\longrightarrow} G \overset{h}{\longrightarrow} \mathbb{Z}_2 \longrightarrow 1.$$ 

However, there is no homomorphism $r: \mathbb{Z}_2 \rightarrow \mathbb{Z}_6$ such that $h \circ r = \text{id}_H$ since any non-constant homomorphism $s: \mathbb{Z}_2 \rightarrow \mathbb{Z}_6$ would have to map 1 to 3 since $s(1) + s(1) = 0$ implies that $s(1) = 3$, but then $h \circ s(1) = h(3) = 0 \neq 1$. So the sequence does not split, and the equivalent conditions from Proposition 1.3.15 do not apply.

\section*{1.4. Congruences and Primitivity}

Recall from Section 1.1.5 that a \textit{congruence} of a permutation group $G$ on a set $D$ is an equivalence relation on $D$ that is preserved by all permutations in $G$.

\textbf{Definition 1.4.1.} Let $G$ be a permutation group on a set $D$. A subset $S$ of $D$ is called a block of $G$ if $g(S) = S$ or $g(S) \cap S = \emptyset$ for all $g \in G$.

\textbf{Lemma 1.4.2.} Let $G$ be a permutation group on a set $D$. Then $S \subseteq D$ is a block of $G$ if and only if $S$ is an equivalence class of a congruence of $G$.

\textbf{Proof.} Suppose that $S$ is an equivalence class of the congruence $\mathcal{C}$, and suppose that $g(S) \cap S$ contains an element $t$. That is, there is an element $s \in S$ such that $g(s) = t \in S$. We will show that $g(S) = S$. Let $r \in S$ be arbitrary. Then $(r,s) \in \mathcal{C}$, and hence $(g(r),g(s)) = (g(r),t) \in \mathcal{C}$. Since $t \in S$, it follows that $g(r) \in S$. Hence, $g(S) \subseteq S$. But also $(r,t) \in \mathcal{C}$, and $(g^{-1}(r),g^{-1}(t)) = (g^{-1}(r),s) \in \mathcal{C}$. Since $s \in S$, it follows that $g^{-1}(r) \in S$ and $r \in g(S)$. Hence, $S \subseteq g(S)$.

Now suppose that $S \subseteq D$ is a block of $G$. Define $C := \{(x,y) \mid \exists g \in G : g(x),g(y) \in S\} \cup \{(x,x) \mid x \in D\}$. This relation is clearly reflexive, symmetric, and preserved by $G$. In order to prove that $C$ is a congruence it remains to verify transitivity. Let $(x,y),(y,z) \in C$. Then there are $g_1,g_2 \in G$ such that $g_1(x),g_1(y),g_2(x),g_2(y) \in S$. Thus, $S \cap g_2(g_1^{-1}(S))$ contains $g_2(y)$, and $(g_2 \circ g_1^{-1})(S) = S$ because $S$ is a block. Since $g_1(x) \in S$ this implies that $g_2(g_1^{-1}(g_1(x))) \in S$. So $g_2(x) \in S$ and $g_2(z) \in S$, and $(x,z) \in C$.

A congruence is \textit{trivial} if each block contains only one element (and \textit{non-trivial} otherwise), and it is called \textit{proper} if it is distinct from the equivalence relation that has only one block.

\textbf{Definition 1.4.3.} A permutation group $G$ is called \textit{primitive} if $G$ is transitive and every proper congruence of $\mathcal{G}$ is trivial, and \textit{imprimitive} otherwise.

Clearly, 2-transitive structures are always primitive. An \textit{orbital} is an orbit of pairs, that is, a set of the form $\{(aa,ab) \mid a \in \mathcal{G}\}$ for $a,b \in B$. The \textit{trivial orbital} is the orbital $\{(a,a) \mid a \in B\}$. When $O$ is an orbital, the \textit{orbital graph} is the directed graph with vertex set $B$ and edges $O$.

\textbf{Theorem 1.4.4 (Higman’s theorem; see e.g. [34]).} A transitive permutation group $G$ is primitive if and only if the graph of all non-trivial orbitals is connected.

Yet another perspective on congruences, blocks and primitivity of transitive permutation groups $G$ comes from the \textit{point stabiliser} $G_a := \{a \in G \mid \alpha(a) = a\}$. The following is Theorem 1.5A in [42].
Theorem 1.4.5. Let $G$ be a group which acts transitively on a set $D$, and let $a \in D$. Let $B$ be the set of all blocks $B$ of $G$ that contain $a$, and let $\mathcal{S}$ be the set of all subgroups $H$ of $G$ that contain $\mathcal{G}_a$. Then there is a bijection $\mu$ between $B$ and $\mathcal{S}$ given by $\mu(B) := G_B$; the inverse mapping is given by $\mu^{-1}(H) = \{g(a) \mid g \in H\}$.

Note that the mapping $\mu$ is order-preserving in the sense that if $B_1, B_2 \in B$ then $B_1 \subseteq B_2 \iff \mu(B_1) \subseteq \mu(B_2)$.

Corollary 1.4.6. Let $G$ be a group acting transitively on a set $D$ with at least two elements. Then $G$ is primitive if and only if each point stabiliser $\mathcal{G}_a$ is a maximal subgroup of $G$.

Proof. The statement follows from Theorem 1.4.5. For a self-contained proof, suppose that $\mathcal{G}_a$ is not maximal, i.e., there exists a proper subgroup $H$ of $G$ that properly contains $\mathcal{G}_a$. Then $B := H(a) := \{h(a) \mid h \in H\}$ is a block of $G$. To see this, let $g \in G$. Suppose that $g(B) \cap B \neq \emptyset$. Then there exist $h_1, h_2 \in H$ such that $g(h_1(a)) = h_2(a)$. Hence, $h_2^{-1} \circ g \circ h_1 \in \mathcal{G}_a$, so $g \in h_2\mathcal{G}_a h_1^{-1} \subseteq H$. But $g(B) = B$ for all $g \in H$, showing that $B$ is a block. Next, observe that $G_B = H$ since for all $g \in G$ we have $g(B) = B$ if and only if $g \in H$, as we have seen in the previous paragraph. If $B = D$, then $G_B = G$, contrary to the assumption that $H = G_B$ is a proper subgroup of $G$. If $B = \{a\}$, then $G_B = \mathcal{G}_a$, contrary to the assumption that $H = G_B$ properly contains $\mathcal{G}_a$. Hence, $G$ must have a non-trivial proper congruence by Lemma 1.4.2, and hence is not primitive.

Conversely, let $B$ be a block of $G$ that strictly contains $a$ and is contained strictly in $D$. Clearly, $G_B$ is a subgroup of $G$ that contains $\mathcal{G}_a$. Also note that $\{g(a) \in G \mid g \in G_B\} = B$. Hence, if $G_B = \mathcal{G}_a$ then $B = \{a\}$, contrary to the assumptions on $B$. If $G_B = G$ then transitivity of $G$ implies that $B = D$, contrary to the assumptions on $B$. So $\mathcal{G}_a$ is in this case not a maximal subgroup of $G$. \qed
CHAPTER 2

Counting Orbits

A permutation group $G$ on a set $A$ is

- **$k$-transitive** if for $s, t \in A^k$ with pairwise distinct entries there is an $g \in G$ such that $g(s) = t$; (recall: the action of $G$ on tuples is componentwise, i.e., $g(s_1, \ldots, s_k) := (g(s_1), \ldots, g(s_k))$)
- **transitive** if it is 1-transitive;
- **$k$-set transitive** if for all $S, T \subseteq A$ of cardinality $k$ there is a $g \in G$ such that $g(S) = \{g(s) | s \in S\} = T$.
- **highly transitive** if it is $k$-transitive for all $k \geq 1$.
- **highly set-transitive** if it is $k$-set transitive for all $k \geq 1$.

An example of a highly transitive structure is $\text{Sym}(\mathbb{N})$, which is of course also highly set-transitive. An example of a highly set-transitive but not highly transitive structure is $\text{Sym}(\mathbb{Q}; <)$; see Section 3.2.

It is easy to see that a 2-set transitive permutation group $G$ on an infinite set is also transitive. We prove the contraposition: assume that $G$ has more than one orbit. There must be an orbit $O$ with two distinct elements $c_1, c_2$. Let $c_3$ be an element not from $O$. Then there is no automorphism that maps $\{c_1, c_2\}$ to $\{c_1, c_3\}$, and hence $G$ is not 2-set transitive. More generally, the following holds. For $B \subset A$, the orbit of $B$ under $G$ is the set $\{g(B) | g \in G\}$ (a set of subsets of $A$).

**Proposition 2.0.1.** Let $G$ be a permutation group. The number $f(n)$ of orbits of $n$-subsets forms a non-decreasing sequence.

We will show this proposition in Section 2.2. Being highly set-transitive is equivalent to $f(n) = 1$ for all $n \in \mathbb{N}$. There is another important sequence attached to a permutation group.

**Definition 2.0.2.** Let $G$ be a permutation group. Then $f^*(n)$ denotes the number of orbits of the componentwise action on $n$-tuples with pairwise distinct entries.

So, $G$ is highly transitive if $f^*(n) = 1$ for all $n \in \mathbb{N}$. Note that

$$f(n) \leq f^*(n) \leq n! f(n)$$

since there are $n!$ different orderings of $n$ elements. These two sequences correspond to two different counting paradigms in combinatorics: labelled and unlabelled enumeration.

**Exercises.**

1. (8) Prove that $(k+1)$-transitivity implies $k$-transitivity, for all $k \geq 1$.
2. (9) Show that $f^*(n) \leq f^*(n+1)$.
3. (10) Show that if there exists a $k$ such that $f^*(k) = f^*(k+1)$, then $f^*(k+1) = 1$.
4. (11) Show that a permutation group $G$ on a set $A$ is highly transitive if and only if $\overline{G} = \text{Sym}(A) = \text{Aut}(A; =)$.
5. (12) Let $(A; E)$ be a countably infinite structure where $E$ denotes an equivalence relation with infinitely many infinite classes. Describe the automorphism group $\text{Aut}(A; E)$. How many orbits of $n$-subsets are there?
(13) (Exercise 3 on page 57 in [33]) Let \((A; E_2)\) be a countably infinite structure where \(E_2\) denotes an equivalence relation with infinitely many classes of size two, and let \((A; E^2)\) be a structure where \(E^2\) denotes an equivalence relation with two infinite classes. Show that \(\text{Aut}(A; E_2)\) and \(\text{Aut}(A; E^2)\) have the same number of orbits of \(n\)-subsets, for all \(n\).

2.1. Combinatorial Tools

In order to prove Proposition 2.0.1, we need a couple of combinatorial tools.

2.1.1. The Pigeon-hole Principle. If \(n\) pigeons fly to less than \(n\) holes, there must be one hole that got more than one pigeon. There is an important infinite version of the statement: if infinitely many pigeons fly to finitely many holes, one hole must have gotten infinitely many pigeons. This will be used in the next tools that we present.

2.1.2. König’s Tree Lemma. A walk in a graph \((V, E)\) (see Example 1.1.1) is a sequence \(x_0, x_1, \ldots, x_n \in V\) with the property that \((x_i, x_{i+1}) \in E\) for all \(i \in \{1, \ldots, n-1\}\). A walk is a path if all its vertices are distinct. A cycle is a walk of length at least three of the form \(x_0, x_1, \ldots, x_n = x_0\) such that \(x_1, \ldots, x_n\) are pairwise distinct. A tree is a connected graph \((V, E)\) (see Section 1.1.4) without cycles. The degree of a vertex \(u \in V\) is the number of vertices \(v \in V\) such that \(\{u, v\} \in E\).

Lemma 2.1.1 (König’s Tree Lemma). Let \((V, E)\) be a tree such that every vertex in \(V\) has finite degree, and let \(v_0 \in V\). If there are arbitrarily long paths that start in \(v_0\), then there is an infinitely long path that starts in \(v_0\).

Proof. Since the degree of \(v_0\) is finite, there exists a neighbour \(v_1\) of \(v_0\) such that arbitrarily long paths start in \(v_0\) and continue in \(v_1\) (by the infinite pigeon-hole principle). We now construct the infinitely long path by induction. Suppose we have already found a sequence \(v_0, v_1, \ldots, v_i\) that can be continued to arbitrarily long paths in \((V, E)\). Since the degree of \(v_i\) is finite, \(v_i\) must have a neighbour \(v_{i+1}\) in \(V \setminus \{v_0, v_1, \ldots, v_i\}\) such that \(v_0, v_1, \ldots, v_{i+1}\) can be continued to arbitrarily long paths in \((V, E)\). In this way, we define an infinitely long path \(v_0, v_1, v_2, \ldots\) in \((V, E)\). □

The degree assumption in Lemma 2.1.1 is necessary, as can be seen from Figure 2.1.

![Figure 2.1](image-url) A tree with arbitrarily long paths, but no infinite paths.
Proofs using König’s tree Lemma are often referred to as compactness arguments – the link with topology will become clear in Section 4.1. The following proposition illustrates one of the many uses of König’s tree Lemma.

**Proposition 2.1.2.** A countably infinite graph $G$ is 3-colourable if and only if every finite subgraph of $G$ is 3-colourable.

**2.1.3. Ramsey’s Theorem.** To prove Proposition 2.0.1 we also use an important tool from combinatorics: Ramsey theory. We denote the set $\{0,\ldots, n-1\}$ also by $[n]$. Subsets of a set of cardinality $s$ will be called $s$-subsets in the following. Let $\binom{M}{s}$ denote the set of all $s$-subsets of $M$. We also refer to mappings $\chi: \binom{M}{s} \to [c]$ as a coloring of $M$ (with the colors $[c]$). In Ramsey theory, one writes $L \to (m)^c$ if for every $\chi: \binom{L}{s} \to [c]$ there exists an $M \subseteq L$ with $|M| = m$ such that $\chi$ is constant on $\binom{M}{s}$. In the following, $\omega$ denotes the cardinality of $\mathbb{N}$. Note the following.

- For all $n \in \mathbb{N}$ we have $[n+1] \to (2)^n$: this is the pigeon-hole principle.
- For all $c \in \mathbb{N}$ we have $\mathbb{N} \to (\omega)^1$: this is the infinite pigeon-hole principle.

We first state and prove a special case of Ramsey’s theorem.

**Theorem 2.1.3.** $\mathbb{N} \to (\omega)^2$.

This statement has the following interpretation in terms of undirected graphs: every countably infinite undirected graph either contains an infinite clique (a complete subgraph) or an infinite independent set (a subgraph without edges).

**Proof.** Let $\chi: \binom{\mathbb{N}}{2} \to [2]$ be a 2-colouring of the edges of $\binom{\mathbb{N}}{2}$. We define an infinite sequence $x_0, x_1, \ldots$ of numbers from $\mathbb{N}$ and an infinite sequence $V_0 \supseteq V_1 \supseteq \cdots$ of infinite subsets of $\mathbb{N}$. Start with $V_0 := \mathbb{N}$ and $x_0 = 0$. By the infinite pigeon-hole principle, there is a $c_0 \in [2]$ such that $\{v \in V_0 \mid \chi(x_0, v) = c_0\} =: V_1$ is infinite. We now repeat this procedure with any $x_i \in V_i$ and $V_i$ instead of $V_0$. Continuing like this, we obtain sequences $(c_i)_{i \in \mathbb{N}}, (x_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}$.

Again by the infinite pigeon-hole principle, there exists $c \in [2]$ such that $c_i = c$ for infinitely many $i \in \mathbb{N}$. Then $P := \{x_i \mid c_i = c\}$ has the desired property. To see this, let $i < j$ be such that $x_i, x_j \in P$. Then $\chi(x_i, x_j) = c_i = c$.

We now state Ramsey’s theorem in its full strength; the proof is similar to the proof of Theorem 2.1.3 shown above.

**Theorem 2.1.4** (Ramsey’s theorem). Let $s, c \in \mathbb{N}$. Then $\mathbb{N} \to (\omega)^c$.

A proof of Theorem 2.1.4 can be found in [58] (Theorem 5.6.1); for a broader introduction to Ramsey theory see [53]. It is easy to derive the following finite version of Ramsey’s theorem from Theorem 2.1.4 via König’s tree lemma.

**Theorem 2.1.5** (Finite version of Ramsey’s theorem). For all $c, m, s \in \mathbb{N}$ there is an $l \in \mathbb{N}$ such that $[l] \to (m)^c$.

**Proof.** A proof by contradiction: suppose that there are positive integers $c, m, s$ such that for all $l \in \mathbb{N}$ there is a $\chi: \binom{[l]}{s} \to [c]$ such that $(*)_{[l]}$ for all $m$-subsets $M$ of $[l]$ the mapping $\chi$ is not constant on $\binom{M}{s}$. We construct a tree as follows. The vertices are the maps $\chi: \binom{[l]}{s} \to [c]$ that satisfy $(*)_{[l]}$. We make the vertex $\chi: \binom{[l]}{s} \to [c]$ adjacent to $\chi: \binom{[l+1]}{s} \to [c]$ if $\chi$ is a restriction of $\chi'$. Clearly, every vertex in the tree has finite degree. By assumption, there are arbitrarily long paths that start in the vertex $\chi_0$ where $\chi_0$ is the map with the empty domain. By Lemma 2.1.1 the tree contains an
infinite path \( x_0, x_1, \ldots \). We use this to define a map \( \chi^N : \left( \begin{smallmatrix} N \\ s \end{smallmatrix} \right) \to [c] \) as follows. For every \( x \in \mathbb{N} \), there exists a \( c_0 \in [c] \) and an \( i_0 \in \mathbb{N} \) such that \( \chi(x) = c_0 \) for all \( i \geq i_0 \). Define \( \chi_N(x) := c_0 \). Then \( \chi_N \) satisfies (\ast)_N, a contradiction to Theorem 2.1.4 \( \square \)

Here comes a variant of Ramsey’s theorem.

**Lemma 2.2.6.** Let \( X \) be an infinite set. Suppose that \( \chi : \left( \begin{smallmatrix} X \\ s \end{smallmatrix} \right) \to [c] \) is surjective. Then there exist infinite sets \( X_1, \ldots, X_c \subseteq X \) and \( k_1, \ldots, k_c \in [c] \) such that \( k_i \in \chi \left( \begin{smallmatrix} X_i \\ s \end{smallmatrix} \right) \) for all \( i \leq c \) and \( k_i \neq \chi \left( \begin{smallmatrix} X_j \\ s \end{smallmatrix} \right) \) for all \( i < j \leq c \).

**Proof.** Ramsey’s theorem states that there exists an infinite set \( X_1 \) such that \( \chi \left( \begin{smallmatrix} X_1 \\ s \end{smallmatrix} \right) \) is constant; we define \( k_1 \) to be this constant. Our proof proceeds by induction. Suppose we have already found \( X_1, \ldots, X_r \) and \( k_1, \ldots, k_r \) such that \( k_i \in \chi \left( \begin{smallmatrix} X_i \\ s \end{smallmatrix} \right) \) for all \( i \leq r \) and \( k_j \neq \chi \left( \begin{smallmatrix} X_i \\ s \end{smallmatrix} \right) \) for all \( i < j \leq c \). Let \( S \in \left( \begin{smallmatrix} X \end{smallmatrix} \right) \) such that \( \chi(S) \notin \{k_1, \ldots, k_r\} \), and let \( Y \subseteq X_r \setminus S \) be infinite. Let \( S_0, S_1, \ldots \) be an enumeration of all the subsets of \( S \) such that \( S_i \subseteq S_{i+1} \Rightarrow i \leq j \) (the enumeration extends the inclusion order). For \( i = 0, 1, \ldots \), define \( \chi_i : \left( \begin{smallmatrix} Y \setminus S_i \end{smallmatrix} \right) \to [c] \) as follows: for \( B \in \left( \begin{smallmatrix} Y \setminus S_i \end{smallmatrix} \right) \), set \( \chi_i(B) = \chi(B \cup S_i) \).

Now by Ramsey’s theorem, there exists an infinite set \( Z_i \) and \( \ell_i \in [c] \) such that \( \chi_i \left( \begin{smallmatrix} Z_i \end{smallmatrix} \right) = \{\ell_i\} \). Note that for \( i = 0 \) we have \( S_0 = \emptyset \) and \( \ell_0 \in \{k_1, \ldots, k_r\} \) since \( Z_1 \subseteq Y \subseteq X_r \). On the other hand, for \( i = 0^* \) we have \( S_1 = S \) and \( \chi \left( \begin{smallmatrix} Y \setminus S_1 \end{smallmatrix} \right) = \chi(S) \notin \{k_1, \ldots, k_r\} \), and \( \ell_1 = \chi \left( \begin{smallmatrix} Y \setminus S_1 \end{smallmatrix} \right) = \chi(S) \notin \{k_1, \ldots, k_r\} \). Let \( i_0 \) be smallest such that \( \ell_0 \notin \{k_1, \ldots, k_r\} \). Then \( X_{r+1} := Z_{i_0} \) and \( k_{r+1} := \ell_{i_0} \) satisfy the desired properties: Clearly, \( k_{r+1} \in \chi \left( \begin{smallmatrix} X_{r+1} \end{smallmatrix} \right) \), and \( k_j \notin \chi \left( \begin{smallmatrix} X_{r+1} \end{smallmatrix} \right) \) for \( r + 1 < j \leq c \) by the minimal choice of \( i_0 \). \( \square \)

### 2.2. On the Number of Orbits of \( n \)-Subsets

Let \( G \) be a permutation group on a countably infinite set \( D \). We want to prove that the number of orbits of \( n \)-subsets in a permutation group \( G \) forms a non-decreasing sequence (Proposition 2.0.1).

**Proof of Proposition 2.0.1.** Let \( O_1, \ldots, O_c \) be distinct orbits of \( n \)-subsets (we do not assume that these are all orbits of \( n \)-subsets, that is, our proof also covers the situation that the group is not oligomorphic). We show that there are at least \( c \) orbits of \( (n+1) \)-subsets, using Ramsey’s theorem in the form of Lemma 2.2.6. Let \( \chi : \left( \begin{smallmatrix} D \\ n \end{smallmatrix} \right) \to [c] \) be the map that assigns to a subset of \( D \) from \( O_i \), the number \( i \in [c] \), and \( i = 1 \) if the subset lies in none of \( O_1, \ldots, O_c \); note that \( \chi \) is surjective. By Lemma 2.2.6 there exist infinite sets \( X_1, \ldots, X_c \subseteq D \) and \( k_1, \ldots, k_c \in [c] \) such that \( k_i \in \chi \left( \begin{smallmatrix} X_i \\ n \end{smallmatrix} \right) \) for all \( i \leq c \) and \( k_j \notin \chi \left( \begin{smallmatrix} X_i \\ n \end{smallmatrix} \right) \) for all \( i < j \leq c \). For each \( i \leq c \), let \( B_i \in \left( \begin{smallmatrix} X_i \\ n+1 \end{smallmatrix} \right) \) be such that there exists an \( S_i \subseteq B_i \) with \( \chi(S_i) = k_i \). The sets \( B_1, \ldots, B_c \) lie in distinct orbits of \( n+1 \)-subsets, because no permutation from \( G \) can map \( B_i \) to \( B_j \) for \( i < j \leq c \) since \( B_i \subseteq X_i \) does not contain \( n \)-subsets of color \( k_i \). This proves that \( G \) has at least as many orbits of \( n+1 \)-subsets as orbits of \( n \)-subsets. \( \square \)

**Exercises.**

(14) Let \( (X; <) \) be a partially ordered set on a countably infinite set \( X \). Show that \( (X; <) \) contains an infinite chain, or an infinite antichain.

(15) Show that \( \operatorname{Aut}(\mathbb{Q}; \text{Cycl}) \) strictly contains \( \operatorname{Aut}(\mathbb{Q}; <) \).

(16) Show that an infinite sequence of elements of a totally ordered set contains one of the following:

- A constant subsequence;
• a strictly increasing subsequence;
• a strictly decreasing subsequence.

Derive the Bolzano-Weierstrass theorem (every bounded sequence in \( \mathbb{R}^n \) has a convergent subsequence), using the completeness property of \( \mathbb{R}^n \).

### 2.3. Highly Set-transitive Permutation Groups

In this section, we present a classification of highly set-transitive closed subgroups of \( \text{Sym}(X) \) for countably infinite \( X \).

**Definition 2.3.1.** A permutation group \( G \) on \( X \) and a permutation group \( H \) on \( X \) are isomorphic (as permutation groups; this is not the same as being isomorphic as groups!) if there exists a bijection \( \alpha \) between \( A \) and \( B \) and a bijection \( \beta \) between \( G \) and \( H \) such that for all \( a \in A \) and \( g \in G \) we have

\[
\alpha(g(a)) = \beta(g)(\alpha(a)).
\]

Note that if \( A \) and \( B \) are isomorphic structures, then \( \text{Aut}(A) \) and \( \text{Aut}(B) \) are isomorphic as permutation groups. For \( x_1, \ldots, x_n \in \mathbb{Q} \) we write \( x_1 \cdots x_n \) when \( x_1 < \cdots < x_n \).

**Theorem 2.3.2 (of [31]).** Let \( G \) be a highly set-transitive permutation group that is closed in \( \text{Sym}(X) \) for a countably infinite set \( X \). Then \( G \) is isomorphic (as a permutation group) to one of the following:

1. \( \text{Aut}(\mathbb{Q}; <) \);
2. \( \text{Aut}(\mathbb{Q}; \text{Betw}) \) where \( \text{Betw} \) is the ternary relation

\[
\{(x, y, z) \in \mathbb{Q}^3 \mid \overline{x} \overline{y} z \lor \overline{z} \overline{y} x \};
\]
3. \( \text{Aut}(\mathbb{Q}; \text{Cycl}) \) where \( \text{Cycl} \) is the ternary relation

\[
\{(x, y, z) \mid \overline{x} \overline{y} z \lor \overline{y} \overline{z} x \lor \overline{z} \overline{x} y \};
\]
4. \( \text{Aut}(\mathbb{Q}; \text{Sep}) \) where \( \text{Sep} \) is the 4-ary relation

\[
\{(x_1, y_1, x_2, y_2) \mid \overline{x_1} \overline{x_2} y_1 y_2 \lor \overline{x_1} \overline{y_1} x_2 y_2 \lor \overline{y_1} \overline{x_2} x_1 y_2 \lor \overline{y_1} \overline{y_2} x_1 x_2 \lor \overline{x_2} \overline{x_1} y_2 y_1 \lor \overline{x_2} \overline{y_2} x_1 y_1 \lor \overline{y_2} \overline{x_1} x_2 y_1 \lor \overline{y_2} \overline{y_1} x_2 x_1 \};
\]
5. \( \text{Aut}(\mathbb{Q}; =) \).

The relation \( \text{Sep} \) is the so-called *separation relation*; note that \( \text{Sep}(x_1, y_1, x_2, y_2) \) holds for elements \( x_1, y_1, x_2, y_2 \in \mathbb{Q} \) iff all four points \( x_1, y_1, x_2, y_2 \) are distinct and the smallest interval over \( \mathbb{Q} \) containing \( x_1, y_1 \) properly overlaps with the smallest interval containing \( x_2, y_2 \) (where properly overlaps means that the two intervals have a non-empty intersection, but none of the intervals contains the other).

To give you some ideas from the proof of Theorem 2.3.2 we give a proof of the following.

**Proposition 2.3.3.** Let \( G \) be a permutation group on a countably infinite set \( D \) such that \( G \) is 3-set transitive but not 2-transitive. Then \( G \) is isomorphic to a permutation group that contains \( \text{Aut}(\mathbb{Q}; <) \).

**Proof.** An *orbital* is an orbit of the componentwise action of \( G \) on \( D^2 \). By Proposition 2.0.1 \( G \) is 2-set transitive, and hence there are at most three orbitals: to see this, fix distinct \( u, v \in D \). Let \( O_1 \) := \( \{(x, y) \in D^2 \mid \exists \alpha \in G : \alpha(u, v) = (x, y)\} \), \( O_2 \) := \( \{(x, y) \in D^2 \mid \exists \alpha \in G : \alpha(v, u) = (x, y)\} \), and \( O_3 \) := \( \{(x, x) \mid x \in D\} \). Then the orbits of pairs are either \( O_1, O_2, O_3 \), or \( O_1 \cup O_2, O_3 \). Since \( G \) is not 2-transitive, there are exactly the three orbitals \( O_1, O_2, O_3 \). The structure \( (D; O_1) \) is a special directed graph called *tournament*: for any two distinct \( x, y \in D \) we have that either
(x, y) ∈ O_1 or (y, x) ∈ O_1. By 3-set transitivity, all of the three-element substructures of (D; O_1) are isomorphic. There are only two possibilities: either these substructures are transitive, or they are oriented 3-cycles. An easy case analysis shows that there is no 4-element tournament such that all 3-element sub-structures are 3-cycles. Hence, the first possibility must hold. Hence, (D; O_1) is transitive, and therefore a linear order. The linear order is dense: for all (a, b) ∈ O_1 there exists a c ∈ D such that O_1(a, c) and O_1(c, b). To see this, let u, v, w ∈ D be such that O_1(a, v) and O_1(v, w) (and hence O_1(a, w)). Then there exists a g ∈ G such that g(a, b) = (u, w). Then c := g(v) has the desired properties since O_1(a, g(v)) and O_1(g(v), b). Similarly, one can show that (D; O_1) is unbounded, that is, for every b ∈ D there exist b, c such that O_1(a, b) and O_1(b, c). We will see later (Proposition 3.2.1) that any countable dense unbounded linear order is isomorphic to (Q; <), which implies the statement.

Exercises.
(17) Verify the claim above that isomorphic structures have isomorphic automorphism groups.
(18) Give an example of two structures that are not isomorphic but have isomorphic automorphism groups.
(19) Show that every permutation group on an infinite set contains an orbital O and points x, y, z such that (x, y) ∈ O, (y, z) ∈ O, and (x, z) ∈ O.

2.4. Highly Transitive Permutation Groups

To do: classify the normal subgroups of Sym(T) (see Chapter 8.1 in [42]).

Lemma 2.4.1. Every normal subgroup of a highly transitive permutation group acting on an infinite set is either highly transitive or trivial.

Proof. Let G be a highly transitive subgroup of Sym(X) for some infinite set X and let H be a normal subgroup of G. The closure K of H in Sym(X) will be a normal subgroup of Sym(X). To see this, let α ∈ K and β ∈ Sym(X). Since K is the closure of H there exists a sequence (α_i)_{i ∈ N} of elements of H that converges against α. Since G is highly transitive, there exists a sequence (β_i)_{i ∈ N} of elements of G that converges against β. Then

βαβ^{-1} = (lim_{i} β_i)(lim_{i} α_i)(lim_{i} β_i)^{-1} = lim_{i} (β_i α_i β_i^{-1}) ∈ K

since K is closed and β_i α_i β_i^{-1} ∈ H because H ⊆ G. The statement now follows from the known fact that Sym(X) has no proper nontrivial closed normal subgroups (the normal subgroups of Sym(T) have been classified).
CHAPTER 3

Oligomorphic Permutation Groups

A permutation group $G$ over a countably infinite set $X$ is oligomorphic if $G$ has only finitely many orbits of $n$-tuples for each $n \geq 1$.

Examples and counterexamples:

- all finite permutation groups;
- $\text{Aut}(\mathbb{Q}; <)$, and all its supergroups;
- $\text{Aut}(\mathbb{Z}; <)$ is a non-example: it has only one orbit, but infinitely many orbitals (orbits of pairs). To see this, note that $(u, v)$ and $(x, y)$ are in the same orbit if and only if $u - v = x - y$.
- $\text{Aut}(\mathbb{Q}; +)$ is another non-example: it has only one orbit and only finitely many orbits of pairs, but infinitely many orbits of triples.

Lemma 3.0.1. Let $G \leq \text{Sym}(\mathbb{N})$ be oligomorphic and $a \in \mathbb{N}^n$ for $n \in \mathbb{N}$. Then $G_a$ is oligomorphic, too.

It follows from this observation and Theorem 1.2.3 that every oligomorphic permutation group must have continuum cardinality.

3.1. sInv-Aut

There is a surprising link between oligomorphicity of permutation groups and first-order logic.

Theorem 3.1.1. Let $\mathbb{A}$ be a countably infinite structure such that $\text{Aut}(\mathbb{A})$ is oligomorphic. Then $R \in \text{sInv}(\text{Aut}(\mathbb{A}))$ if and only if $R$ is first-order definable over $\mathbb{A}$.

One direction of the equivalence holds for general relational structures.

Proposition 3.1.2. Let $\mathbb{A}$ be a structure. If $R$ is first-order definable in $\mathbb{A}$, then $R \in \text{sInv}(\text{Aut}(\mathbb{A}))$.

Proof. Straightforward induction over the syntactic structure of first-order formulas and their semantics. □

Corollary 3.1.3. There is no linear order that is first-order definable over $(\mathbb{C}; +, *)$.

Proof. Let $<$ be a linear order on $\mathbb{C}$, and suppose without loss of generality that $-i < i$. The map $a \mapsto \overline{a}$ (complex conjugation) is an automorphism of $\mathbb{C}$, and exchanges $-i$ to $i$. We thus found an automorphism that violates $<$, and Proposition 3.1.2 implies that $<$ is not first-order definable. □

Corollary 3.1.4. Let $\mathbb{A}$ be a structure such that $\text{Aut}(\mathbb{A})$ is oligomorphic. Let $\mathbb{A}'$ be a structure with domain $\mathbb{A}' = \mathbb{A}$ such that all relations of $\mathbb{A}'$ are first-order definable in $\mathbb{A}$. Then $\text{Aut}(\mathbb{A}')$ is oligomorphic, too.

It follows for example that $\text{Aut}(\mathbb{Q}; \text{Betw})$ is oligomorphic.

Exercises.
(20) Show that the relation \( \{(x, y) \in \mathbb{R}^2 \mid x = y^2\} \) is not first-order definable over the structure \((\mathbb{R}; +)\).

In order to show Theorem 3.1.1 we first show the following.

**Lemma 3.1.5.** Let \( A \) be a structure with an oligomorphic permutation group, and let \( s, t \in A^k \) be tuples that satisfy the same first-order formulas in \( A \). Then for every \( a \in A \) there exists \( b \in A \) such that all first-order formulas that hold for \( (s, a) \in A^{k+1} \) also hold for \( (t, b) \in A^{k+1} \).

**Proof.** Let \( \Psi \) be the set of all first-order formulas satisfied by \( (s, a) \) in \( A \). Since \( \operatorname{Aut}(A) \) is oligomorphic, there are only finitely many inequivalent first-order formulas in \( \Psi \), and in particular \( \Psi \) contains a formula \( \psi \) such that \( A \models \psi(s, a) \) if and only if \( A \models \phi(s, a) \) for all \( \phi \in \Psi \). Then \( A \models \exists y. \psi(s, y) \), and \( A \models \exists y. \psi(t, y) \) by assumption. So there exists \( b \in A \) such that \( A \models \psi(t, b) \). This concludes the proof because \((t, b)\) satisfies all first-order formulas satisfied by \((s, a)\). \( \square \)

**Proof of Theorem 3.1.1.** One implication of the statement has been shown in Proposition 3.1.2. Conversely, suppose that \( R \in \operatorname{sInv}(\operatorname{Aut}(A)) \). Let \( \tau \) be the signature of \( A \). Then \( R \) is a union of orbits of \( \operatorname{Aut}(A) \) on \( A^k \); since \( \operatorname{Aut}(A) \) is oligomorphic, there exists an \( n \in \mathbb{N} \) such that \( R = O_1 \cup \cdots \cup O_n \). We show that orbits of \( k \)-tuples are first-order definable in \( A \); this is sufficient because if \( \psi_1 \) defines \( O_1 \) over \( A \), then \( \psi_1 \lor \cdots \lor \psi_n \) defines \( R \). So let \( O \) be such an orbit and let \( a = (a_1, \ldots, a_k) \in O \), and let \( \Psi \) be the set of all first-order \( \tau \)-formulas \( \psi(x_1, \ldots, x_k) \) in the language of \( A \) such that \( A \models \psi(a_1, \ldots, a_k) \). We prove that if a tuple \( b = (b_1, \ldots, b_k) \) satisfies every formula in \( \Psi \) then \( b \in O \) by constructing an automorphism of \( A \) that maps \( a \) to \( b \). This is done by a back-and-forth argument, using Lemma 3.1.5 for going forth, and again using Lemma 3.1.5 for going back (see the proof of Proposition 3.2.1). As in the proof of Lemma 3.1.5 we see that \( \Phi \) contains a formula \( \psi \) such that

\[
A \models \psi(s) \iff (A \models \phi(s) \text{ for all } \phi \in \Phi)
\]

Therefore, \( O \) is first-order definable in \( A \). \( \square \)

**Corollary 3.1.6.** Let \( A \) be \( \omega \)-categorical and let \( P \subseteq \operatorname{Sym}(A) \). Then \( \overline{\operatorname{Aut}(A)} = \overline{\operatorname{Aut}(P)} \) if and only if the set of relations that are definable in \( A \) equals \( \operatorname{sInv}(P) \).

**Proof.** Suppose that \( \overline{\operatorname{Aut}(A)} = \overline{\operatorname{Aut}(P)} \). Every relation that is first-order definable in \( A \) is strongly preserved by \( \operatorname{Aut}(A) \), and hence in particular by \( P \subseteq \overline{\operatorname{Aut}(P)} = \overline{\operatorname{Aut}(A)} \). Conversely, if \( R \) is strongly preserved by \( P \), then it is also strongly preserved by \( \overline{\operatorname{Aut}(P)} \), and hence first-order definable by Theorem 3.1.1.

Conversely, suppose that the set of relations that are definable in \( A \) equals \( \operatorname{sInv}(P) \). We then have

\[
\overline{\operatorname{Aut}(P)} = \operatorname{Aut}(\operatorname{sInv}(P)) = \operatorname{Aut}(A)
\]

(by Proposition 1.2.6)

(by assumption). \( \square \)

It follows from Theorem 3.1.1 that if \( B \) is a structure with an oligomorphic automorphism group \( G \), then the congruences of \( G \) are exactly the first-order definable equivalence relations in \( B \). Another application of Theorem 3.1.1 can be found in the following example.
3.2. Countably Categorical Structures

Example 3.1.7. The center of $G$ is the set $C(G) := \{ \alpha \mid \alpha \beta = \beta \alpha \text{ for all } \beta \in G \}$. If $G = \text{Aut}(A)$ is oligomorphic, then the center contains precisely those automorphisms of $A$ that are preserved by all automorphisms of $A$, and hence, by Theorem 3.1.3 precisely the automorphisms of $A$ that are first-order definable in $A$.

Remarks.

- Note that when $G_1$ and $G_2$ act oligomorphically on $A$ and $B$, respectively, then the natural intransitive action of $G_1 \times G_2$ is also oligomorphic: when $a(n)$ is the number of orbits of the componentwise action of $G_1$ on $A^n$, and $b(n)$ is the number of orbits of the componentwise action of $G_2$ on $B$, then the number of orbits of the componentwise of $G_1 \times G_2$ on $A \cup B$ is $\sum_{0 \leq i \leq n} a(i)b(n-i)$, and hence finite for all $n$.

- When $A$ and $B$ have the same signature $\tau$, then the automorphism group of the $\tau$-structure $A \times B$ (see Definition 1.1.4) contains the image of the product action of $\text{Aut}(A) \times \text{Aut}(B)$ on $A \times B$. The number of orbits of $n$-tuples of this action can be bounded by $a_nb_n$ where $a_n$ is the number of orbits of $n$-tuples in $A$ and $b_n$ is the number of orbits on $n$-tuples in $B$. Hence $\text{Aut}(A \times B)$ is oligomorphic.

- Let $B, C$ be structures on the same domain such that $\text{Aut}(B)$ and $\text{Aut}(C)$ are oligomorphic. Then $\text{Aut}(B) = \text{Aut}(C)$ if and only if $B$ and $C$ are first-order interdefinable in the sense that all relations of $B$ have a first-order definition in $C$ and vice-versa.

3.2. Countably Categorical Structures

A (first-order) theory is a set of first-order sentences. When the first-order sentences are over the signature $\tau$, we also say that $T$ is a $\tau$-theory. A model of a $\tau$-theory $T$ is a $\tau$-structure $A$ such that $A$ satisfies all sentences in $T$. Theories that have a model are called satisfiable. For every $\tau$-structure $A$, we denote by $\text{Th}(A)$ the theory of $A$, that is, the set of all $\tau$-sentences that are satisfied by $A$.

A satisfiable first-order theory $T$ is called $\omega$-categorical (or $\aleph_0$-categorical, which we use interchangeably) if all countable models of $T$ are isomorphic. A structure is called $\omega$-categorical if its first-order theory is $\omega$-categorical. Note that the theory of a finite structure does not have countable models, and hence is $\omega$-categorical.

Cantor [37] proved that the linear order of the rational numbers $(\mathbb{Q} ; <)$, which we will use as a running example in this section. We will see many more examples of $\omega$-categorical structures later. One of the standard approaches to verify that a structure is $\omega$-categorical is via a so-called back-and-forth argument. To illustrate, we give the back-and-forth argument that shows that $(\mathbb{Q} ; <)$ is $\omega$-categorical; much more about this important concept in model theory can be found in [58,97].

Proposition 3.2.1. The structure $(\mathbb{Q} ; <)$ is $\omega$-categorical.

Proof. Let $A$ be a countable model of the first-order theory $T$ of $(\mathbb{Q} ; <)$. It is easy to verify that $T$ contains (and, as this argument will show, is uniquely given by)

- $\exists x. x = x$ (no empty model)
- $\forall x, y, z ((x < y \land y < z) \Rightarrow x < z)$ (transitivity)
- $\forall x. \neg(x < x)$ (irreflexivity)
- $\forall x, y (x < y \lor y < x \lor x = y)$ (totality)
- $\forall x \exists y. x < y$ (no largest element)
- $\forall x \exists y. y < x$ (no smallest element)
• \( \forall x, z \exists y (x < y \land y < z) \) (density).

An isomorphism between \( \mathcal{A} \) and \((\mathbb{Q}; <)\) can be defined inductively as follows. Suppose that we have already defined \( f \) on a finite subset \( S \) of \( \mathbb{Q} \) and that \( f \) is an embedding of the structure induced by \( S \) in \((\mathbb{Q}; <)\) into \( \mathcal{A} \). Since \( <_{\mathcal{A}} \) is dense and unbounded, we can extend \( f \) to any other element of \( \mathbb{Q} \) such that the extension is still an embedding from a substructure of \( \mathbb{Q} \) into \( \mathcal{A} \) (going forth). Symmetrically, for every element \( v \) of \( \mathcal{A} \) we can find an element \( u \in \mathbb{Q} \) such that the extension of \( f \) that maps \( u \) to \( v \) is also an embedding (going back). We now alternate between going forth and going back; when going forth, we extend the domain of \( f \) by the \( n \)ext element of \( \mathbb{Q} \), according to some fixed enumeration of the elements in \( \mathbb{Q} \). When going back, we extend \( f \) such that the image of \( \mathcal{A} \) contains the next element of \( \mathcal{A} \), according to some fixed enumeration of the elements of \( \mathcal{A} \). If we continue in this way, we have defined the value of \( f \) on all elements of \( \mathbb{Q} \). Moreover, \( f \) will be surjective, and an embedding, and hence an isomorphism between \( \mathcal{A} \) and \((\mathbb{Q}; <)\).

A second important running example of this section is the countable random graph \((\mathbb{V}; E)\). This (simple and undirected) graph has the following extension property: for all finite disjoint subsets \( U, U' \) of \( \mathbb{V} \) there exists a vertex \( v \in \mathbb{V} \setminus (U \cup U') \) such that \( v \) is adjacent to all vertices in \( U \) and to no vertex in \( U' \).

**Proposition 3.2.2.** The random graph \((\mathbb{V}; E)\) is \( \omega \)-categorical.

**Proof.** Note that the extension property of \((\mathbb{V}; E)\) given above is a first-order property; a back-and-forth argument similar to the one given in the proof of Proposition 3.2.1 shows that every countably infinite graph with this property is isomorphic to \((\mathbb{V}; E)\).

The reason why we treat \( \omega \)-categoricity in this course is the following theorem. An accessible proof can be found in Hodges’ book (Theorem 6.3.1 in [58]).

**Theorem 3.2.3 (Engeler, Ryll-Nardzewski, Svenonius).** A countably infinite structure \( \mathcal{B} \) is \( \omega \)-categorical if and only if \( \text{Aut}(\mathcal{B}) \) is oligomorphic.

If the signature of \( \mathcal{B} \) is countable, there is another characterisation of \( \omega \)-categoricity of \( \mathcal{B} \) via the property of Theorem 3.1.1.

**Corollary 3.2.4.** Let \( \mathcal{B} \) be a structure with a countable signature. Then \( \mathcal{B} \) is \( \omega \)-categorical if and only if \( \text{slInv}(\text{Aut}(\mathcal{B})) \) equals the set of relations with a first-order definition over \( \mathcal{B} \).

**Proof.** The forwards implication is the content of Theorem 3.1.1. Conversely, suppose that \( \text{Aut}(\mathcal{B}) \) are infinitely many orbits of \( n \)-tuples, for some \( n \). Then the union of any subset of the set of all orbits of \( n \)-tuples is preserved by all automorphisms of \( \mathcal{B} \), but there are only countably many first-order formulas over a countable language, so not all the invariant sets of \( n \)-tuples can be first-order definable in \( \mathcal{B} \).

**Exercises.** The following exercises are taken from Peter Cameron’s book “Oligomorphic permutation groups”.

1. Write down sentences \( \phi_n, \psi_n \) (over the signature \( \{=\} \)) such that
   (a) any model of \( \phi_n \) has at least \( n \) elements;
   (b) any model of \( \psi_n \) has exactly \( n \) elements.

2. The diameter of a graph \( G = (V, E) \) is defined as the maximal distance in \( G \) between any two vertices \( u, v \in V \); if there is no path, the diameter is defined to be infinite. Use a compactness argument to show that a theory (over the signature \( \{E\} \) of graphs) having models with arbitrarily large diameter has a model with infinite diameter.
3.3. Homogeneous Structures and Amalgamation Classes

A relational structure $A$ is called homogeneous (sometimes also called ultrahomogeneous \cite{58}) if every isomorphism between finite substructures of $A$ can be extended to an automorphism of $A$.

Examples.

- $(\mathbb{Q}, \subset)$.
- when $G$ is a permutation group, then every canonical structure for $G$ is homogeneous.
- every expansion of an $\omega$-categorical structure by all first-order definable relations is homogeneous.

A versatile tool to construct countable homogeneous structures from classes of finite structures is the amalgamation technique à la Fraïssé. We present it here for the special case of relational structures; this is all that is needed in the examples we are going to present. For a stronger version of Fraïssé-amalgamation for classes of structures that might involve function symbols, see \cite{58}.

In the following, let $\tau$ be a countable relational signature. The age of a $\tau$-structure $A$ is the class of all finite $\tau$-structures that embed into $A$. A class $C$ has the joint embedding property (JEP) if for any two structures $B_1, B_2 \in C$ there exists a structure $C \in C$ that embeds both $B_1$ and $B_2$.

Proposition 3.3.1. Let $C$ be a class of $\tau$-structures. Then $C$ is the age of a (countable) relational structure if and only if $C$ is closed under isomorphisms and substructures and has the JEP.

The union of two relational $\tau$-structures $B_1, B_2$ is the $\tau$-structure $C$ with domain $B_1 \cup B_2$ and relations $R^C := R^{B_1} \cup R^{B_2}$ for all $R \in \tau$. The intersection of $B_1$ and $B_2$ is defined analogously. Let $B_1, B_2$ be $\tau$-structures such that $A$ is an induced substructure of both $B_1$ and $B_2$ and all common elements of $B_1$ and $B_2$ are elements of $A$; note that in this case $A = B_1 \cap B_2$. Then we call $B_1 \cup B_2$ the free amalgam of $B_1, B_2$ over $A$. More generally, a $\tau$-structure $C$ is an amalgam of $B_1$ and $B_2$ over $A$ if for $i \in \{1, 2\}$ there are embeddings $f_i$ of $B_i$ to $C$ such that $f_1(a) = f_2(a)$ for all $a \in A$. We refer to $(A, B_1, B_2)$ as an amalgamation diagram.

Definition 3.3.2. An isomorphism-closed class $C$ of $\tau$-structures

- has the free amalgamation property if for every amalgamation diagram $(A, B_1, B_2)$ the free amalgam of $B_1$ and $B_2$ over $A$ is contained in $C$;
- has the amalgamation property if every amalgamation diagram $(A, B_1, B_2)$ has an amalgam $C \in C$;
- is an amalgamation class if it contains at most countably many non-isomorphic structures, has the amalgamation property, and is closed under taking induced substructures.

Note that since we only look at relational structures here (and since we allow structures to have an empty domain), the amalgamation property of $C$ implies the joint embedding property.

\footnote{The entire theory can be adapted to general signatures that might also contain function symbols; to keep the exposition simple, we restrict our focus to relational signatures in this section.}
Example 3.3.3. Let $\mathcal{C}$ be the class of all linear orders. Then $\mathcal{C}$ is clearly closed under isomorphisms and induced substructures, and has countably many isomorphism types. To show that it also has the amalgamation property, let $B_1, B_2 \in \mathcal{C}$, and let $A$ be an induced substructure of both $B_1$ and $B_2$. Let $\mathcal{C}$ be the free amalgam of $B_1$ and $B_2$ over $A$. Then $\mathcal{C}$ is an acyclic finite graph; therefore, any depth-first traversal of $\mathcal{C}$ leads to a linear ordering of the elements that is an amalgam (even a strong amalgam, but not a free amalgam) in $\mathcal{C}$ of $B_1$ and $B_2$ over $A$. It follows that $\mathcal{C}$ is an amalgamation class.

Theorem 3.3.4 (Fraïssé [49,50; see 58]). Let $\tau$ be a countable relational signature and let $\mathcal{C}$ be an amalgamation class of $\tau$-structures. Then there is a homogeneous and at most countable $\tau$-structure $\mathcal{C}$ whose age equals $\mathcal{C}$. The structure $\mathcal{C}$ is unique up to isomorphism, and called the Fraïssé-limit of $\mathcal{C}$.

Example 3.3.5. Let $\mathcal{C}$ be the class of all finite partially ordered sets. Amalgamation can be shown by computing the transitive closure: when $\mathcal{C}$ is the free amalgam of $B_1$ and $B_2$ over $A$, then the transitive closure of $\mathcal{C}$ gives an amalgam in $\mathcal{C}$. The Fraïssé-limit of $\mathcal{C}$ is called the homogeneous universal partial order.

Example 3.3.6. Let $\mathcal{C}$ be the class of all finite graphs. It is even easier than in the previous examples to verify that $\mathcal{C}$ is an amalgamation class, since here the free amalgam itself shows the amalgamation property. The Fraïssé-limit of $\mathcal{C}$ is also known as the countable random graph, and will be denoted by $(V;E)$.

Proposition 3.3.7. For every $n$, there is a permutation group which is $n$-transitive but not $(n+1)$-transitive.

Proof. Let $R$ be a relation symbol of arity $(k+1)$. Let $\mathcal{C}$ be the class of all $\{R\}$-structures where $R$ denotes a relation that only contains tuples with pairwise distinct entries. The class $\mathcal{C}$ is clearly closed under isomorphism, substructures, and has only countably many isomorphism classes of structures. It also has the free amalgamation property. Note that any two structures with at most $k$ elements are isomorphic, since $R$ denotes the empty relation in those structures. Since the Fraïssé-limit is homogeneous, it is therefore $k$-transitive. On the other hand, the class contains non-isomorphic structures of size $k+1$ (e.g., a structure where $R$ denotes the empty relation and a structure where $R$ is non-empty), and hence the structure is not $k+1$-transitive.

Example 3.3.8. Let $\mathcal{C}$ be the class of all finite triangle-free graphs, that is, all graphs that do not contain $K_3$ as a subgraph. Again, we have the free amalgamation property. The Fraïssé-limit is up to isomorphism uniquely described as the triangle-free graph $\mathcal{A}$ such that for any finite $S, T \subseteq A$ such that $S$ is stable (i.e., induces a graph with no edges; such a vertex subset is sometimes also called an independent set) there exists $v \in A \setminus (S \cup T)$ which is connected to all points in $S$, but to no point in $T$.

We now introduce a convenient tool to describe classes of finite $\tau$-structures. When $\mathcal{N}$ is a class of $\tau$-structures, we say that a structure $\mathcal{A}$ is $\mathcal{N}$-free if no $\mathcal{B} \in \mathcal{N}$ embeds into $\mathcal{A}$. The class of all finite $\mathcal{N}$-free structures we denote by $\text{Forb}(\mathcal{N})$.

Example 3.3.9. Henson [54] used Fraïssé limits to construct $2^\omega$ many homogeneous directed graphs. A tournament is a directed graph without self-loops such that for all pairs $x, y$ of distinct vertices exactly one of the pairs $(x, y)$, $(y, x)$ is an arc in the graph. Note that for all classes $\mathcal{N}$ of finite tournaments, $\text{Forb}(\mathcal{N})$ is an amalgamation class, because if $\mathcal{A}_1$ and $\mathcal{A}_2$ are directed graphs in $\text{Forb}(\mathcal{N})$ such that
\( A = A_1 \cap A_2 \) is an induced substructure of both \( A_1 \) and \( A_2 \), then the free amalgam \( A_1 \cup A_2 \) is also in \( \text{Forb}(N) \).

Henson in his proof specified an infinite set \( T \) of tournaments \( T_1, T_2, \ldots \) with the property that \( T_i \) does not embed into \( T_j \) if \( i \neq j \). Draw the family on the board. Note that this property implies that for two distinct subsets \( N_1 \) and \( N_2 \) of \( T \) the two sets \( \text{Forb}(N_1) \) and \( \text{Forb}(N_2) \) are distinct as well. Since there are \( 2^\omega \) many subsets of the infinite set \( T \), there are also that many distinct homogeneous directed graphs; they are often referred to as Henson digraphs.

The structures from Example 3.3.9 can be used to prove various negative results about homogeneous structures with finite signature. A better behaved class of homogeneous structures are those whose age is finitely bounded (this is the same terminology as in [82]).

**Definition 3.3.10.** We say that a class \( \mathcal{C} \) of finite relational \( \tau \)-structures (or a structure with age \( \mathcal{C} \)) is finitely bounded if \( \tau \) is finite and there exists a finite set of finite \( \tau \)-structures \( \mathcal{N} \) such that \( \mathcal{C} = \text{Forb}(\mathcal{N}) \).

Finally, we should also mention that Fraïssé’s theorem (Theorem 3.3.4) has a converse: the age of every homogeneous relational structure has the amalgamation property (this is not hard to show; see [58]).

**Exercises.**

(24) Show the age \( \mathcal{C} \) of a structure has the amalgamation property if and only if it has the 1-point amalgamation property, i.e., if for all \( A, B_1, B_2 \in \mathcal{C} \) and embeddings \( e_1 : A \hookrightarrow B_1 \) and \( e_2 : A \hookrightarrow B_2 \) such that \( |B_1| = |B_2| = |A| + 1 \) there are a \( C \in \mathcal{C} \) and embeddings \( f_1 : B_i \hookrightarrow C \) for \( i \in \{1, 2\} \) such that \( f_1 \circ e_1 = f_2 \circ e_2 \).

(25) Let \( D \) be the tournament obtained from the directed cycle \( C_3 \) of length three by adding a new vertex \( u \), and adding the edges \( (u, v) \) for every vertex \( v \) of \( C_3 \). Let \( D' \) be the tournament obtained from \( D \) by flipping the orientation of each edge. Show that \( \text{Forb}(\{D, D'\}) \), the class of all finite tournaments that embeds neither \( D \) nor \( D' \), is an amalgamation class.

(26) Let \( P \) be a unary relation symbol. Let \( \mathcal{D} \) be the class of all finite \( \{P, <\}\)-structures \( A \) such that \( \langle A \rangle \) is a linear order.

(a) Show that \( \mathcal{D} \) is an amalgamation class.

(b) Let \( B \) be the Fraïssé-limit of the class \( \mathcal{D} \), and define \( E \subseteq B^2 \) by \( (u, v) \in E \) if

- \( u < v \) and \( (u \in P \Leftrightarrow v \in P) \), or
- \( u > v \) and not \( (u \in P \Leftrightarrow v \in P) \).

Show that \( (B; E) \) is a tournament.

(c) Show that the class \( \text{Age}(B; E) \) equals the class of tournaments that can be obtained from tournaments \( T \) in \( \text{Age}(\mathcal{Q}; <) \) by performing the following operation: pick \( u \in T \) and reverse all edges between \( u \) and other elements of \( T \) (we ‘switch edges at \( u \)’).

(d) Show that \( (B; E) \) is homogeneous.

(e) Show that \( \text{Age}(B; E) \) equals the class \( \mathcal{C} \) from Exercise 25.

(f) Show that \( (B; E) \) is isomorphic to the tournament whose vertices are a countable dense subset \( S \subseteq \mathbb{R}^2 \) of the unit circle without antipodal points, and where the edges are oriented in clockwise order, i.e., put \( ((u_1, u_2), (v_1, v_2)) \in E \) if and only if \( u_1v_2 - u_2v_1 > 0 \).
3.4. Strong Amalgamation and Algebraicity

A strong amalgam of $B_1, B_2$ over $A$ is an amalgam of $B_1, B_2$ over $A$ where $f_1(B_1) \cap f_2(B_2) = f_1(A)(= f_2(A))$. We say that a class $C$ has the strong amalgamation property if all amalgamation diagrams have a strong amalgam in $C$.

Examples:

- The age of the Random Graph.
- The age of $(\mathbb{Q}; <)$.
- The age of all other structures we have seen so far.
- An example of an amalgamation class that does not have strong amalgamation: let $P$ be a unary relation symbol, and let $C$ be the class of all finite $\{P\}$-structures where $P$ contains only one element. Then $C$ is an amalgamation class, and the Fraïssé-limit is a countably infinite structure where $P$ contains only one element. But $C$ does not have strong amalgamation.

Picture with an amalgamation diagram that fails to have a strong amalgam.

3.4.1. Algebraicity. When $C$ is a strong amalgamation class, then the Fraïssé-limit of $C$ has a remarkable property. Let $A$ be a structure, and $B \subseteq A$ finite. In this course, $acl_A(B)$ denotes the set of elements of $A$ that lie in finite orbits in $Aut(A)|_B$.

In $\omega$-categorical structures $A$, this coincides precisely with the model-theoretic algebraic closure of $B$ in $A$, i.e., the elements of $A$ that lie in finite sets that are first-order definable over $A$ with parameters from $B$. We say that $A$ has no algebraicity if $acl_A(B) = B$ for all finite sets of parameters $B$.

**Theorem 3.4.1** (See (2.15) in [33]). Let $A$ be a homogeneous structure. Then the age of $A$ has strong amalgamation if and only if $A$ has no algebraicity.

**Proof.** We first show that strong amalgamation of the age implies no algebraicity of $A$. Let $F \subseteq A$ be finite, and $u \in A \setminus F$. We want to show that the orbit of $u$ in $Aut(B)|_F$ is infinite. Let $n \in \mathbb{N}$ and let $B$ be the structure induced by $F \cup \{u\}$ in $A$. Then there exists a strong amalgam $\overline{B} \in \text{Age}(A)$ of $B$ with $\overline{B}$ over $A$. We iterate this, taking a strong amalgam of $\overline{B}$ with $\overline{B}'$ over $A$, showing that, because of homogeneity, there are $n$ distinct elements in $A \setminus F$ that lie in the same orbit as $u$ in $Aut(A)|_F$. Since $n \in \mathbb{N}$ and $u \in A \setminus F$ were chosen arbitrarily, the group $Aut(A)|_F$ has no finite orbits outside $F$.

For the other direction, we rely on the following lemma of Peter Neumann.

**Lemma 3.4.2.** Let $G$ be a permutation group on $D$ without finite orbits, and let $A, B \subseteq D$ be finite. Then there exists a $g \in G$ with $g(A) \cap B = \emptyset$.

**Proof.** The proof here is from Cameron [33], and is a nested induction. The outer induction is on $|A|$. We assume the result for any set $A'$ with $|A'| < |A|$. The induction base $A = \emptyset$ is trivial. Suppose for contradiction that no $g \in G$ with the required property exists.

**Claim.** For any $C$ with $|C| \leq |A|$, there are only finitely many translates $g(A)$ of $A$ that contain $C$. The proof of the claim is by induction on $|A| - |C|$. When $|A| - |C| = 0$ then the only translate of $A$ that contains $C$ is $C$, and the statement holds. So suppose that $|C| < |A|$ and that the claim holds for all $C'$ with $|C'| > |C|$. By the outer induction hypothesis, we may assume that $C \cap B = \emptyset$. By the inner induction hypothesis, for each of the finitely many points $b \in B$, only finitely many translates of $A$ contain $C \cup \{b\}$. So only finitely many translates of $A$ contain $C$ and have non-empty intersection with $B$. Since we assumed that every translate of $A$ has non-empty intersection with $B$, we have shown the claim.
For $C = \emptyset$, the claim implies that $A$ has only finitely many translates, a contradiction to the assumption that $G$ has no finite orbits.

We now continue with the reverse implication of Theorem 3.4.1. Let $A$ be without algebraicity, and let $(A_0, B_1, B_2)$ be an amalgamation diagram with $A_0, B_1, B_2 \in \text{Age}(A)$. By homogeneity of $A$ we can furthermore assume that $A_0, B_1, B_2$ are substructures of $A$; that is, the structure induced by $B_1 \cup B_2$ in $A$ is an amalgam, but possibly not a strong one. Since $A$ has no algebraicity, $\text{Aut}(A)_{|A_0}$ has no finite orbits outside $A_0$. By the lemma, there exists $g \in \text{Aut}(A)_{|A_0}$ such that $(B_1 \setminus A_0) \cap (B_2 \setminus A_0) = \emptyset$. Then the structure induced by $g(B_1 \cup B_2)$ is a strong amalgam of $B_1$ and $B_2$ over $A_0$.

3.4.2. Free superpositions. For strong amalgamation classes there is a powerful construction to obtain new strong amalgamation classes from known ones.

Definition 3.4.3. Let $C_1$ and $C_2$ be classes of finite structures with disjoint relational signatures $\tau_1$ and $\tau_2$, respectively. Then the free superposition of $C_1$ and $C_2$, denoted by $C_1 \ast C_2$, is the class of $(\tau_1 \cup \tau_2)$-structures $A$ such that the $\tau_i$-reduct of $A$ is in $C_i$, for $i \in \{1, 2\}$.

The following lemma has a straightforward proof by combining amalgamation in $C_1$ with amalgamation in $C_2$.

Lemma 3.4.4. If $C_1$ and $C_2$ are strong amalgamation classes, then $C_1 \ast C_2$ is also a strong amalgamation class.

When $\Gamma_1$ and $\Gamma_2$ are homogeneous structures with no algebraicity, then $\Gamma_1 \ast \Gamma_2$ denotes the (up to isomorphism unique) Fraïssé limit of the free superposition of the age of $\Gamma_1$ and the age of $\Gamma_2$.

Example 3.4.5. For $i \in \{1, 2\}$, let $\tau_i = \{<, i\}$, let $C_i$ be the class of all finite $\tau_i$-structures where $<$ denotes a linear order, and let $\Gamma_i$ be the Fraïssé limit of $C_i$. Then $\Gamma_1 \ast \Gamma_2$ is known as the random permutation (see e.g. [29][80][108]).

Exercises.

(27) Show that there are permutation groups $G_1, G_2$ on a countably infinite set such that both $G_1$ and $G_2$ are isomorphic (as permutation groups) to $(\mathbb{Q}, <)$, but $G_1 \cap G_2 = \{\text{id}\}$.

(28) Construct an permutation group $G$ on a set $X$ with precisely $n!$ orbits of $n$-element subsets. Extra question (I don’t know the answer to it): does this property characterise $G$ uniquely up to isomorphism?

(29) Show that the random graph can be partitioned into two subsets so that both parts are isomorphic to the random graph. Show that the same is not true for all partitions of the random graph into two infinite subsets.

(30) Give an example of a homogeneous structure with a transitive automorphism group whose age does not have strong amalgamation.

(31) Let $G$ be a permutation group on a set $A$. Show that the following are equivalent.

(a) There exists a structure $A$ with finite relational signature such that $G = \text{Aut}(A)$;

(b) There exists a relation $R \subseteq A^n$ such that $G = \text{Aut}(A, R)$;

(c) There exists a structure $A$ with finite relational signature such that $G = \text{Aut}(A)$, and all relations in $A$ have pairwise distinct entries.

If additionally the domain $A$ is finite, we can combine the two conditions, and the above items are equivalent to
(d) There exists a relation $R \subseteq A^n$ such that $G = \text{Aut}(A, R)$ and $R$ has pairwise distinct entries.

(32) Show that $(\mathbb{Z}_2)^\omega$ cannot be isomorphic to an automorphism group of a countable structure, but not to an automorphism group of an $\omega$-categorical structure.

3.5. First-order Interpretations

Many $\omega$-categorical structures can be derived from other $\omega$-categorical structures via first-order interpretations (our definition follows [58]).

**Definition 3.5.1.** A relational $\sigma$-structure $B$ has a (first-order) interpretation $I$ in a $\tau$-structure $A$ if there exists a natural number $d$, called the dimension of $I$, and

- a $\tau$-formula $\delta_I(x_1, \ldots, x_d)$ — called the domain formula,
- for each atomic $\sigma$-formula $\phi(y_1, \ldots, y_k)$ a $\tau$-formula $\phi_I(x_1, \ldots, x_k)$ where the $x_i$ denote disjoint $d$-tuples of distinct variables — called the defining formulas,
- a surjective map $h : D \to B$, where

$$D := \{ (a_1, \ldots, a_d) \in A^d \mid A \models \delta_I(a_1, \ldots, a_d) \}$$

— called the coordinate map,

such that for all atomic $\sigma$-formulas $\phi$ and all tuples $\overline{x}_i \in D$

$$B \models \phi(h(\overline{x}_1), \ldots, h(\overline{x}_k)) \iff A \models \phi_I(\overline{x}_1, \ldots, \overline{x}_k).$$

Note that the coordinate map $h$ determines the defining formulas up to logical equivalence; hence, we sometimes identify $I$ with $h$. Note that the kernel of $h$ coincides with the relation defined by $(x = y)_I$, for which we also write $=_I$, the defining formula for equality.

We say that $B$ is interpretable in $A$ with finitely many parameters if there are $a_1, \ldots, a_n \in A$ such that $B$ is interpretable in the expansion of $A$ by the singleton relations $\{a_i\}$ for all $1 \leq i \leq n$.

**Lemma 3.5.2** (see Theorem 7.3.8 in [57]). Let $A$ be an $\omega$-categorical structure. Then every structure $B$ that is first-order interpretable in $A$ with finitely many parameters is $\omega$-categorical or finite.

**Proof.** An easy consequence of Theorem 3.2.3.

Note that in particular all reducts of an $\omega$-categorical structure and all expansions of an $\omega$-categorical structure by finitely many constants are again $\omega$-categorical.

**Example 3.5.3.** Allen’s interval algebra: studied in temporal reasoning in computer science. Domain: closed bounded intervals over the rational numbers. Relations: containment, disjointness, precedence, etc. Formally a structure $A$ that is best described by a first-order interpretation $I$ in $(\mathbb{Q}; <)$:

- the dimension of the interpretation is two;
- the domain formula $\delta_I(x, y)$ is $x < y$;
- for each inequivalent $\{<\}$-formula $\phi$ with four variables a binary relation $R$ such that $(a_1, a_2, a_3, a_4)$ satisfies $\phi$ if and only if $((a_1, a_2), (a_3, a_4)) \in R$.

By Lemma 3.5.2, $A$ is $\omega$-categorical.
### 3.5.1. Composing Interpretations

First-order interpretations can be composed. In order to conveniently treat these compositions, we first describe how an interpretation of a \( \sigma \)-structure \( B \) gives rise to interpreting formulas for arbitrary \( \sigma \)-formulas \( \psi(x_1, \ldots, x_n) \). Replace each atomic \( \sigma \)-formula \( \phi(y_1, \ldots, y_n) \) in \( \psi \) by \( \phi_I(y_1, \ldots, y_1, d, \ldots, y_{n, 1}, \ldots, y_{n, d}) \); we write \( \psi_I(x_1, \ldots, x_{1, d}, \ldots, x_{n, 1}, \ldots, x_{n, d}) \) for the resulting \( \tau \)-formula, and call it the interpreting formula for \( \psi \). Note that if \( \psi \) defines the relation \( R \) in \( B \), then \( \phi_I \) defines \( I^{-1}(R) \) in \( A \). For all \( d \)-tuples \( a_1, \ldots, a_k \in I^{-1}(B) \)

\[
B \models \phi(I(a_1), \ldots, I(a_k)) \iff A \models \phi_I(a_1, \ldots, a_k).
\]

**Definition 3.5.4.** Let \( C, B, A \) be structures with the relational signatures \( \rho, \sigma, \) and \( \tau \). Suppose that

- \( C \) has a \( d \)-dimensional interpretation \( I \) in \( B \), and
- \( B \) has an \( e \)-dimensional interpretation \( J \) in \( A \).

Then \( C \) has a natural \( de \)-dimensional first-order interpretation \( I \circ J \) in \( A \): the domain of \( I \circ J \) is the set of all \( de \)-tuples in \( A \) that satisfy the \( \tau \)-formula \((\top)_I \circ J \), and we define

\[
I \circ J(a_{1, 1}, \ldots, a_{1, e}, \ldots, a_{d, 1}, \ldots, a_{d, e}) := I(J(a_{1, 1}, \ldots, a_{1, e}), \ldots, J(a_{d, 1}, \ldots, a_{d, e})).
\]

Let \( \phi \) be a \( \tau \)-formula which defines a relation \( R \) over \( A \). Then the formula \((\phi_I)_J \) defines in \( A \) the preimage of \( R \) under \( I \circ J \).

### 3.5.2. Bi-interpretations

Two interpretations \( I_1 \) and \( I_2 \) of \( B \) in \( A \) are called **homotopic** if the relation \( \{ (x, y) \mid I_1(x) = I_2(y) \} \) is first-order definable in \( D \). Note that \( \text{id}_C \) is an interpretation of \( C \) in \( C \), called the **identity interpretation** of \( C \) (in \( C \)).

**Definition 3.5.5.** Two structures \( A \) and \( B \) with an interpretation \( I \) of \( B \) in \( A \) and an interpretation \( J \) of \( A \) in \( B \) are called mutually interpretable. If both \( I \circ J \) and \( J \circ I \) are homotopic to the identity interpretation (of \( A \) and of \( B \), respectively), then we say that \( A \) and \( B \) are **bi-interpretable** (via \( I \) and \( J \)).

**Example 3.5.6.** The directed graph \( C := (N^2; E) \) where

\[
E := \{(u_1, u_2), (v_1, v_2) \mid u_2 = v_1\}
\]

and the structure \( D := (N; =) \) are bi-interpretable. The interpretation \( I_1 \) of \( C \) in \( D \)

is 2-dimensional, the domain formula is true, and the coordinate map is the identity.

The interpretation \( I_2 \) of \( D \) in \( C \) is 1-dimensional, the domain formula is true, and the coordinate map sends \((x, y)\) to \( x \).

Then \( I_2(I_1(x, y)) = z \) is definable by the formula \( x = z \), and hence \( I_1 \circ I_2 \) is homotopic to the identity interpretation of \( D \). Moreover, \( I_1(I_2(w), i_2(v)) = w \) is first-order definable by

\[
E(w, v) \land \exists p (E(p, u) \land E(p, w)) ,
\]

so \( I_2 \circ I_1 \) is also homotopic to the identity interpretation of \( C \).

**Example 3.5.7.** It is easy to see that Allen’s interval algebra (Example 3.5.3) is bi-interpretable with \((Q; <)\).

We show that the property to have essentially infinite language is preserved by bi-interpretability.

**Proposition 3.5.8.** Let \( B \) and \( C \) be \( \omega \)-categorical structures that are first-order bi-interpretable. Then \( B \) has essentially infinite signature if and only if \( C \) has.

\footnote{We follow the terminology from [3].}
Proof. Let $\tau$ be the signature of $B$. We have to show that if $C$ has finite signature, then $B$ is interdefinable with a structure $B'$ with a finite signature. Let $\sigma \subseteq \tau$ be the set of all relation symbols that appear in all the formulas of the interpretation of $C$ in $B$. Since the signature of $C$ is finite, the cardinality of $\sigma$ is finite as well.

We will show that there is a first-order definition of $B$ in the $\sigma$-reduct $B'$ of $B$. Suppose that the interpretation $I_1$ of $C$ in $B$ is $d_1$-dimensional, and that the interpretation $I_2$ of $B$ in $C$ is $d_2$-dimensional. Let $\theta(x, y_{1,1}, \ldots, y_{d_1,d_2})$ be the formula that shows that $I_2 \circ I_1$ is homotopic to the identity interpretation. That is, $\theta$ defines in $B$ the $(d_1d_2 + 1)$-ary relation that contains a tuple $(a, b_{1,1}, \ldots, b_{d_1,d_2})$ iff $a = h_2(h_1(b_{1,1}, \ldots, b_{1,d_2}), \ldots, h_1(b_{d_1,1}, \ldots, b_{d_1,d_2}))$.

Let $\phi$ be an atomic $\tau$-formula with $k$ free variables $x_1, \ldots, x_k$. We will specify a $\sigma$-formula that is equivalent to $\phi$ over $B'$.

$$\exists y_{1,1}^{1}, \ldots, y_{1,1}^{k}, y_{d_1,d_2}^{1}, \ldots, y_{d_1,d_2}^{k} \left( \bigwedge_{i \leq k} \theta(x, y_{1,1}^{i}, \ldots, y_{d_1,d_2}^{i}) \right. \\
\left. \land \phi_{I_1I_2}(y_{1,1}^{1}, \ldots, y_{d_1,d_2}^{1}, y_{2,2}^{1}, \ldots, y_{d_1,d_2}^{k}) \right)$$

is equivalent to $\phi(x_1, \ldots, x_k)$ over $B$. Indeed, by surjectivity of $h_2$, for every element $a_i$ of $B$ there are elements $c_{1,i}^{1}, \ldots, c_{d_2}^{1}$ of $C$ such that $h_2(c_{1,i}^{1}, \ldots, c_{d_2}^{1}) = a_i$, and by surjectivity of $h_1$, for every element $c_{j}^{1}$ of $C$ there are elements $b_{1,j}^{1}, \ldots, b_{d_1,j}^{1}$ of $B$ such that $h_1(b_{1,j}^{1}, \ldots, b_{d_1,j}^{1}) = c_{j}^{1}$. Then

$$B \models R(a_1, \ldots, a_k) \iff C \models \phi_{I_2}(c_{1}^{1}, \ldots, c_{d_2}^{1}, \ldots, c_{1}^{k}, \ldots, c_{d_2}^{k})$$
$$\iff B' \models \phi_{I_1I_2}(b_{1,1}^{1}, \ldots, b_{1,d_2}^{1}, b_{2,2}^{1}, \ldots, b_{1,d_2}^{k}, \ldots, b_{d_1,d_2}^{k})$$

□

We will come back to bi-interpretability in Section 5.3.
CHAPTER 4

Topological Groups

We start with a very brief introduction to concepts from topology that will be relevant in what follows.

4.1. Topological Spaces

Topological spaces are mathematical structures that are used to capture the notions of “closeness” and “continuity” on a very general level. A topological space is a set $S$ together with a collection of subsets of $S$, called the open sets of $S$, such that

1. the empty set and $S$ are open;
2. arbitrary unions of open sets are open;
3. the intersection of two open sets is open.

Complements of open sets are called closed. Note that $S$ and the empty set are both open and closed.

Example 4.1.1. On every set $S$, there is the trivial topology where the only open sets are $S$ and the empty set.

Example 4.1.2. Every set $S$ can be equipped with the discrete topology: in this topology, every subset of $S$ is open (and hence also closed).

Example 4.1.3. The standard topology on $\mathbb{R}$: a set $U \subseteq \mathbb{R}$ is open exactly if for every $x \in U$ there exists an $\epsilon > 0$ such that the every $y \in \mathbb{R}$ with $x - \epsilon < y < x + \epsilon$ is contained in $U$. The standard topology on $\mathbb{Q}$ is defined analogously.

Example 4.1.4. The standard topology on $\mathbb{R}^d$, $d \in \mathbb{N}$: a set $U \subseteq \mathbb{R}^d$ is open exactly if for every $x \in U$ there exists an $\epsilon > 0$ such that the $\epsilon$-ball around $x$ is contained in $U$.

Example 4.1.5. Every set $S$ can be equipped with the cofinite topology: in this topology, the open sets are the empty set and every cofinite subset of $S$. The only closed subsets in this topology are the finite sets and the entire set $S$.

Example 4.1.6. The topology of pointwise convergence on $\text{Sym}(\mathbb{N})$; see Proposition 1.2.2.

For $E \subseteq S$, the closure of $E$, denoted by $\overline{E}$, is the intersection over all closed sets over $S$ that contain $E$. A subset $E$ of $S$ is called dense (in $S$) if its closure is the full space $S$. The subspace of $S$ induced on $E$ is the topological space on $E$ where the open sets are exactly the intersections of $E$ with the open sets of $S$. When we work with permutation groups $G \subseteq \text{Sym}(X)$ then we will always work with the topology on $G$ inherited from $\text{Sym}(X)$ in this way.

4.1.1. Countability, separation, and connection. A basis (or base) of $S$ is a collection of open subsets of $S$ such that every open set in $S$ is the union of sets from the collection. Clearly, a topology is uniquely given by any of its bases. For $p \in S$, a collection of open subsets of $S$ is called a basis at $p$ if each set from the
collection contains \( p \), and every open set containing \( p \) also contains an open set from the collection. A topological space \( S \) is called
- **first-countable** if for all \( s \in S \) there exists a countable basis at \( s \);
- **second-countable** if it has a countable basis;
- **separable** if it contains a countable dense set.

Note that being second-countable implies first-countable (to obtain a countable basis at \( s \in S \), select all members of the countable basis of \( S \) that contain \( s \).) Every second-countable space is also separable: if \( \{ U_1, U_2, \ldots \} \) is a countable basis of nonempty sets, choosing any \( x_n \in U_n \) gives a countable dense set \( \{ x_1, x_2, \ldots \} \). (Suppose for contradiction that there exists \( p \in S \setminus \{ x_1, x_2, \ldots \} \). Then there is a closed set \( V \) containing \( \{ x_1, x_2, \ldots \} \) but not \( p \). Hence, there is an open set \( U \) that contains \( p \) but no point in \( \{ x_1, x_2, \ldots \} \). Since \( \{ U_1, U_2, \ldots \} \) is a basis, there exists an \( i \in \mathbb{N} \) such that \( U_i \subseteq U \). But then \( x_i \in U_i \subseteq U \), a contradiction.)

**Example 4.1.7.** We revisit the examples that we have seen above.

- The standard topology on \( \mathbb{R} \) and, more generally, on \( \mathbb{R}^d \) for \( d \in \mathbb{N} \) are second-countable (and in particular first-countable), a countable basis being the set of all open balls with rational center and rational radius.
- The topology of pointwise convergence on \( \text{Sym}(\mathbb{N}) \) is second-countable: there are countably many basic open sets \( S(a, b) \) with \( a, b \in \mathbb{N}^n \) for \( n \in \mathbb{N} \).
- The discrete topology on any set \( S \) is first-countable (for every \( p \in S \), the set \( \{ \{ p \} \} \) is a basis at \( p \)), but if \( S \) is uncountable, \( S \) is not separable (we have \( \bar{U} = U \) for all \( U \subseteq X \), and therefore also not second-countable.
- The cofinite topology on an uncountably infinite set \( S \) is not first-countable, but separable: the closure of any countably infinite subset of \( S \) is \( S \).

A topological space \( S \) is called **Hausdorff** if for any two distinct points \( u, v \) of \( S \) there are disjoint open sets \( U \) and \( V \) that contain \( u \) and \( v \), respectively.

**Example 4.1.8.** The standard topology on \( \mathbb{R}, \mathbb{R}^d, \mathbb{Q} \), and the topology of pointwise convergence on \( \text{Sym}(\mathbb{N}) \) are Hausdorff. The trivial topology on a set with at least two elements is not Hausdorff. The cofinite topology on an infinite set is a more interesting example that is not Hausdorff: the intersection of any two non-empty open sets is infinite, so in particular not disjoint. So we cannot separate two distinct points with open disjoint sets.

In what concerns countability and separation properties, \( \mathbb{R} \) and \( \text{Sym}(\mathbb{N}) \) share many properties. But in some other respects, these spaces are very different. A topological space \( S \) is called **disconnected** if it is the union of two or more disjoint nonempty open subsets, and **connected** otherwise. A subset of \( S \) is said to be **connected** if it is connected under its subspace topology. The inclusion-wise maximal connected subsets of a non-empty topological space are called the **connected components** of that space. Picture in \( \mathbb{R}^2 \)!

**Example 4.1.9.** The standard topology on \( \mathbb{Q} \) is disconnected: the two sets
\[ \{ x \in \mathbb{Q} \mid x < \pi \} \] and \( \{ x \in \mathbb{Q} \mid x > \pi \} \)
are open, and for any irrational \( \pi \) they partition \( \mathbb{Q} \). On the other hand, \( \mathbb{R} \) and \( \mathbb{R}^d \) are connected\(^1\).

A topological space \( S \) is **totally disconnected** if all connected subsets of \( X \) are one-element sets.

\(^1\)This relies on the fact that the real numbers are (by definition!) **Dedekind-complete**: every non-empty subset \( S \) of \( \mathbb{R} \) with an upper bound in \( \mathbb{R} \) has a least upper bound.
Example 4.1.10. The topology of pointwise convergence on $\text{Sym}(\mathbb{N})$ is totally disconnected: if $f, g \in \text{Sym}(\mathbb{N})$ are distinct, there exists an $i \in \mathbb{N}$ such that $f(i) \neq g(i)$. Then $S(i, f(i))$ is an open set that contains $f$ and $S(i, g(i))$ is a disjoint open set that contains $g$.

Exercises.

(33) Show the claim from Example 4.1.9 that the standard topology on $\mathbb{R}$ is connected.

(34) On which sets $X$ is the cofinite topology separable?

(35) Show that $\mathbb{R}$ with the standard topology is not homeomorphic to a closed subgroup of $\text{Sym}(\mathbb{N})$.

(36) Show that the cofinite topology on an uncountably infinite set is not first-countable.

4.1.2. Continuity and convergence. A mapping between two topological spaces is called continuous if the pre-images of open sets are open, and open if images of open sets are open. A bijective open and continuous map is called a homeomorphism.

Exercises.

4.1.11. Example

Finally, the implication from (3) to (1) again holds in arbitrary topological spaces. Let $V \subseteq T$ be open. We want to show that $U := f^{-1}(V)$ is empty. When $s$ is a point from $U$, then because $f$ is continuous at $s$, and $V$ contains $f(s)$ and is open, there is an open set $U_s \subseteq S$ containing $s$ whose image $f(U_s)$ is contained in $V$. Then $\bigcup_{s \in U} U_s = U$ is open as a union of open sets. □
4.1.3. **Product spaces.** The product $\prod_{i \in I} S_i$ of a family of topological spaces $(S_i)_{i \in I}$ is the topological space on the cartesian product $\prod_{i \in I} S_i$ where the open sets are unions of sets of the form $\prod_{i \in I} U_i$ where $U_i$ is open in $S_i$ for all $i \in I$, and $U_i = S_i$ for all but finitely many $i \in I$. When $I$ has just two elements, say 1 and 2, we also write $S_1 \times S_2$ for the product (this operation is clearly associative and commutative). We denote by $S^k$ for the $k$-th power $S \times \cdots \times S$ of $S$, equipped with the product topology as described above.

We also write $S^I$ to a $|I|$-th power of $S$, where the factors are indexed by the elements of $I$. In this case, we can view each element of $T := S^I$ as a function from $I$ to $S$ in the obvious way. The product topology on $T$ is also called the topology of pointwise convergence, due to the following.

**Proposition 4.1.12.** Let $S$ be a topological space, and $I$ be a set. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of elements of the product space $T := S^I$. Then $\lim_{n \to \infty} f_n = f$ if and only if $\lim_{n \to \infty} f_n(j) = f(j)$ in $S$ for all $j \in I$.

**Proof.** Suppose first that $\lim_{n \to \infty} f_n = f$ in $T$. Let $j \in I$ be arbitrary and let $V$ be an open set that contains $f(j)$. Then the set $U := \prod_{i \in I} T_i$ where $T_i = V$ if $i = j$, and $T_i = S_i$ otherwise, is open in $T$ and contains $f$. So there is an $n_0$ such that $f_n \in U$ for all $n \geq n_0$. But then $f_n(j) \in V$ for all $n \geq n_0$, and so $\lim_{n \to \infty} f_n(j) = f(j)$.

Now suppose that $\lim_{n \to \infty} f_n(j) = f(j)$ in $S$ for all $j \in I$, and let $V$ be an open set of $T$ that contains $f$. Then there exists a finite $J \subseteq I$ and open subsets $(V_j)_{j \in J}$ of $S$ such that $f \in \prod_{i \in I} T_i$ where $T_i = V_i$ if $i \in J$ and $T_i = S_i$ otherwise. For each $j \in J$ there exists an $n_j$ so that $f_n(j) \in V_j$ for all $n \geq n_j$. Then $f_n \in V$ for all $n \geq \max_{j \in J} n_j$, and hence $\lim_{n \to \infty} f_n = f$. \qed

Since a topological space is Hausdorff if and only if limits of converging sequences are unique, it follows from Proposition 4.1.12 that products of Hausdorff spaces are Hausdorff.

**Example 4.1.13.** When we equip the natural numbers $\mathbb{N}$ with the discrete topology, then $\mathbb{N}^k$ with the topology of pointwise convergence is called the *Baire space*. Note that the open sets are exactly the unions of sets of the form

$$S(\bar{a}, \bar{b}) := \{ g \in \mathbb{N} \to \mathbb{N} \mid g(\bar{a}) = \bar{b} \}$$

for some $\bar{a}, \bar{b} \in \mathbb{N}^k$, $k \in \mathbb{N}$. Exercise: prove this. \qed

**Theorem 4.1.14 (Baire).** $\mathbb{N}^\ast$ is homeomorphic to the irrational numbers $\mathbb{P}$.

**Proof.** It suffices to construct a mapping from $\mathbb{Z}^N$ to $\mathbb{P}$. Note that the sets of the form $\{ x \in \mathbb{Z}^N \mid x_1 \ldots x_n = s \}$ for $s \in \mathbb{Z}^\ast$ form a basis of open sets for $\mathbb{Z}^N$. Let $(q_n)_{n \in \mathbb{N}}$ be an enumeration of the rational numbers. Inductively construct a sequence of open intervals $(I_w)_{w \in \mathbb{Z}^\ast}$ satisfying the following:

1. $I_\epsilon = \mathbb{R}$;
2. if $s \in w \in \mathbb{Z}^\ast \setminus \{ \epsilon \}$ then $I_s$ is an open interval in $\mathbb{R}$ with rational endpoints,
3. for every $s \in \mathbb{Z}^\ast$ and $n \in \mathbb{Z}$ we have $I_{sn} \subseteq I_s$,
4. the right end point of $I_{sn}$ is the left end point of $I_{s(n+1)}$,
5. $\{ I_n \mid n \in \mathbb{Z} \}$ covers $I_\epsilon \cap \mathbb{P}$,
6. the length of $I_s$ is less than $1/|s|$ for $s \neq \epsilon$, and
7. the $s$-th rational $q_s$ is an endpoint of $I_s$ for some $s \in \mathbb{Z}^\ast$ with $|s| \leq n + 1$.

Define the function $f: \mathbb{Z}^N \to \mathbb{P}$ as follows. Given $x \in \mathbb{Z}^N$ the set $\bigcap_{n \in \mathbb{N}} I_{x_1 \ldots x_n}$ must consist of a singleton irrational:

- it is nonempty because $I_{x_1 \ldots x_{n+1}} \subseteq I_{x_1 \ldots x_n}$;
- it is a singleton because the length of $I_{x_1 \ldots x_n}$ tends to zero for $n \to \infty$. 

\[ \text{□} \]
So we can define \( f \) by

\[
\{ f(x) \} := \bigcap_{n \in \mathbb{N}} I_{x_1 \ldots x_n}.
\]

The function \( f \) is injective because if \( s \) and \( t \) are not prefixes of each other then \( I_s \) and \( I_t \) are disjoint, and \( f \) is surjective because for every \( u \in \mathcal{P} \) and \( n \in \mathbb{N} \) there is a unique \( s \in \mathbb{Z}^* \) of length \( n \) with \( u \in I_s \). Finally, \( f \) is a homeomorphism because

\[
f(\{x \in \mathbb{Z}^n | x_1 \ldots x_n = s\}) = I_s \cap \mathcal{P}
\]

and the sets of the form \( I_s \cap \mathcal{P} \) form a basis for \( \mathcal{P} \). □

### 4.1.4. Metric spaces

Important examples of topologies come from metric spaces. A metric space is a pair \((M, d)\) where \( M \) is a set and \( d \) is a metric on \( M \), i.e., a function

\[
d : M \times M \to \mathbb{R}
\]

such that for any \( x, y, z \in M \), the following holds:

1. \( d(x, y) \geq 0 \) (non-negativity)
2. \( d(x, y) = 0 \) if and only if \( x = y \) (indiscernibility)
3. \( d(x, y) = d(y, x) \) (symmetry)
4. \( d(x, z) \leq d(x, y) + d(y, z) \) (subadditivity or triangle inequality)

When \( M' \subseteq M \) then the restriction of \( d \) to \( M' \) is clearly a metric, too. Every metric on \( M \) gives rise to a topology on \( M \), namely the topology with the basis

\[
\{ \{ y \in M | d(x, y) < \epsilon \} | 0 < \epsilon \in \mathbb{R}, x \in M \}.
\]

A topological space \( S \) is **metrisable** if there exists a metric \( d \) on \( S \) which is compatible with the topology, i.e., the topology equals the topology that arises from the metric as described above.

**Example 4.1.15.** The discrete metric \( \rho \) on \( X \) is defined by

\[
\rho(x, y) = \begin{cases} 
 1 & \text{if } x \neq y, \\
 0 & \text{if } x = y
\end{cases}
\]

for any \( x, y \in X \). In this case \((X, \rho)\) is called a **discrete metric space** or a **space of isolated points**.

**Example 4.1.16.** The distance function \( d(x, y) = |x - y| \) (absolute difference) defines a metric on \( \mathbb{R} \), \( \mathbb{R}^d \), and on \( \mathbb{Q} \). The topology that arises from this metric is precisely the standard topology on \( \mathbb{R} \), \( \mathbb{R}^d \), and on \( \mathbb{Q} \).

A sequence \((s_n)_{n \in \mathbb{N}}\) of elements of a metric space \((M, d)\) is called a **Cauchy sequence** if

\[
\forall \epsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n, m > n_0 : d(s_n, s_m) < \epsilon ;
\]

in this case, we write

\[
\lim_{n, m \to \infty} d(s_n, s_m) = 0.
\]

A metric space \((M, d)\) is called **complete** if every Cauchy sequence converges against an element of \( M \).

**Example 4.1.17.** The standard distance metric on \( \mathbb{R} \) is complete. The same metric on \( \mathbb{Q} \) is not complete.

A topological space \( S \) is called **completely metrizable** if it has a compatible complete metric. It is called **Polish** if \( S \) is separable and completely metrizable.
Example 4.1.18. The symmetric group \( \text{Sym}(D) \) on a countably infinite set \( D \) has the following compatible metric \( d \). Let \( b_1, b_2, \ldots \) be an enumeration of \( D \). Then for elements \( f, g \in \mathcal{G} \) we define \( d(f, g) := 0 \) if \( f = g \), and otherwise \( d(f, g) := 1/2^{n+1} \) where \( n \) is the least natural number such that \( f(b_n) \neq g(b_n) \). In fact, \( d \) is an \textit{ultrametric}, that is, it satisfies

\[
d(x, z) \leq \max(d(x, y), d(y, z))
\]

for all \( x, y, z \). This metric is not complete: to see this, let \( f \) be an arbitrary injective non-surjective mapping from \( D \to D \). For each \( n \), there exists a permutation \( h_n \) of \( D \) such that \( h_n(b_i) = f(b_i) \) for all \( i \leq n \). Hence, the sequence \( (h_n)_{n \geq 1} \) is Cauchy, but it does not converge to a permutation.

Example 4.1.19. Similarly to the previous example, the Baire space (Example 4.1.13) can be equipped with a left-invariant metric: define \( d(f, g) := 0 \) if \( f = g \), and otherwise \( d(f, g) := 1/2^{n+1} \) where \( n \) is the least natural number such that \( f(n) \neq g(n) \). For the Baire space, this is a complete metric.

Example 4.1.20. The topology on \( \text{Sym}(D) \) on a countably infinite set \( D \) is also completely metrizable. Again, let \( b_1, b_2, \ldots \) be an enumeration of \( D \). We define a compatible complete metric \( d' \) on \( \text{Sym}(D) \) by setting \( d'(f, g) := 0 \) if \( f = g \), and otherwise \( d'(f, g) = 1/2^{n+1} \) where \( n \) is the least natural number such that \( f(b_n) \neq g(b_n) \) or \( f^{-1}(b_n) \neq g^{-1}(b_n) \). Since \( \text{Sym}(D) \) is also separable, we have that \( \text{Sym}(D) \) is Polish.

Let \( (M_1, d_1) \) and \( (M_2, d_2) \) be two metric spaces. An \textit{isometry} between \( (M_1, d_1) \) and \( (M_2, d_2) \) is a function \( i: M_1 \to M_2 \) such that \( d_1(x_1, x_2) = d_2(i(x_1), i(x_2)) \) (note that \( i \) must be injective, but it is not required to be surjective). Two metrics are called \textit{isometric} if there exists a bijective isometry between them.

Metric spaces have the advantage that we can use Cauchy sequences to talk about points that aren’t really there. More formally:

Definition 4.1.21. A completion of a metric space \( (M, d) \) is a complete metric space \( (M^*, d^*) \) together with an isometry \( i: M \to M^* \) such that \( i(M) \) is dense in \( M^* \).

Proposition 4.1.22. Every metric space has a completion.

Proof. Let \( (M, d) \) be a metric space. Let \( C \) be the collection of all Cauchy sequences in \( M \). Define an equivalence relation \( \sim \) on \( C \) as follows:

\[
(x_n) \sim (y_n) \iff \lim_{n \to \infty} d(x_n, y_n) = 0
\]

Define

- \( M^* \) to be the set of all equivalence classes of \( \sim \),
  \[ X^* := \{ [(x_n)] \mid (x_n) \in C \} \]
- \( d^*: X^* \times X^* \to \mathbb{R} \) by
  \[
d^*([(x_n)], [(y_n)]) := \lim_{n \to \infty} d(x_n, y_n)
\]
  for \( [(x_n)], [(y_n)] \in M^* \), and
- \( i: M \to M^* \) by
  \[
  \phi(x) := [(x, x, \ldots)]
  \]
  for \( x \in M \).
Claim 1. \(d^*\) is well-defined. Let \((x'_n)\) and \((y'_n)\) be two Cauchy sequences such that \((x'_n) \sim (x_n)\) and \((y'_n) \sim (y_n)\). By the triangle inequality
\[
d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n)\\
d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n)
\]
and thus
\[
|d(x_n, y_n) - d(x'_n, y'_n)| \leq |d(x'_n, y'_n) - d(x_n, y_n)|
\]
which tends to 0 for \(n \to \infty\), and proves that \(\lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(x'_n, y'_n)\).

Claim 2. \(d^*\) is a metric on \(M^*\). This is straightforward.

Claim 3. \(d^*\) is an isometry:
\[
d^*(x, y) = \lim_{n \to \infty} d(x_n, y_n) = d(x, y).
\]

Claim 4. \(i(M)\) is dense in \(M^*\). Let \([x_1] \in M^*\) and \(\epsilon > 0\). Since \((x_n)\) is Cauchy, there exists an \(n_0 \in \mathbb{N}\) such that \(d(x_m, x_n) < \epsilon\) for all \(m, n \geq n_0\). For \(z := i(x_{n_0})\) we have
\[
d^*([x_1], i(z)) = \lim_{n \to \infty} d(x_n, x_{n_0}) \leq \epsilon.
\]

Claim 5. \(d^*\) is complete. Let \([x_1^n]\), \([x_2^n]\), \ldots be Cauchy in \((M^*, d^*)\). Diagonal argument: We define a function \(k : \mathbb{N} \to \mathbb{N}\) as follows. Set \(k(1) = 1\), and \(k(2)\) such that \(d(x_2^l, x_2^m) < 1/l\) whenever \(l \geq k(2)\). For \(s \in \mathbb{N}\), choose \(k(s)\) such that
\[
N(s) \geq N(s - 1)\\
d(x_k^{k(s)}, x_k^{k(t)}) < 1/k \quad \text{whenever} \quad l \geq k(s).
\]
Then \((x_k^{k(s)})\) is a Cauchy sequence:
\[
d(x_k^{k(s)}, x_k^{k(t)}) \leq \lim_{j \to \infty} d(x_k^{k(s)}, x_k^{s}) + d(x_k^{s}, x_k^{j}) + d(x_k^{j}, x_k^{k(t)})
\]
which tends to 0 for \(n, m \to \infty\).

Moreover, \(\lim_{m \to \infty} [x_m^n] = [x_{n_0}^n]\): let \(\epsilon > 0\) and choose \(n_0 \in \mathbb{N}\) such that \(1/n_0 < \epsilon/2\) and if \(m, n \geq n_0\) then \(d(x_m^n, x_m^n) < \epsilon/2\). Now, for \(m \geq n_0\):
\[
d^*([x_m^n], [x_{n_0}^n]) = \lim_{n \to \infty} d(x_m^n, x_{n_0}^n)
\]
\[
= \limsup_{n \to \infty} d(x_m^n, x_{k(m)}^n) + \limsup_{n \to \infty} d(x_{k(m)}^n, x_{n_0}^n)
\]
\[
\leq 1/n_0 + \epsilon/2 < \epsilon
\]
\(\square\)

We will in the following we refer to the completion of \((M, d)\) because completions are essentially unique:

Proposition 4.1.23. Let \((M_1^*, d_1^*, i_1)\) and \((M_2^*, d_2^*, i_2)\) be two completions of \((M, d)\). Then there is a unique bijective isometry \(f\) between \(M_1^*\) and \(M_2^*\) such that \(f \circ i_1 = i_2\).

Proof. Let \(x \in M_1^*\). Since \(i_1(M)\) is dense in \(M_1^*\) for each \(n \in \mathbb{N}\) there exists \(x_n \in i_1(M)\) with \(d_1^*(x_n, x) \leq 1/n\). Let \(y_n := i_2(i_1^{-1}(x_n))\). Since \(i_1\) and \(i_2\) are isometries, we have \(d_2^*(y_n, y_m) = d_1^*(x_n, x_m)\) for all \(m, n \in \mathbb{N}\). The sequence \((x_n)\) converges against \(x\), so it is Cauchy, and it follows that \((y_n)\) is Cauchy, too. Since \(M_2^*\) is complete the sequence \((y_n)\) must converge to some \(y \in M_2^*\).
Claim 1. The map \( f : M^*_1 \to M^*_2 \) defined by \( f(x) := y \) is well-defined. Suppose that \((x'_n)_{n \in \mathbb{N}}\) is another sequence of elements of \( M_1 \) that converges to \( x \). For \( n \in \mathbb{N} \), let \( y'_n := i_2(i_1^{-1}(x'_n)) \); we have to show that \( \lim_{n \to \infty} y'_n = y \).

Let \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} y_n = x \) there exists \( m \in \mathbb{N} \) such that \( d^*_1(y_n, y) < \varepsilon/2 \) for all \( n \geq m \). There is also a \( k \in \mathbb{N} \) such that for all \( n \geq k \) we have \( d^*_1(x, x'_n) < \varepsilon/4 \) and \( d^*_1(x'_n, x) < \varepsilon/4 \). Hence, \( d_1(x_n, x'_n) \leq d^*_1(x, x'_n) + d^*_1(x'_n, x) + d^*_1(y_n, y) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2 \). Since \( i_1 \) and \( i_2 \) are isometries we have \( d_2(y_n, y'_n) = d_1(x_n, x'_n) \). Hence, \( d_2(y_n, y'_n) < \varepsilon/2 \) for all \( n \geq k \). So for \( n \geq \max(k, m) \) we have \( d_2^*(y'_n, y) \leq d_2^*(y'_n, y_n) + d_2(y_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon \), showing that \( \lim_{n \to \infty} y'_n = y \).

Claim 2. \( f \) is an isometry. Let \( x, x' \in M^*_1 \) and let \((x_n)_{n \in \mathbb{N}}\) and \((x'_n)_{n \in \mathbb{N}}\) be sequences of elements of \( i_1(M) \) that converge to \( x \) and \( x' \), respectively. Define \( y_n := i_2(i_1^{-1}(x_n)) \) and \( y'_n := i_2(i_1^{-1}(x'_n)) \), and we have seen that \( \lim_{n \to \infty} y_n = f(x) \) and \( \lim_{n \to \infty} y'_n = f(x') \). Then
\[
d^*_2(f(x), f(x')) = \lim_{n \to \infty} d_2(y_n, y'_n) = \lim_{n \to \infty} d_1(x_n, x'_n) = d(x, x').
\]

Claim 3. \( f \) is surjective. Let \( y \in M^*_2 \). Since \( i_2(M) \) is dense in \( M^*_2 \) there is a sequence \((y_n)_{n \in \mathbb{N}}\) of elements of \( i_2(M) \) converging to \( y \). Similarly as above it can be shown that \((i_1(i_2^{-1}(y_n)))_{n \in \mathbb{N}}\) is a sequence in \( i(M_1) \) that converges to some \( x \in M^*_1 \), and that \( f(x) = y \). \( \square \)

If the isometry \( i \) of a metric completion \((M^*, d^*)\) of \((M, d)\) is not specified, we typically assume that \( M \subseteq M^* \) and \( i \) is the identity. Clearly, the completion of a separable metric space is separable, too.

Example 4.1.24. \((\mathbb{R}; d)\) is the completion of \((\mathbb{Q}; d)\).

Example 4.1.25. Let \( d \) be the ultrametric on \( \text{Sym}(\mathbb{N}) \) from Example 4.1.18. Then the metric completion of \((\text{Sym}(\mathbb{N}), d)\) is (isometric to) the Baire space with the left-invariant complete metric introduced in Example 4.1.19.

We finally state an important property of completely metrizable spaces.

Theorem 4.1.26 (The Baire Category Theorem). Every Polish\(^2\) space \( S \) is Baire, i.e., has the property that countable intersections of dense open sets are dense.

Proof. Let \((U_n)_{n \in \mathbb{N}}\) be a sequence of open dense sets. We want to show that \( U := \bigcap U_n \) is dense. It is sufficient to show that any non-empty open set \( W \) in \( S \) contains an element of \( U \). Since \( U_1 \) is dense, there is \( x_1 \in U_1 \cap W \). Hence, there is an \( r_1 \) with \( 0 < r_1 < 1 \) such that \( \{ z \in S \mid d(x_1, z) \leq r_1 \} \subseteq U_1 \cap W \) where \( d \) is the compatible complete metric. We can continue recursively to find a sequence \((x_n)_{n \in \mathbb{N}}\) of elements of \( S \) and a sequence \((r_n)_{n \in \mathbb{N}}\) of elements of \( \mathbb{R} \) such that
\[
\{ z \in S \mid d(z, x_n) \leq r_n \} \subseteq U_n \cap B_{r_{n+1}}(x_{n+1})
\]
as follows: if we have defined \( x_1, \ldots, x_n \) and \( r_1, \ldots, r_n \) satisfying (1), then density of \( U_{n+1} \) guarantees the existence of an element \( x_{n+1} \) in the open set \( B_{r_n}(x_n) \cap U_{n+1} \).

Then there is \( r_{n+1} \) such that \( \{ z \in S \mid d(z, x_{n+1}) \leq r_{n+1} \} \subseteq U_{n+1} \cap B_{r_n}(x_n) \). Since \( x_n \in B_{r_n}(x_n) \) for all \( n > m \), we have that \( (x_n)_{n \in \mathbb{N}} \) is Cauchy, and hence converges to some limit \( x \) by completeness of \( d \). For any \( n \), the set \( \{ z \in S \mid d(z, x_n) \leq r_n \} \) is closed and hence contains \( x \). Therefore, \( x \in W \) and \( x \in U_n \) for all \( n \).

\(^2\)Using the axiom of dependent choices (DC), this assumption can be weakened from Polish to completely metrisable (in fact, the modified statement is then equivalent to DC).

\(^3\)Here we use the axiom of choice.
4.1.5. Uniform continuity. Given metric spaces \((X,d_1)\) and \((Y,d_2)\), a function \(f: X \to Y\) is called uniformly continuous if
\[
\forall \epsilon > 0 \exists \delta > 0 \forall x,y \in X : (d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon).
\]
For comparison: continuity of \(f\) with respect to the topologies induced by \(d_1\) and \(d_2\) only requires that
\[
\forall \epsilon > 0, \exists \delta > 0 \forall x,y \in X : (d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon).
\]

Example 4.1.27. The function \(x \mapsto x^2\) from \(\mathbb{R} \to \mathbb{R}\) is continuous, but not uniformly continuous: given an arbitrarily small positive real \(\epsilon\), uniform continuity requires the existence of a positive number \(\delta\) such that for all \(x_1, x_2\) with \(|x_1 - x_2| < \delta\) we have \(|f(x_1) - f(x_2)| < \epsilon\). But \((x + \delta')^2 - x^2 = 2x\delta' + (\delta')^2\) is larger than \(\epsilon\) for sufficiently large \(x\).

Example 4.1.28. An endomorphism \(\xi\) of the Baire space (Example 4.1.19) is uniformly continuous if for every finite \(F \subseteq \mathbb{N}\) there exists a finite \(G \subseteq \mathbb{N}\) such that for all \(f, g \in \mathbb{N}^\mathbb{N}\) if \(f|_G = g|_G\) then \(\xi(f)|_F = \xi(g)|_F\).

For comparison: an endomorphism of the Baire space is continuous if and only if for every finite \(F \subseteq \mathbb{N}\) and every \(f \in \mathbb{N}^\mathbb{N}\) there exists a finite \(G \subseteq \mathbb{N}\) such that if \(g \in \mathbb{N}^\mathbb{N}\) is such that \(f|_G = g|_F\) then \(\xi(f)|_F = \xi(g)|_F\).

Proposition 4.1.29. A uniformly continuous map \(f\) between metric spaces maps Cauchy sequences to Cauchy sequences.

Proof. Let \((s_n)_{n \in \mathbb{N}}\) be a Cauchy sequence, and let \(\epsilon > 0\). By uniform continuity of \(f\) there exists \(\delta > 0\) such that \(d(f(x) - f(y)) < \epsilon\) for \(d(x - y) < \delta\). Since \((s_n)\) is Cauchy, there exists an \(n_0 > 0\) such that \(d(s_n - s_m) < \delta\) for all \(n, m > n_0\). Hence, \(d(f(s_n) - f(s_m)) < \epsilon\) for all \(n, m > n_0\). Therefore, \((f(s_n))_{n \in \mathbb{N}}\) is Cauchy.

4.1.6. Compactness. A topological space \(S\) is called compact if for an arbitrary collection \(\mathcal{U} = \{U_i\}_{i \in A}\) of open subsets of \(S\) with \(S = \bigcup_{i \in A} U_i\) (also called an open cover) there is a finite subset \(B\) of \(A\) such that \(S = \bigcup_{i \in B} U_i\) (the collection \(\{U_i\}_{i \in B}\) is called a subcover of \(\mathcal{U}\)). Clearly, finite spaces are compact. We state some closure properties for compactness.

Proposition 4.1.30. Closed subsets of compact spaces are compact.

Proof. Let \(T\) be a compact space and let \(C\) be a closed subspace of \(T\). Let \(\mathcal{U}\) be an open cover of \(C\). By assumption, \(T \setminus C\) is open in \(T\). Hence, \(\mathcal{U} \cup \{T \setminus C\}\) is an open cover of \(T\). As \(T\) is compact, there is a finite subcover of \(\mathcal{U}\), say \(\{U_{i_1}, U_{i_2}, \ldots, U_{i_r}\}\). This also covers \(C\) by the fact that it covers \(T\). If \(T \setminus C\) is among \(U_{i_1}, U_{i_2}, \ldots, U_{i_r}\), then it can be removed and the remaining sets still cover \(C\). Thus we have found a finite subcover of \(\mathcal{U}\) which covers \(C\), and hence \(C\) is compact.

The following is more substantial (actually, equivalent to the axiom of choice).

Theorem 4.1.31 (Tychonoff). Products of compact spaces are compact.

Proof. We only prove the statement for countable products of compact spaces; this is all that will be needed in this text anyway. We first show that if \(X\) and \(Y\) are compact, then so is \(X \times Y\). Let \(\mathcal{U}\) be a collection of open subsets of \(X \times Y\) such that no finite subset of \(\mathcal{U}\) covers \(X \times Y\); we will show that \(\mathcal{U}\) does not cover \(X \times Y\).

Claim 1. There exists \(x_0 \in X\) such that for every open \(U \subseteq X\) that contains \(x_0\) the set \(U \times Y\) is not finitely covered by \(\mathcal{U}\). Suppose otherwise that for every \(x \in X\) there exists an open set \(U_x \subseteq X\) that contains \(x\) such that \(U_x \times Y\) is covered by finitely many elements of \(\mathcal{U}\). By compactness of \(X\), finitely many of the \(U_x\) cover \(X\), so finitely many sets of the form \(U_x \times Y\) cover \(X \times Y\), contradicting the assumptions.
Claim 2. There exists \( y_0 \in Y \) such that for every open \( U \subseteq X \) that contains \( x_0 \) and every open \( V \subseteq Y \) that contains \( y_0 \) no finite subset of \( U \) covers \( U \times V \). Otherwise, for every \( y \in Y \) there is an open \( U_y \subseteq X \) containing \( x_0 \) and an open \( V_y \subseteq Y \) containing \( y \) such that \( U_y \times V_y \) is covered by finitely many elements of \( U \). By compactness of \( Y \), there is a finite subset \( F \subseteq Y \) such that \( Y = \bigcup_{y \in F} V_y \). Set \( U := \bigcap_{u \in F} U_y \). Then \( U \) is open and contains \( x_0 \), and

\[
U \times Y = \bigcup_{y \in F} U \times V_y \subseteq \bigcup_{y \in F} U_y \times V_y
\]

is covered by finitely many elements of \( U \), contradicting Claim 1.

It follows that no basic open set containing \((x_0, y_0)\) is covered by finitely many elements of \( U \). In particular, no basic open set containing \((x_0, y_0)\) can be contained in an element of \( U \), so \((x_0, y_0)\) is not covered by \( U \). This finishes the proof that \( X \times Y \) is compact. To prove the statement for countable products, we first slightly generalise the proof of Claim 2 to get the following.

Claim 3. Suppose that \( U \) is a family of open subsets of \( X \times Y \times Z \) where \( Y \) is compact, and suppose that there is an \( x_0 \in X \) such that for every open \( U \subseteq X \) that contains \( x_0 \) the set \( U \times Y \times Z \) is covered covered by finitely many elements of \( U \). Then there exists an \( y_0 \in Y \) such that for every open \( U \subseteq X \) that contains \( x_0 \) and every open \( V \subseteq Y \) that contains \( y_0 \), the set \( U \times V \times Z \) is finitely covered by \( U \).

We finally prove that if \( X_1, X_2, \ldots \) are compact, then \( X = \prod_{i \in \mathbb{N}} X_i \) is compact. Let \( U \) be a family of open sets that no finite subset of \( U \) covers \( X \). We will construct an element \( x = (x_1, x_2, \ldots) \) of \( X \) that is not covered by \( U \). Note first that there is an \( x_1 \in X_1 \) such that for every open \( U_1 \subseteq X_1 \) that contains \( x_1 \) the set \( U_1 \times X_2 \times X_3 \times \cdots \) is not finitely covered; the proof is as the proof of Claim 1, with \( X_2 \times X_3 \times \cdots \) playing the role of \( Y \). Next, we can find \( x_2 \in X_2 \) such that such that for every open \( U_1 \subseteq X_1 \) that contains \( x_1 \) and every open \( U_2 \subseteq X_2 \) that contains \( x_2 \) the set \( U_1 \times U_2 \times X_3 \times X_4 \times \cdots \) is not covered by finitely many elements of \( U \); this follows from Claim 3 applied to \( X_1 \times X_2 \times (X_3 \times X_4 \times \cdots) \). Continuing in this way, we inductively define \( x_1, x_2, x_3, \ldots \) such that for each \( n \) and all open \( U_i \subseteq X_i \) for \( i \leq n \) such that \( U_i \) contains \( x_i \), the set \( U_1 \times \cdots \times U_n \times X_{n+1} \times \cdots \) is not covered by finitely many elements of \( U \). The element \((x_1, x_2, \ldots) \in X \) is then not covered by \( U \). □

Exercises.

(37) Prove that a finite union of compact sets is compact.

In order to discuss which subsets of \( \mathbb{R} \) and of \( \text{Sym}(\mathbb{N}) \) are compact (with respect to the subspace topology), we need the following definition for metric spaces.

Definition 4.1.32. A subset \( S \) of a metric space \((M, d)\) is bounded if it is contained in an open ball of finite radius, i.e., if there exists \( x \in M \) and a real \( \epsilon > 0 \) such that for all \( s \in S \), we have \( d(x, s) < \epsilon \).

The open ball of radius \( \epsilon \) and center \( x \) will be denoted by \( B_x(\epsilon) \) in the following.

Example 4.1.33. A subset of \( \mathbb{R}^d \) is compact if and only if it is closed and bounded – this is the theorem of Heine-Borel.

Which subsets of \( \text{Sym}(\mathbb{N}) \) are compact?

Proposition 4.1.34. Any compact subset \( S \) of a Hausdorff topological space \( X \) is closed in \( X \).

Proof. If \( S \) is compact but not closed then there exists \( a \in \overline{S} \setminus S \). For each \( x \in S \) there exists an open set \( U_x \) that contains \( x \) but does not intersect an open set \( V_x \) that contains \( a \), because \( X \) is Hausdorff. Then \( U := \{ U_x \mid x \in S \} \) is an open cover
We claim that \( \bar{U} \text{ intersection. Then} \) a finite subcover of \( X \) is bounded, but the converse is not true: the discrete metric is bounded, but not totally bounded: for every real \( \epsilon > 0 \) there exists a finite collection of open balls in \( M \) of radius \( \epsilon \) whose union contains \( S \). Clearly, a totally bounded space is bounded, but the converse is not true: the discrete metric is bounded, but not totally bounded: for \( \epsilon = 1/2 \), we need infinitely many open \( \epsilon \)-balls (points!) to cover the infinite set.

**Proposition 4.1.36.** If a metric space is totally bounded, then it is separable. A metric space is second countable if and only if it is separable.

**Proof.** If \( X \) is totally bounded then for each positive \( n \in \mathbb{N} \) there exists a finite \( A_n \subseteq X \) such that \( X = \bigcup_{x \in A_n} B_x(1/n) \). Let \( A := \bigcup_{n \geq 0} A_n \). Clearly \( A \) is countable.

We claim that \( A = X \). Let \( x \in X \). For any \( n \in \mathbb{N} \) there is some \( y_n \in A_n \) such that \( x \in B_{y_n}(1/n) \). This gives a sequence \( (y_n) \) with \( d(x, y_n) < 1/n \). Thus \( \lim_{n \to \infty} y_n = x \) which proves the claim, and separability of \( X \).

For the second part, suppose first that \( X \) is separable, and let \( A \) be the countable dense set. Then open balls with rational radii and centres from \( A \) form a countable basis. Why? Conversely, when \( X \) is second-countable, we choose for every \( U \) from a countable base of \( X \) one element; this will give a countable dense subset of \( X \).

**Theorem 4.1.37.** For a metric space \( (X, d) \), the following are equivalent.

1. \( X \) is compact;
2. Every collection of closed sets in \( X \) with the finite intersection property (every finite subcollection has a nonempty intersection) has a nonempty intersection;
3. If \( F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots \) is a decreasing sequence of nonempty closed sets in \( X \), then \( \bigcap_{n \geq 1} F_n \) is nonempty;
4. \( X \) is sequentially compact, that is, every sequence in \( X \) has a convergent subsequence;
5. \( X \) is totally bounded and complete.

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( \mathcal{C} \) is a collection of closed sets with empty intersection. Then \( \mathcal{U} := \{ X \setminus C \mid C \in \mathcal{C} \} \) is an open cover of \( X \), and hence contains a finite subcover of \( X \). The complements of the members of the subcover in \( X \) give the collection with the finite intersection property.

(2) \( \Rightarrow \) (3). A decreasing sequence of non-empty closed sets obviously has the finite intersection property.
(3) ⇒ (4). Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of points in \(X\), and let \(F_n\) be the closure of the set \(\{x_n, x_{n+1}, \ldots\}\). Then \(F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots\) and all the sets \(F_n\) are nonempty and closed. Therefore, by (3), the set \(\bigcap_{n \geq 1} F_n\) contains at least one point \(a\). Then \((x_n)_{n \in \mathbb{N}}\) contains a subsequence converging to \(a\): to see this, let \(d\) be the compatible metric, and set \(n_0 = 1\). Now suppose that \(n_k\) has already been defined for \(k \in \mathbb{N}\). Since \(a\) is in the closure of \(\{x_{n_k+1}, x_{n_k+2}, \ldots\}\) there exists an \(n \in \{n_k+1, n_k+2, \ldots\}\) such that \(d(x_n, a) < 1/(k + 1)\). Let \(n_{k+1}\) the the smallest such \(n\). Then \(\lim_{n \to \infty} x_{n_k} = a\).

(4) ⇒ (5). To prove that \(X\) is complete, let \((x_n)\) be any Cauchy sequence in \(X\). By (4), there is a subsequence converging to some point \(a \in X\). But then the whole sequence \((x_n)\) converges to \(a\). This shows that \(X\) is complete.

Now suppose that \(X\) is not totally bounded, i.e., there exists a number \(\epsilon > 0\) such that \(X\) has no finite covering by open balls of radius \(\epsilon\). Then we can define a sequence \((x_n)_{n \geq 1}\) of points in \(X\) having \(d(x_i, x_j) \geq \epsilon\) for all \(i \neq j\), by the following inductive construction: First let \(x_1\) be any point in \(X\). Then, supposing that \(x_1, \ldots, x_{n-1}\) have been chosen, we know \(B_{x_1}(\epsilon) \cup \cdots \cup B_{x_{n-1}}(\epsilon)\) is not the whole space. Hence we can choose a point \(x_n\) satisfying \(d(x_i, x_n) \geq \epsilon\) for all \(i < n\). On the other hand, the sequence \((x_n)\) cannot have any convergent subsequence; for if it had a subsequence \((x_{n_k})\) converging to \(a\), then there would exist an integer \(k_0\) such that \(d(x_{n_k}, a) < \epsilon/2\) for all \(k \geq k_0\), and hence by the triangle inequality \(d(x_{n_k}, x_{n_j}) < \epsilon\) for all \(k, k' \geq k_0\), contrary to the definition of the sequence \((x_n)\).

(5) ⇒ (4). Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of elements from \(X\). Let \(S = \{x_n \mid n \in \mathbb{N}\}\). If \(S\) is finite then the statement is trivial so assume that \(S\) is infinite. Since \(X\) is totally bounded, there exists a finite cover of \(X\) with open balls of radius \(\epsilon_1 := 1\). One of these balls, call it \(B_{x_1}\), must contain infinitely many elements from \(S\). Again by total boundedness, there exists a finite cover of \(X\) with open balls of radius \(\epsilon_2 := 1/2\), and again, one of those balls must contain infinitely many elements from \(B_{x_1} \cap S\); this ball we call \(B_2\). We continue this process, producing a sequence of balls \((B_k)\) of radius \(1/k\) so that \(B_k \cap B_{k-1} \cap \cdots \cap B_1\) contains infinitely many elements of the sequence \(x_n\). Pick now indices \(n_1 < n_2 < n_3 < \cdots\) such that \(y_k := x_{n_k} \in B_k\). It is easy to see that \((y_n)\) is Cauchy and so by the completeness assumption on \(X\) it must have a convergent subsequence.

(4) ⇒ (1). Let \(U\) be an open cover of \(X\). From the implication (4) ⇒ (5) we have that \(X\) is totally bounded, and Proposition 4.1.35 implies that \(X\) is second-countable. By Proposition 4.1.35 (Lindelöf) we can therefore assume that \(U\) is countable, \(U = \{U_1, U_2, \ldots\}\). Suppose for contradiction that \(U\) does not have a finite subcover. Pick \(x_n \in X \setminus (U_1 \cup \cdots \cup U_n)\) arbitrarily. Then by assumption, the sequence \((x_n)\) has a subsequence \((y_n)\) which converges to some \(y_0 \in X\). Since \(U\) is a cover of \(X\) there is some \(m \in \mathbb{N}\) with \(y \in U_m\). But then \(y_j \notin U_m\) for all \(j \geq m\), which is a contradiction.

We mention that sequential compactness and compactness are not equivalent in general topological spaces; for example, \([0, 1]^\mathbb{R}\) is compact by Theorem 4.1.31 but it can be shown that it is not sequentially compact.

**Local compactness.** A topological space \(S\) is called **locally compact** if every \(p \in S\) is contained in an open set which itself contained in a compact subset of \(S\). Clearly, every compact space \(S\) is also locally compact (take \(S\) itself as compact open set that contains \(p\)).

**Example 4.1.38.** \(\mathbb{R}\) is locally compact, but not compact.

**Example 4.1.39.** The discrete space on \(S\) is locally compact, but only compact if \(S\) is finite.
4.2. Topological Groups

A topological group is an (abstract) group $G$ together with a topology on the elements $G$ of $G$ such that $(x, y) \mapsto xy$ is continuous from $G^2$ to $G$ and $x \mapsto x^{-1}$ is continuous from $G$ to $G$. Two topological groups are said to be isomorphic if the groups are isomorphic, and the isomorphism is a homeomorphism between the respective topologies.

Example 4.2.1. The groups $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$ are topological groups with respect to their standard topology. (Why?)

Example 4.2.2. The elements of the group $G := \text{Sym}(\mathbb{N})$ form a (non-closed!) subset of the Baire space $\mathbb{N}^\mathbb{N}$ (Example 1.1.13), and the topology induced by the Baire space on $\text{Sym}(\mathbb{N})$ is also called the topology of pointwise convergence. Observe that a set of permutations of a set $X$ is a closed subset of $\text{Sym}(X)$ if and only if it is locally closed as defined in Proposition 1.2.1.

Composition is continuous as a map from $G^2 \to G$. If $U \subseteq G$ is a basic open set $S(\bar{a}, \bar{c})$ for $\bar{a}, \bar{c} \in \mathbb{N}^\mathbb{N}$, then the preimage of $U$ is
\[
\{(f, h) \in G^2 \mid f \circ h \in S(\bar{a}, \bar{c})\} = \{(f, h) \in G^2 \mid \exists b (h \in S(\bar{a}, \bar{b}) \text{ and } f \in S(\bar{b}, \bar{c}))\}
\]
which is open as a union of open sets. The preimage of $S(\bar{a}, \bar{b})$ under the inverse map is $S(\bar{b}, \bar{a})$, which is open, too.

Proposition 4.2.3. Let $G$ be a topological group, $g \in G$, and $U \subseteq G$ open. Then $gU$ is open, too. If $U$ is an open subgroup, then it is also closed.

Proof. As a consequence of Proposition 1.1.12, for every $g \in G$ the function $t_g \colon G \to G$ defined by $t_g(x) := hx$ is continuous. The pre-image of $U$ under the function $t_{g^{-1}}$ is $gU$. Therefore, this set is open as the pre-image of an open set under a continuous function. The second part follows since the complement of $U$ in $G$ equals $\bigcup_{g \in G \setminus U} gU$, a union of open sets, hence open.

Proposition 4.2.3. also implies that the topology on $G$ is given by a basis at $1^G$.

Exercises.

(38) Show that a group $G$ with a topology on $G$ is a topological group if and only if the map $(x, y) \mapsto xy^{-1}$ is continuous from $G^2$ to $G$.

(39) Show that for all $n \in \mathbb{N}$ the groups $\text{GL}(n, \mathbb{R})$ and $\text{GL}(n, \mathbb{C})$ of invertible real or complex matrices are topological groups with respect to the standard topology.

A topological group $G$ is Hausdorff (first-countable, metrizable, Polish) if the topology of $G$ is Hausdorff (first-countable, metrizable, Polish, respectively). Note that $G$ is first-countable if and only if $G$ has a countable basis at the identity: if $B$ is a basis of open sets at the identity, and $g \in G$, then $\{g^{-1}(U) \mid U \in B\}$ is a basis at $g$.

4.2.1. Continuous group actions. Recall from Section 1.3.4 that an action of a group $G$ on a set $S$ is a homomorphism from $G$ to $\text{Sym}(S)$.

Definition 4.2.4. An action $\xi$ of a topological group $G$ on a topological space $S$ is called continuous if $(g, s) \mapsto \xi(g)(s)$ is continuous as a map from $G \times S \to S$.

Example 4.2.5. Recall the faithful action of $G$ on $G$ by left multiplication from the proof of Cayley’s theorem, Theorem 1.3.3. This is the special case of Example 1.3.4 where $H = \{1\}$. This action is continuous since it equals the group composition which is continuous by definition.
If $S$ is a topological space, then $\text{Homeo}(S) \subseteq \text{Sym}(S)$ denotes the set of all homeomorphisms of $S$. We view $\text{Homeo}(S)$ as a topological space with the subspace topology inherited from $S^S$ which carries the product topology.

**Proposition 4.2.6.** Every continuous action of a topological group $G$ on a topological space $S$ is a continuous homomorphism from $G$ into $\text{Homeo}(S)$.

**Proof.** Suppose first that $\xi: G \to \text{Sym}(S)$ is a continuous action of $G$ on $S$, so the map $\chi(g,s) := \xi(g)(s)$ is continuous from $G \times S$ to $S$. For every $g \in G$, the map $t_g$ defined by $s \mapsto \chi(g,s)$ is continuous. The inverse of $t_g$ is $s \mapsto \chi(g^{-1},s)$, which is also continuous. Hence, $t_g$ is a homeomorphism. To show that $\xi$ is continuous, let $U$ be a basic open subset of $\text{Homeo}(S)$, i.e., $U = \biguplus_{s \in S} U_s$ where $U_s$ is open in $S$ for all $s \in S$, and there exists a finite set $F$ such that $U_s = S$ for all $s \in S \setminus F$. Note that for fixed $s$, the map $t_s: G \to S$ given by $g \mapsto \xi(g)(s)$ is continuous, and hence for all $s \in F$ the set $\{g \in G \mid \xi(g)(s) \in U_s\}$ is open. Therefore,

$$\xi^{-1}(U) = \{ (g, s) \in G \times S \mid \xi(g)(s) \in U_s \text{ for all } s \in F \} = \bigcap_{s \in F} \{g \in G \mid \xi(g)(s) \in U_s\}$$

is a finite intersection of open sets and hence open. \[\square\]

If $S$ carries the discrete topology (in which case $\text{Homeo}(S) = \text{Sym}(S)$), the statement of Proposition 4.2.6 can be strengthened to obtain an equivalent characterisation of continuity of actions.

**Lemma 4.2.7.** Let $G$ be a topological group and $\xi$ an action of $G$ on a set $S$ equipped with the discrete topology. Then the action $\xi$ is continuous if and only if $\xi$ is continuous as a map from $G$ to $\text{Sym}(S)$.

**Proof.** The forward implication follows from Proposition 4.2.6. For the converse implication, we show that the function $\chi: G \times S \to S$ given by $(g, s) \mapsto \xi(g)(s)$ is continuous. Let $S' \subseteq S$ and $s' \in S'$; it suffices to show that there exists an open $U \subseteq G$ and an open $T \subseteq S'$ such that $\chi(U, T)$ contains $s'$. Since $S$ is discrete, in particular $T := \{s'\}$ is open. Let $U := \xi^{-1}(\text{Sym}(S)|_{s'})$ which is by assumption an open subset of $G$. Then $\chi(U, T)$ contains $s'$. \[\square\]

An important example of a continuous action is the action by conjugation from Example 1.3.5.

**Example 4.2.8.** Let $G$ be a topological group. Then the conjugation action $\xi: G \to G$ given by $\xi(g)(h) := ghg^{-1}$ is continuous since composition and inverse in a topological group are continuous.

Further important examples of continuous actions of a topological group arise from the action by left translation (Example 1.3.4). We also view $G/H$ as a topological space, with the quotient topology. We first define $p: G \to G/H$ by setting $p(g) = gH$ (the projection map). Define $U \subseteq G/H$ to be open if and only if $p^{-1}(U)$ is open in $G$ (in this way, $p$ will necessarily be continuous). In other words, a set of left-cosets is open if their union is open in $G$.

**Proposition 4.2.9.** Let $H$ be an open subgroup of a topological group $G$. Then the action of $G$ on $G/H$ by left translation is continuous.

\[\text{Note that it is not clear (and depends on $S$) whether Homeo}(S) \text{ with this topology is a topological group.}\]
**Proof.** Let \( \xi \) be the action of \( G \) on \( G/H \) by left translation. Let \( S \subseteq G/H \) be open, and let \( gH \in S \). It suffices to show that there are open subsets \( U \subseteq G \) and \( T \subseteq G/H \) such that \( gH \in \{ \xi(u)(t) \mid u \in U, t \in T \} \subseteq S \). By the definition of the quotient topology \( p^{-1}(S) \) is open in \( G \). Since composition in \( G \) is continuous the set \( \{(g_1, g_2) \in G^2 \mid g_1 g_2 \in p^{-1}(S)\} \) is open in \( G^2 \). This set contains \((1, g)\) since \( p(1g) = gH \in S \). So there exists an open \( U \subseteq G \) containing 1 and an open \( V \subseteq G \) containing \( g \) such that \( \{uv \mid u \in U, v \in V\} \subseteq p^{-1}(S) \). Then \( T := p(V) = \{vH \mid v \in V\} \) is open in \( G/H \), and \( gH \in \{\xi(u)(t) \mid u \in U, t \in T\} = \{uvH \mid u \in U, v \in V\} \subseteq S \). \( \square \)

Also, if \( G \leq \text{Sym}(X) \) then for every \( n \in \mathbb{N} \) the componentwise action of \( G \) on \( X^n \) and the setwise action of \( G \) on \( (X^n)_\# \) are continuous. A general result about the continuous actions of a permutation group is Theorem [5.2.1] in Chapter 3. We present an example of a discontinuous group action of an oligomorphic permutation group.

**Example 4.2.10.** Let \( K \) be the class of all finite structures \( (A; E_1, E_2, \ldots) \) where \( E_i \) denotes an equivalence relation on injective \( \mathbb{N} \)-tuples in \( A^i \) with at most two equivalence classes. Clearly, \( K \) is closed under substructures and isomorphism. It is easy to verify that it also has the amalgamation property (Section 3.3). Let \( A \) be the Fraïssé-limit of \( K \). Then \( \text{Aut}(A) \) has a continuous homomorphism \( \xi_1 \) to \( (\mathbb{Z}_2)^\mathbb{N} \) (which is equipped with the product topology): for \( \alpha \in \text{Aut}(A) \), we define \( \xi_1(\alpha) = (\alpha_i)_{i \in \mathbb{N}} \) where \( \alpha_i = 0 \) if \( \alpha \) fixes the equivalence classes of \( E_{i+1} \) and \( \alpha_i = 1 \) otherwise. This map is clearly a (continuous) group homomorphism.

To construct a discontinuous group homomorphism, let \( \mathcal{U} \) be an ultrafilter on \( \mathbb{N} \), and let \( \xi_2 \colon (\mathbb{Z}_2)^\mathbb{N} \rightarrow \mathbb{Z}_2 \) be the function that maps \((\alpha_i)_{i \in \mathbb{N}} \) to 0 if \( \{ i \mid \alpha_i = 0 \} \in \mathcal{U} \), and to 1 otherwise. Again, it is straightforward to verify that \( \xi_2 \) is a group homomorphism. It is continuous if and only if \( \mathcal{U} \) is principal. For a non-principal ultrafilter \( \mathcal{U} \) the map \( \xi_2 \circ \xi_1 \) is a discontinuous group homomorphism from an oligomorphic permutation group to \( \mathbb{Z}_2 \).

\[ \square \]

**Exercises.**

(40) Show that \((\mathbb{Z}_2)^\mathbb{N}\) cannot be isomorphic (as an abstract group) to an oligomorphic permutation group.

**Proposition 4.2.11 (Proposition 13 and Proposition 14 in [30].)** Let \( G \) be a topological group, and let \( H \) be a subgroup of \( G \). Then

- \( H \) is open in \( G \) if and only if \( G/H \) is discrete;
- \( H \) is closed in \( G \) if and only if \( G/H \) is Hausdorff.

**Proof.** Part 1: \( G/H \) is discrete if each left coset \( gH \) is open, which is the case if and only if \( H \) is open.

Part 2: If \( G/H \) is Hausdorff, then every coset of \( H \) is contained in an open set \( O \) that does not intersect \( H \). The union of all those opens is open, and defines the complement of \( H \) in \( G/H \). Hence, \( H \) is closed.

If \( H \) is closed, then the equivalence relation \( R := \{(x, y) \mid x^{-1}y \in H\} \) on \( G \) is closed in \( G \times G \), since it is the inverse image of \( H \) under the continuous mapping \((x, y) \mapsto x^{-1}y \). Let \( g_1H, g_2H \subseteq G/H \) be distinct, that is, \((g_1, g_2) \notin R \). Since \((G/H)^2 \setminus R \) is open and contains \((g_1H, g_2H)\), then by the definition of the product topology there exist open sets \( U_1, U_2 \) such that \((g_1H, g_2H) \subseteq U_1 \times U_2 \subseteq (G/H)^2 \setminus R \). Since \( R \) contains \((C, C)\) for all \( C \subseteq G/H \), the sets \( U_1 \) and \( U_2 \) are disjoint, and this proves that \( G/H \) is Hausdorff. \( \square \)

\(^5\)A tuple is called injective if its entries are pairwise distinct.
4.2.2. Topologically faithful actions. Recall that an action on \( S \) is called *faithful* if it is an injective homomorphism to \( \text{Sym}(S) \). A faithful action of a closed subgroup of \( \text{Sym}(\mathbb{N}) \) on a discrete space \( S \) is called *topologically faithful* if it is continuous and its image is closed in \( \text{Sym}(S) \).

**Example 4.2.12.** Let \( H \) be an open subgroup of \( G \). By Proposition 4.2.11 the quotient space \( G/H \) is discrete, so in particular the image of the action of \( G \) on \( G/H \) by left translation is closed.

We also give an example of a continuous injective homomorphism from a closed subgroup \( G \) of \( \text{Sym}(\mathbb{N}) \) to \( \text{Sym}(\mathbb{N}) \) whose image is not closed. That is, we have a faithful continuous action of \( G \) which is not topologically faithful; \( G \) is even oligomorphic.

**Example 4.2.13.** This example is due to Dugald Macpherson and can be found in Hodges’ model theory [57] (on page 354). Let \( Q \) be the structure \( (Q, <, P) \) where

- \( < \) is the usual strict order of the rational numbers, and
- \( P \subseteq Q \) is such that both \( P \) and \( O := Q \setminus P \) are dense in \( (Q, <) \).

Let \( P \) be the substructure induced by \( P \) in \( Q \). It is easy to see (and follows from more general principles that will be presented in Corollary 5.3.3) that the mapping which sends \( f \in \text{Aut}(Q) \) to \( f|_P \) induces a continuous homomorphism \( \xi \) from \( \text{Aut}(Q) \) to \( \text{Aut}(P) \) whose image is dense in \( \text{Aut}(P) \). We claim that \( \xi \) is not surjective. To prove this, we consider Dedekind cuts \( (S, T) \) of \( P \); that is, partitions of \( P \) into subsets \( S, T \) with the property that for all \( s \in S \) and \( t \in T \) we have \( s < t \). Note that for each element \( o \in O \) we obtain a Dedekind cut \( (S, T) \) with \( S := \{ a \in P \mid a < o \} \) and \( T := \{ a \in P \mid a > o \} \). But since there are uncountably many Dedekind cuts and only countably many elements of \( O \), there also exists a Dedekind cut \( (S', T') \) which is not of this form. By a standard back-and-forth argument, there exists an \( \alpha \in \text{Aut}((P, <)) \) that maps \( S \) to \( S' \) and \( T \) to \( T' \). Suppose for contradiction that there is \( \beta \in \text{Aut}(Q) \) with \( \beta|_P = \alpha \). Then \( s < \beta(a) < t \) for all \( s \in S', t \in T' \), in contradiction to the assumptions on \( (S', T') \).

Some other groups have the remarkable property that *every* faithful continuous action is topologically faithful; this is for example known for \( \text{Sym}(\mathbb{N}) \) (due to [52]; see Theorem 1.3 in [118] for a more recent and more powerful result in this context).

4.2.3. Metrics on topological groups. The (ultra-)metric \( d \) on \( \text{Sym}(D) \) from Example 4.1.18 is left-invariant, i.e., \( d(gh_1, gh_2) = d(h_1, h_2) \) for all \( g, h_1, h_2 \in G \), because if \( n \in \mathbb{N} \) is smallest such that \( h_1(n) \neq h_2(n) \), then \( n \) is also smallest such that \( g(h_1(n)) \neq g(h_2(n)) \).

**Theorem 4.2.14** (Birkhoff, Kakutani; see Theorem 9.1 in [65]). A topological group \( G \) is metrisable if and only if \( G \) is Hausdorff and first-countable. Every metrisable topological group has a compatible left-invariant metric.

**Lemma 4.2.15.** Let \( G \) be a group with a left-invariant metric \( d \). Then for all \( g, h \in G \)

\[
d(gh, 1_G) \leq d(g, 1_G) + d(h, 1_G).
\]

**Proof.** \( d(gh, 1_G) \leq d(h, g^{-1}) \leq d(h, 1_G) + d(1_G, g^{-1}) = d(g, 1_G) + d(h, 1_G). \)

**Proposition 4.2.16.** Let \( \xi : G \to H \) be a continuous homomorphism between topological groups with compatible left-invariant metrics \( d_1 \) and \( d_2 \). Then \( f \) is uniformly continuous.
Let \( G \) be a topological group with a compatible left-invariant metric \( d \).

**Proof.** Let \( \epsilon > 0 \). Since \( \xi \) is continuous, there exists a \( \delta > 0 \) such that for all \( g \in G \) with \( d_1(1, g) < \delta \) we have \( d_2(1, \xi(g)) < \epsilon \). Let \( g_1, g_2 \in G \) be such that \( d_1(g_1, g_2) < \delta \). Then \( d_1(1, g_i^{-1} g_2) < \delta \), and hence

\[
  d_2(\xi(g_1), \xi(g_2)) = d_2(1, \xi(g_1)^{-1} \xi(g_2)) = d_2(1, \xi(g_1^{-1} g_2)) < \epsilon
\]

which shows uniform continuity of \( \xi \).

We have seen in Example 4.1.18 an example of a left-invariant metric \( d \) on \( \text{Sym}(D) \) which is not complete.

**Lemma 4.2.17.** Let \( G \) be a topological group with a compatible left-invariant metric \( d \). Then

\[
d'(g, h) := d(g, h) + d(g^{-1}, h^{-1})
\]

is a compatible metric, too.

**Proof.** Clearly, \( d' \) is non-negative, indiscernible, symmetric, and subadditive. We have to show that \( d' \) induces the same topology on \( G \) as \( d \). Let \( \epsilon > 0 \). The set \( S := \{ g \in G \mid d(1, g) < \epsilon \} \) is open with respect to \( d \) and contains the identity \( 1 \in G \). This set contains the set \( S' := \{ g \in G \mid d'(1, g) < \epsilon \} \) which is open with respect to \( d' \) and also contains 1. Conversely, the set \( S' \) contains the set \( \{ g \in G \mid d(1, g) < \epsilon/2 \} \) which is open with respect to \( d \) and contains 1: if \( g \) is such that \( d(1, g) < \epsilon \), the by left-invariance of \( d \) we have that \( d(g^i, 1) < \epsilon \), and hence \( d'(1, g) < d(1, g) + d(1, g^{-1}) < \epsilon/2 + \epsilon/2 < \epsilon \), so \( g \in S' \). Since the topology on \( G \) is given by a basis of open sets at 1, the statement follows.

**Lemma 4.2.18.** Let \( G \) be a topological group, let \( d \) be a compatible left-invariant metric, and let \( d' \) be the compatible metric defined in Lemma 4.2.17. Let \( (g_i)_{i \in \mathbb{N}} \) and \( (h_i)_{i \in \mathbb{N}} \) be Cauchy sequences in \( (G, d') \). Then \( (g_i^{-1} h_i)_{i \in \mathbb{N}} \) is Cauchy in \( (G, d) \) and in \( (G, d') \).

**Proof.** Let \( \epsilon > 0 \). Then there exists an \( n_0 \in \mathbb{N} \) such that \( d'(h_n, h_m) < \epsilon/3 \) for all \( n, m \geq n_0 \). By the continuity of the multiplication operation there exists a \( \delta > 0 \) such that for all \( k \in G \) with \( d(k, 1) < \delta \)

\[
d(h_{n_0}^{-1} k h_{n_0}, 1) < \epsilon/3.
\]

Let \( n_1 \geq n_0 \) be such that \( d'(g_n, g_m) < \delta \) for all \( n, m \geq n_1 \). Then for all \( n, m \geq n_1 \)

\[
d(g_n, g_m^{-1}) = d(g_n^{-1}, g_m^{-1}) < \delta
\]

and hence

\[
d(h_{n_0}^{-1} g_m g_n^{-1} h_{n_0}, 1) < \epsilon/3.
\]

Therefore

\[
  d(g_n^{-1} h_n, g_m^{-1} h_m) = d(h_{n_0}^{-1} g_m g_n^{-1} h_{n_0}, 1) \\
  \leq d(h_{n_0}^{-1} h_{n_0}, 1) + d(h_{n_0}^{-1} g_m g_n^{-1} h_{n_0}, 1) + d(h_{n_0}^{-1} h_{n_0}, 1)
\]

\[
  \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\]

which proves that \( (g_i^{-1} h_i)_{i \in \mathbb{N}} \) is Cauchy with respect to \( d \). Note that by symmetry also the sequence \( (h_i^{-1} g_i)_{i \in \mathbb{N}} \) is Cauchy in \( (G, d') \), and it follows that both sequences are also Cauchy in \( (G, d') \).

**Lemma 4.2.19.** Let \( G \) be a topological group with a compatible left-invariant metric \( d \), and let \( d' \) be the metric defined in Lemma 4.2.17. Let \( (G^*, d^*) \) be the metric completion of \( (G, d') \). Then the group multiplication can be extended uniquely to \( G^* \) such that \( G^* \) becomes a topological group with the compatible metric \( d^* \).
To define an extension of the multiplication operation of $G$ to $G^*$, pick representatives $(g_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ of elements of $G^*$ (recall our construction of $G^*$ in Proposition 4.1.22). Then $(g_n h_n)_{n \in \mathbb{N}}$ is Cauchy with respect to $d^*$ by Lemma 4.2.18.

Define 

$$[(g_n)_{n \in \mathbb{N}}] \cdot [(h_n)_{n \in \mathbb{N}}] := [(g_n h_n)_{n \in \mathbb{N}}].$$

To show that this is well-defined, let $(g'_n)_{n \in \mathbb{N}}$ and $(h'_n)_{n \in \mathbb{N}}$ be Cauchy sequences in $(G, d^*)$ such that $\lim_{n \to \infty} d(g_n, g'_n) = 0$ and $\lim_{n \to \infty} d(h_n, h'_n) = 0$.

Let $\varepsilon > 0$. Then there exists $n_0$ such that $d(h_n^{-1}, h_m) < \varepsilon/3$ for all $n, m \geq n_0$. By the continuity of the multiplication operation there exists a $\delta > 0$ such that for all $k \in G$ with $d(k, 1_G) < \delta$

$$d(h_n^{-1}kh_{n_0}, 1_G) < \varepsilon/3.$$

Let $n_1 \geq n_0$ be such that $d'(g_n^{-1}g'_n) < \delta$ for all $n \geq n_1$. Then for $n \geq n_1$ and by Lemma 4.2.15

$$d(g_n h_n, g'_n h'_n) = d(h_n^{-1}g_n, g'_n h'_n, 1_G) \leq d(h_n^{-1}h_{n_0}, 1_G) + d(h_{n_0}^{-1}g_n h_{n_0}, 1_G) + d(h_n^{-1}h_n, 1_G) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

which shows that $[(g'_n, h'_n)_{n \in \mathbb{N}}] = [(g_n h_n)_{n \in \mathbb{N}}]$. The multiplication operation defined on $G^*$ is associative and has the neutral element $[(1_G)_{n \in \mathbb{N}}]$. The inverse of $((g_n)_{n \in \mathbb{N}})$ is $[(g_n^{-1})_{n \in \mathbb{N}}]$ (Lemma 4.2.15). Multiplication on $G^*$ is continuous with respect to the topology induced by $d^*$: if $\lim_{n \to \infty} \lim_{m \to \infty} g_{n,m} = g$ and $\lim_{m \to \infty} \lim_{n \to \infty} h_{n,m} = h$ then

$$\lim_{m \to \infty} \left( \lim_{n \to \infty} g_{n,m} \right) \cdot \left( \lim_{n \to \infty} h_{n,m} \right) = \lim_{m \to \infty} \lim_{n \to \infty} g_{n,m} \cdot \lim_{n \to \infty} h_{n,m} = \left[ \lim_{m \to \infty} \lim_{n \to \infty} g_{n,m} \cdot \lim_{n \to \infty} h_{n,m} \right] = \left[ \lim_{n \to \infty} \lim_{m \to \infty} g_{n,m} h_{n,m} \right] = [h^{-1} \cdot g]$$

so that we indeed obtain a topological group $G^*$.

**Lemma 4.2.20 (Proposition 2.2.1 in [51]).** Let $G$ be a Polish group and $U$ a subgroup. Then $U^{-1}$ is Polish if and only if $U$ is closed in $G$.

**Proof.** Suppose first that $U$ is closed in $G$. Let $S \subseteq G$ be a countable set that is dense in $G$. Hence, $S \cap U$ is countable and dense in $U$, showing that $U$ is separable. The restriction of a complete compatible metric for $G$ metric to $U$ is certainly also a compatible metric for $U$, and it is complete since $U$ is closed.

Conversely, we assume that $U$ is Polish. Let $\overline{U}$ be the closure of $U$ in $G$. Suppose for contradiction that there exists a $g \in \overline{U} \setminus U$. Since $H$ is dense in $\overline{H}$, the set $gH$ is dense in $\overline{H}$, too. By the Baire category theorem (Theorem 4.1.26), the intersection of $gH$ and $H$ is dense in $\overline{H}$, and in particular non-empty, which is impossible.

**Lemma 4.2.21.** Let $G$ be a Polish group with a compatible left-invariant metric $d$. Then $d'(g, h) := d(g, h) + d(g^{-1}, h^{-1})$ is a compatible complete metric on $G$.

**Proof.** Let $(G^*, d^*)$ be the metric completion of $(G, d)$ (see Proposition 4.1.22). By Lemma 4.2.19 $G^*$ can be viewed as a topological group $G^*$ with the compatible complete metric $d^*$, and since $G^*$ is also separable we conclude that $G^*$ is Polish.

Since $G$ is Polish, too, Lemma 4.2.20 implies that $G$ is closed in $G^*$. Therefore, $G = G^*$. This shows that $d'$ is a compatible complete metric on $G$.
4.3. CLOSED SUBGROUPS

Exercises.
(41) Show that \( \text{Sym}(\mathbb{N}) \) does not admit a compatible complete and left-invariant metric.
(42) True or false: the closure of an open ball of radius \( r \) in a metric space is the closed ball of radius \( r \) in that metric space.

4.3. Closed subgroups

In this section we give a topological characterisation of those topological groups that appear as automorphism groups of countable structures. We have already seen in Section 4.1.4 that such a group must be Polish. But also \((\mathbb{R}; +)\) is Polish, and it certainly isn’t the automorphism group of a countable structure, so we need to identify more properties of closed subgroups of \( \text{Sym}(\mathbb{N}) \).

A topological group is non-archimedian if it has a basis at the identity consisting of open subgroups.

Example 4.3.1. The group \((\mathbb{R}; +)\) is archimedian: for all \( a, b \in \mathbb{R} \) with \( 0 < a \leq b \) there exists an \( n \in \mathbb{N} \) such that \( na := a + a + \cdots + a \geq b \). Hence, the open interval \((a, b)\) does not contain any non-trivial open subgroup, since if the subgroup contains \( a \in (0, b) \), then it also contains elements larger than \( b \). This implies that \((\mathbb{R}; +)\) is not non-archimedian in the sense above and motivates the terminology.

Example 4.3.2. The group \( \text{Sym}(D) \) is non-archimedian: the point stabilisers \( G_a \) for \( a \in D^n, n \in \mathbb{N} \), are open subgroups of \( G \) and they form a basis at the identity.

Automorphism groups of countable structures can be characterised in topological terms, as demonstrated in Proposition 4.3.3 below.

Proposition 4.3.3 (Section 1.5 in [9]; also see Theorems 2.4.1 and 2.4.4 in [51]). Let \( G \) be a topological group. Then the following are equivalent.

1. \( G \) is topologically isomorphic to the automorphism group of a countable relational structure.
2. \( G \) is topologically isomorphic to a closed subgroup of \( \text{Sym}(\mathbb{N}) \).
3. \( G \) is Polish and admits a compatible left-invariant ultrametric.
4. \( G \) is Polish and non-archimedian.

Proof. The equivalence of (1) and (2) has been shown in Proposition 1.2.1. The implication from (1) to (3) has been explained in the paragraphs preceding the statement of the proposition. So it suffices to show (3) \( \Rightarrow \) (4) \( \Rightarrow \) (2).

For the implication from (3) to (4), let \( d \) be a left-invariant ultrametric on \( G \). Let \( U_n = \{ g \in G \mid d(g, 1) < 2^{-n} \} \), for \( n \in \mathbb{N} \). We claim that the set of all those \( U_n \) forms a basis at the identity consisting of open subgroups. Since \( d \) is a left-invariant ultra-metric, for \( g, h \in U_n \) we have
\[
d(g^{-1}h, 1) = d(h, g) \leq \max(d(h, 1), d(1, g)) < 2^{-n}.
\]
Hence, \( g^{-1}h \in U_n \) and \( U_n \) is indeed a subgroup.

For the implication from (4) to (1), let \( \{B_1, B_2, \ldots\} \) be an at most countable basis at the identity (which exists since \( G \) is metrizable). Each \( B_i \) has an open subset \( V_i \) which is a subgroup, since \( G \) has a basis at the identity consisting of open subgroups. Then \( \{V_1, V_2, \ldots\} \) is a countable basis of the identity consisting of open subgroups. Each \( V_i \) has at most countably many cosets since \( G \) is separable. So the set of all cosets of those groups gives an at most countable basis \( \mathcal{B} := \{U_1, U_2, \ldots\} \) that is closed under left multiplication. If \( \mathcal{B} \) is infinite, we define the map \( \xi: G \to \text{Sym}(\mathbb{N}) \) by setting
\[
\xi(g)(n) = m \iff gU_n = U_m.
\]
If $|B| = n_0$ is finite, we define the map $\xi : G \to \text{Sym}(\mathbb{N})$ similarly, but set $\xi(g)(n) = n$ for all $n > n_0$.

**Claim 1.** $\xi$ is a homomorphism. We have $\xi(fg) = \xi(f)\xi(g)$ since 
\[ \xi(f)\xi(g)(n) = m \iff f(gU_n) = U_m \iff fgU_n = U_m \iff \xi(fg)(n) = m. \]

**Claim 2.** $\xi$ is injective: when $f, g \in G$ are distinct, then there are disjoint open subsets $U$ and $V$ with $f \in U$ and $g \in V$, because the topology is metrizable and therefore Hausdorff; since $B$ is a basis, we can assume that $U = U_n$ for some $n \geq 1$. If $fU_n = gU_n$, then $g \in U_n = U$ since $f \in U_n$, contradicting the assumption that $U$ and $V$ are disjoint. Hence, $\xi(f)(n) \neq \xi(g)(n)$, and so $\xi(f) \neq \xi(g)$. So $\xi$ is indeed an isomorphism between $G$ and a subgroup of $\text{Sym}(\mathbb{N})$.

**Claim 3.** $\xi$ is continuous. Let $V \subseteq \text{Sym}(\mathbb{N})$ be an open set. Then $V$ is a union of basic open sets $S(\bar{a}, b)$ for some $\bar{a}, b \in \mathbb{N}^\infty$. Let $i \leq n$ and $g, h \in G$ be such that $g \circ h \in U_b$. Since composition in $G$ is continuous and $U_b$ is open, there is an open subset $G_{g,h}$ of $G$ containing $g$ and an open set $H_{g,h}$ of $G$ containing $h$ such that $(g, h) \in G_{g,h} \times H_{g,h} \subseteq \xi^{-1}(U_b)$. We then have 
\[ \xi^{-1}(S(\bar{a}, b)) = \bigcap_{i \leq n} \{ g \in G \mid gU_{a_i} = U_b \} = \bigcap_{i \leq n} \bigcup_{g \in G, h \in U_{a_i}} G_{g,h}. \]
This set is a finite intersection of a union of open sets and thus open. Hence, $\xi^{-1}(V)$ is a union of open sets and therefore open as well, which concludes the proof that $\xi$ is continuous.

**Claim 4.** The map $\xi$ is open and the image of $\xi$ is closed in $\text{Sym}(\mathbb{N})$. Let 
- $d_1$ be the left-invariant compatible metric on $G$ (Theorem 4.1.14),
- $d_2'$ be the compatible complete metric on $G$ defined as $d_2'(g, h) = d_1(g, h) + d_1(g^{-1}, h^{-1})$ (see Lemma 4.2.21), and
- $d_2''$ be the compatible complete metric on $\text{Sym}(\mathbb{N})$ from Example 4.1.20.

We will show that $\xi^{-1}$ is Cauchy-continuous as a map from $(\xi(G), d_2')$ to $(G, d_1)$. This clearly implies both parts of the claim.

Let $g_1, g_2, \ldots$ be a sequence in $G$ such that $\xi(g_1), \xi(g_2), \ldots$ converges against $h \in \text{Sym}(\mathbb{N})$. We have to show that $g_1, g_2, \ldots$ is $d_2''$-Cauchy. Since $d_1$ is left-invariant, $\lim_{n,m \to \infty} d_1(g_n, g_m) = 0$ if and only if $\lim_{n,m \to \infty} d_1(g_m^{-1}g_n, 1) = 0$. Let $\epsilon > 0$ be arbitrary. Since $B$ is a basis, there exists $U_k \in B$ such that 
\[ U_k \subseteq \{ g \in G \mid d_1(g, 1) < \epsilon/2 \} \]
and $U_k U_k^{-1} \subseteq \{ g \in G \mid d_1(g, 1) < \epsilon \}$. Since $\lim_{n \to \infty} \xi(g_n) = h$, there exists an $n_0$ such that $\xi(g_n)(k) = \xi(g_m)(k) = h(k)$ for all $n, m > n_0$. Then $g_n U_k = g_m U_k$, and so 
\[ g_m^{-1} g_n \in U_k U_k^{-1} \subseteq \{ g \in G \mid d_1(g, 1) < \epsilon \}. \]
Hence, $d_1(g_m^{-1}g_n, 1) < \epsilon$, and $\lim_{n,m \to \infty} d_1(g_m^{-1}g_n, 1) = 0$. Similarly one can show that $\lim_{n,m \to \infty} d_1(g_m^{-1}g_n, 1) = 0$, using the fact that $\xi(g_n^{-1}) = \xi(g_n)^{-1}$, and hence $\lim_{n \to \infty} \xi(g_n^{-1}) = h^{-1}$. Thus, $\lim_{n,m \to \infty} d_1(g_n, g_m) = 0 = \lim_{n,m \to \infty} d_1(g_m^{-1}, g_m^{-1})$, and therefore $\lim_{n,m \to \infty} d_1(g_n, g_m) = 0$. \hfill \Box

### 4.4. Open Subgroups

Let $G$ be a permutation group over the base set $B$. For a sequence $\bar{a}$ of elements of $B$, the point stabiliser $G_{\bar{a}}$ of $G$ is the set of all elements of $G$ that fix $\bar{a}$. For $A \subseteq B$, the set stabilizer $G_A$ of $G$ is the set of all $\alpha \in G$ that fix $A$ setwise, that is, $\alpha A = A$.
4.4. OPEN SUBGROUPS

Lemma 4.4.1. Let \( G \) be subgroup of \( \text{Sym}(\mathbb{N}) \) and let \( U \) be a subgroup. Then the following are equivalent.

1. \( U \) is open in \( G \);
2. \( U \) contains the point stabiliser of some finite set of elements of \( \mathbb{N} \);
3. \( U \) contains an open subset of \( G \);

Proof. 1 \( \Rightarrow \) 2: Since \( U \) is open in \( G \) it must contain \( G \cap S(a, b) \) for some \( a, b \in \mathbb{N}^n \) since these sets form a basis of the topology of \( \text{Sym}(\mathbb{N}) \). Every element of \( G_s \) can be written as \( \alpha \beta \) with \( \alpha \in G \cap S(b, a) \subseteq U \) and \( \beta \in G \cap S(a, b) \subseteq U \). Hence, \( U \) contains \( G_s \).

2 \( \Rightarrow \) 3 is trivial.

3 \( \Rightarrow \) 1: Let \( H \) be an open subgroup of \( G \). Then \( U = \bigcup_{\beta \in \mathcal{H}} \beta H \). Since \( H \) and \( \beta H \) are open, it follows that \( U \) is open, too. \( \square \)

We have seen in Proposition 4.2.3 that every open subgroup of a topological group is closed. The converse is false: for example, when \( E \) is an equivalence relation on a countably infinite set \( B \) with two infinite classes, then \( \text{Aut}(B, E) \) is a closed subgroup of \( \text{Sym}(B) \) (we already saw in Proposition 1.2.1 that the closed subgroups of the automorphism group of a structure \( A \) correspond precisely to arbitrary expansions of \( A \)), but does not contain the point stabiliser of some finite subset of \( B \).

Lemma 4.4.2. Let \( G \) be a subgroup of \( \text{Sym}(B) \). Then \( U \) is an open subgroup of \( G \) if and only if \( U = G_S \) is the set stabilizer of a block \( S \) of the componentwise action of \( G \) on \( B^n \) for some \( n \in \mathbb{N} \).

Proof. Let \( S \subseteq B^n \) be a block and let \( C \) be the corresponding congruence of \( G \). We first prove that the subgroup \( U := G_S \) of \( G \) is open in \( G \). Arbitrarily pick an \( s \in S \). To show that \( G_S \) is open it suffices by Lemma 4.4.2 that \( G_S \) contains \( G_s \). Let \( \alpha \in G_s \) and \( t \in S \). Then \( (s, t) \in C \) and hence \( \langle \alpha s, t \rangle \in C \). Since \( \alpha s = s \in S \) we conclude that \( \alpha t \in S \). So \( \alpha \in G_S \).

Conversely, let \( U \) be an open subgroup of \( G \). Then \( U \) must contain \( G_1 \) for some \( n \in \mathbb{N} \) and some \( t \in B^n \). We claim that \( S := \{ \beta t \mid \beta \in U \} \) is a block of the componentwise action of \( G \) on \( B^n \). By Lemma 4.4.2 it suffices to verify that \( \alpha(S) = S \) or \( \alpha(S) \cap S = \emptyset \) for all \( \alpha \in G \). Suppose that \( \alpha(S) \cap S \neq \emptyset \). Then there are \( \beta_1, \beta_2 \in U \) such that \( \beta_1(t) = \alpha \beta_2 t \). Thus, \( \beta_1^{-1} \circ \alpha \circ \beta_2 \in G_1 \), and \( \alpha \in \beta_1 G_1 \beta_2^{-1} \subseteq U \). But then \( \alpha(S) = S \) and we are done.

We verify that \( U = G_S \). Let \( g \in U \) and \( s \in S \). Then \( s = \beta t \) for some \( \beta \in U \). Hence, \( \alpha s = \alpha \beta t \) by the definition of \( S \), because \( \alpha \beta \in U \). Therefore, \( U \subseteq G_S \). Now suppose that \( \alpha \in G_S \). Since \( \alpha \) preserves \( S \), we have \( \alpha t \in S \), and thus there exists a \( \beta \in U \) with \( \beta \alpha t = t \). So \( \beta \alpha \in G_1 \subseteq U \), and thus \( \alpha \in U \). \( \square \)

Lemma 4.4.2 has the following consequence.

Corollary 4.4.3. Let \( G \) be a closed oligomorphic subgroup of \( \text{Sym}(\mathbb{N}) \) and let \( U \) be an open subgroup of \( G \). Then \( U \) is contained in only finitely many subgroups of \( G \).

Proof. By Lemma 4.4.2, \( U \) contains the stabiliser of some \( s \in \mathbb{N}^n \) for some \( n \in \mathbb{N} \). Every subgroup \( V \) of \( G \) that contains \( U \) will be open by Lemma 4.4.2 and hence will be of the form \( G_S \) for some block \( S \) of the componentwise action of \( G \) on \( B^n \) by Lemma 4.4.2. Since \( V \) contains \( U \) we must have \( m \leq n \). As \( G \) is oligomorphic, there are finitely many congruence relations of \( G \) on \( B^m \) for each \( m \leq n \) and the statement follows. \( \square \)

Lemma 4.4.2 has another consequence.
4. TOPOLOGICAL GROUPS

Corollary 4.4.4. Every closed oligomorphic subgroup of Sym(\(\mathbb{N}\)) has countably many open subgroups.

Proof. An oligomorphic group \(G\) has for each \(n\) finitely many congruences of the componentwise action of \(G\) on \(B^n\), and at most countably many congruence classes for each congruence. 

So in particular Sym(\(\mathbb{N}\)) itself has only countably many open subgroups.

4.5. Compact Subgroups

Proposition 4.5.1. Let \(G\) be a compact subgroup of Sym(\(\mathbb{N}\)). Then all orbits of \(G\) are finite.

Proof. Let \(O\) be an infinite orbit of \(G\), and fix \(a \in O\). Then the sets \(S(a,b)\) for \(b \in O\) form an open partition of \(G\), and hence no finite sub-collection of those sets can cover \(G\). Hence, if \(O\) has infinite orbits, then \(G\) is not compact. 

Proposition 4.5.2. Let \(G\) be a closed subgroup of Sym(\(\mathbb{N}\)). Then \(G\) is compact if and only if all orbits of \(G\) are finite.

Proof. The forward implication follows from Proposition 4.5.1. For the other direction, suppose that the orbits \(O_1, O_2, \ldots\) of \(G\) are finite. We write \(G|_{O_i}\) for the group of restrictions of \(G\) to \(O_i\), which is a finite subgroup of \(G\). Then \(G = \prod_{i=1}^{\infty} G|_{O_i}\) is a closed subset of a product of finite subgroups of \(G\), and hence compact by Tychonoff’s theorem (Theorem 4.1.31) and Proposition 4.1.30. We do not use the entire strength of Tychonoff’s theorem, and show two alternative proofs.

Second Proof. Let \(\{U_i\}_{i \in A}\) be an open cover of \(G\). Since Sym(\(\mathbb{N}\)) and hence \(G\) are second-countable, we can assume that \(A = \mathbb{N}\) (Proposition 4.1.35). Suppose for contradiction that for no finite \(B \subseteq A\) we have that \(G \subseteq \bigcup_{i \in B} U_i\). Consider the following rooted tree. The vertices on level \(n\) are the restrictions of the permutations in \(G\) to \(\{1, \ldots, n\}\). Adjacency between a vertex on level \(n\) and vertices on level \(n+1\) is defined by restriction. Clearly, for every \(n\) there are finitely many vertices on level \(n\) since the orbit of \((1, \ldots, n)\) with respect to the componentwise action of \(G\) is finite. A vertex on level \(n\) is good if it is the restriction of a function from \(G \setminus \bigcup_{i \leq n} U_n\). Clearly, the restriction of a good vertex is good. Moreover, by assumption there are good vertices on all levels. By König’s tree lemma, there is an infinite branch of good vertices in the tree. This branch defines an injection from \(\mathbb{N}\) to \(\mathbb{N}\). In fact, it must be a bijection since the finiteness of the orbits implies that the map is surjective onto each orbit. This map is from \(G\) since \(G\) is closed, but it does not lie in any of the \(U_i\), a contradiction.

Third Proof. Our final proof uses Theorem 4.1.37. Clearly \(G\) is complete since it is closed in Sym(\(\mathbb{N}\)). To prove total boundedness of \(G\), let \(\epsilon > 0\). Choose \(n \in \mathbb{N}\) such that \(1/2^n < \epsilon\). Since all orbits of \(G\) are finite, the orbit \(O\) of \((1, \ldots, n)\) with respect to the componentwise action of \(G\) on \(\mathbb{N}^n\) is finite too; choose \(f_1, \ldots, f_k \in G\) so that \(O = \{f_1(1, \ldots, n), \ldots, f_k(1, \ldots, n)\}\). Then by construction \(B_{f_1}(\epsilon), \ldots, B_{f_k}(\epsilon)\) covers all of \(G\).

It follows from Proposition 4.5.1 that a compact subgroup \(G\) of Sym(\(\mathbb{N}\)) must have infinitely many orbits, and in particular cannot be oligomorphic. In fact, oligomorphicity is already ruled out by local compactness; this can be seen from Lemma 3.0.1 and the following.
Corollary 4.5.3. Let $G \leq \text{Sym}(\mathbb{N})$ be locally compact. Then there exists $a \in \mathbb{N}^n$, $n \in \mathbb{N}$ such that $G_a$ has only finite orbits. If $G$ is additionally closed, then also the converse holds.

Proof. If $G$ is locally compact there exists an open set $U \subseteq G$ that contains $1_G$ and that is contained in a compact set $K \subseteq G$. Since $U$ is open and contains $1_G$ there exists a finite tuple $\bar{a} \in \mathbb{N}^n$, $n \in \mathbb{N}$, such that $G_{\bar{a}} \subseteq U$. Then $G_{\bar{a}}$ is a closed subset of the compact set $K$ and hence compact by Proposition 4.1.30. The statement then follows from Proposition 4.5.1.

Conversely, if $G$ is closed, then so is $G_{\bar{a}}$, and hence $G_{\bar{a}}$ is compact by Theorem 1.5.2. To show local compactness of $G$, let $\alpha \in G$. Then $\alpha G_{\bar{a}}$ is an open and compact set which contains $\alpha$, and hence $G$ is locally compact. \(\square\)

Also note that locally compact subgroups of $\text{Sym}(\mathbb{N})$ contain all countable subgroups of $\text{Sym}(\mathbb{N})$ by Theorem 1.2.3; see Figure 4.1.

Exercises.

(43) Prove that every countable compact subgroup of $\text{Sym}(\mathbb{N})$ is finite.

4.6. Closed Normal Subgroups

Example 4.6.1. Let $E$ be an equivalence relation on a countably infinite set $D$ such that all equivalence classes $B_1, B_2, \ldots$ of $E$ have size two. Then the oligomorphic group $\text{Aut}(D; E)$ has the (closed!) normal subgroup $\text{Aut}(D; B_1, B_2, \ldots)$ (which is not oligomorphic).

Proposition 4.6.2. Let $G$ be a closed subgroup of $\text{Sym}(B)$. If $E$ is a $G$-invariant equivalence relation on $B^n$, for some $n$, then the subgroup of $G$ that preserves each equivalence class of $E$ is closed and normal. Conversely, every closed normal subgroup of $G$ is the intersection of closed normal subgroups that arise in this way.

Proof. Let $C$ be the expansion of $B$ by a unary relation for each equivalence class of $E$. Then $\text{Aut}(C)$ is closed by Proposition 1.2.1 and it is a normal subgroup of $\text{Aut}(B)$: when $g \in \text{Aut}(B)$ and $h \in \text{Aut}(C)$, then $g \circ h \circ g^{-1}$ preserves each.
4. TOPOLOGICAL GROUPS

<table>
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<th>Structural description</th>
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<tr>
<td>((A, R)) is expansion of (A)</td>
<td>Intersection of set-stabilisers of (G \cap B^n)</td>
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<td>Expansion ((A, r))</td>
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<td>(E) is congruence of (A) (Blocksystem)</td>
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**Figure 4.2.** Informal summary of the connections between concepts for structures, their automorphism groups as permutation groups, and their automorphism groups as topological groups (from Sections 4.3, 4.4, 4.5, and 4.6).

equivalence class of \(E\), and thus is an automorphism of \(C\). Normality of \(\text{Aut}(C)\) follows from Proposition 1.3.12.

For the second part, suppose that \(C\) has a closed normal subgroup \(N\). Consider the relation

\[ R_n := \{(x, y) \mid x, y \in B^n \text{ and there is } h \in N \text{ such that } h(x) = y\} . \]

This relation is obviously an equivalence relation, and it is preserved by all the automorphisms of \(B\). For this, we have to show that for all \(g \in G\) and all \((x, y) \in R_n\) we have that \((g(x), g(y)) \in R_n\). So suppose that \(x, y \in B^n\) such that \(h(x) = y\) for some \(h \in N\). Then \(g(y) = g(h(x)) \in (gN)(x) = (Ng)(x) = N(g(x))\) by normality of \(N\). Hence there exists an \(h' \in N\) such that \(h'(g(x)) = g(y)\), which shows that \((g(x), g(y)) \in R_n\).

Let \(C\) be the structure that contains for all \(n\) the \(n\)-ary relations given by the equivalence classes of the relations \(R_n\) for all \(n \geq 0\). We claim that \(N\) is precisely the automorphism group of \(C\). As in the first part we can verify that every \(h \in N\) is an automorphism of \(C\). The converse follows by local closure as follows. Let \(g\) be an automorphism of \(C\), and let \(x, y\) be from \(B^n\) so that \(g(x) = y\). Since \(g\) preserves the equivalence classes of \(R_n\), there exists an \(h \in N\) such that \(h(x) = y\). Hence, \(g\) lies in the closure of \(N\), which implies that \(g\) is from \(N\) since \(N\) is closed.

**Example 4.6.3.** The automorphism group \(G\) of the structure \(B = (\mathbb{Q}; \text{Betw})\), where \(\text{Betw} = \{(x, y, z) \mid (x < y < z) \lor (z < y < x)\}\), is 2-transitive and therefore primitive. However, the relation

\[ \{(x_1, x_2, y_1, y_2) \mid (x_1 < x_2 \land y_1 < y_2) \lor (x_1 > x_2 \land y_1 > y_2) \lor (x_1 = x_2 \land y_1 = y_2)\} \]

is a \(G\)-invariant equivalence relation on \(\mathbb{Q}^2\). And indeed, \(G\) has a closed normal subgroup \(N\) that is isomorphic to the automorphism group of \((\mathbb{Q}; <)\), and \(G/N\) has two elements, corresponding to the automorphisms that reverse the order <, and the automorphisms that preserve the order.

\(\square\)
CHAPTER 5

Birkhoff’s Theorem and Permutation Groups

Often one would like to understand the various ways in which a given group can act on a given set. Birkhoff’s theorem from universal algebra, specialised to permutation groups, may help to understand this question better (Section 5.1). In this chapter we also present a topological generalisation where we will be interested in topological groups and continuous actions (Section 5.2). Finally, as an application, this gives rise to a topological characterisation of bi-interpretable for \( \omega \)-categorical structures (Section 5.3; bi-interpretability has already been defined in Section 3.5.2). More specifically, we obtain the following corollary which has been credited to Coquand by Ahlbrandt and Ziegler [3].

**Theorem 5.0.1.** Let \( A \) and \( B \) be \( \omega \)-categorical structures. Then \( \text{Aut}(A) \) and \( \text{Aut}(B) \) are isomorphic as topological groups if and only if \( A \) and \( B \) are bi-interpretable.

### 5.1. Birkhoff’s Theorem

In order to apply Birkhoff’s theorem to permutation groups, we represent permutation groups as \( G \)-sets as in Example 1.1.3 and 1.1.6. If \( K \) is a class of \( \tau \)-structures, then we write
- \( P(K) \) for the class of all powers of structures in \( K \);
- \( S(K) \) for the class of all substructures of structures in \( K \);
- \( H(K) \) for the class of all homomorphic images of structures in \( K \).

If \( A \) is a \( G \)-set, then we also write \( \text{Gr}(A) \) for \( G \); recall that by definition \( G \) equals the permutation group on \( A \) consisting of all unary term functions over \( A \).

**Theorem 5.1.1 (Birkhoff’s theorem for permutation groups).** Let \( G \) be a subgroup of \( \text{Sym}(B) \) and let \( B \) be a \( G \)-set with signature \( \tau \).

- For every homomorphism \( \xi : G \to \text{Sym}(A) \) there is an \( A \in HSP(\{B\}) \) such that \( t^A = \xi(t^B) \) for every \( \tau \)-term \( t \).
- If \( A \in HSP(\{B\}) \) then the function \( \xi \) that maps \( t^B \) to \( t^A \) for every \( \tau \)-term \( t \) is well-defined and a homomorphism from \( G \) onto \( \text{Gr}(A) \).

**Proof.** Let \( \xi : G \to \text{Sym}(A) \) be a homomorphism. Let \( C \) be \( B^A \). Let \( I \) be a well-ordered set such that \( C = \{ c^i \mid i \in I \} \). For \( a \in A \), define \( c_a := (c^i(a))_{i \in I} \). Let \( S \) be the smallest substructure of \( B^A \) that contains \( \{ c_a \mid a \in A \} \). So the elements of \( S \) are precisely those that can be written as \( t^S(c_a) \) for some \( \tau \)-term \( t \) and some \( a \in A \). Define \( \mu : S \to A \) by

\[
\mu(t^S(c_a)) := \xi(t^B)(a).
\]

**Claim 1.** \( \mu \) is well-defined. Suppose that \( t^S(c_a) = r^S(c_a') \). We first show that \( t^B = r^B \). Let \( b \in B \). Note that there is some \( i \in I \) such that \( c^i(a) = b \) and \( c^i(a') = b \). Hence,

\[
\begin{align*}
t^B(b) &= t^B(c^i(a)) = t^S(c_a)_i \\
&= r^S(c_a')_i = r^B(c^i(a')) = r^B(b).
\end{align*}
\]
Hence, \( t^B = r^B \) and therefore \( t^S = r^S \). Thus, \( t^S(c_a) = r^S(c_{a'}) \) implies that \( c_a = c_{a'} \) since \( r^S = r^S \) is injective, and hence \( a = a' \). Therefore

\[
\xi(t^B)(a) = \xi(r^B)(a) = \xi(r^B)(a')
\]

and hence \( \mu \) is well-defined.

**Claim 2.** \( \mu \) is surjective. For every \( a \in A \), choose the \( \tau \)-term \( t := x \) (just a variable); then \( \mu(c_a) = \mu(t^S(c_a)) = \xi(t^B)(a) = a \) since \( \xi(1^S) = 1^{\text{Sym}(A)} \).

Let \( \Delta \) be the \( \tau \)-algebra where \( g \in \tau \) denotes \( \xi(g^B) \).

**Claim 3.** \( \mu \) is a homomorphism from \( S \) to \( A \). Let \( f \in \tau \) and let \( s \in S \), and let \( s = t^S(c_a) \) for some \( \tau \)-term \( t \) and some \( a \in A \). Then

\[
\mu(f^S(s)) = \mu(f^S(t^S(c_a))) = \mu(f(t^S(c_a))) = f(t^A(a)) = f^A(t^A(a)) = f^A(t^S(c_a)) = f^A(\mu(s))
\]

Hence, \( A \) is the homomorphic image of the subalgebra \( S \) of \( B^A \), so \( A \in \text{HSP}(B) \).

For the second statement, we first show that \( \xi \) is well-defined. Suppose that \( r \) and \( t \) are \( \tau \)-terms such that \( r^B = t^B \), and let \( \mu : S \rightarrow A \) be a homomorphism from a subalgebra \( S \) of a power of \( B \) to \( A \). Then \( r^S = t^S \). Hence, for all \( s \in S \) we have \( \mu(r^S(s)) = \mu(t^S(s)) \). Therefore, since \( \text{Gr}(B) \) is a permutation group, there exists a \( \tau \)-term \( t \) such that \( t^B = (g^B)^{-1} \). Then

\[
\xi(g^B)(\Delta) = \xi((gh)^B) = (gh)^A = g^A h^A = \xi(g^B)\xi(h^B).
\]

To show (3), consider the \( \tau \)-term \( x \) that just consists of a variable. Then

\[
\xi(1^S) = \xi(\text{id}_B) = \xi(x^B) = x^A = \text{id}_A = 1^{\text{Sym}(A)}.
\]

To show (4), let \( g \) and \( h \) be \( \tau \)-terms. Then

\[
\xi(g^B h^B) = \xi((gh)^B) = (gh)^A = g^A h^A = \xi(g^B)\xi(h^B).
\]

Finally, since \( \text{Gr}(B) \) is a permutation group, there exists a \( \tau \)-term \( t \) such that

\[
\mu = t^B = (g^B)^{-1}.
\]

and hence \( \Delta = (g^A)^{-1} \). We then have

\[
\xi(g^B)^{-1} = \xi(t^B) = \Delta = (g^A)^{-1} = \xi(g^B)^{-1}.
\]

\( \square \)

**Corollary 5.1.2.** Let \( C \leq \text{Sym}(B) \) and \( H \leq \text{Sym}(C) \). Then the following are equivalent:

1. \( G \) and \( H \) are isomorphic as abstract groups;
2. There exists a \( G \)-set \( B \) and an \( H \)-set \( C \) such that \( \text{HSP}(B) = \text{HSP}(C) \).

**Proof.** (2) \( \Rightarrow \) (1):
5.2. Topological Birkhoff

In the following, uniform continuity will refer to the left-invariant ultrametric of $\text{Sym}(A)$.

**Theorem 5.2.1** (Topological Birkhoff for permutation groups). Let $G$ be a subgroup of $\text{Sym}(B)$ and let $B$ be a $G$-set with signature $\tau$.

- For every continuous homomorphism $\xi: G \to \text{Sym}(A)$ such that $\xi(G)$ has finitely many orbits, there is an $A \in \text{HSP}_G^\text{fin}(\{B\})$ such that $t^A = \xi(t^B)$ for every $\tau$-term $t$.

- If $A \in \text{HSP}_G^\text{fin}(\{B\})$ then the function $\xi$ that maps $t^B$ to $t^A$ for every $\tau$-term $t$ is well-defined and a (uniformly) continuous surjective homomorphism from $G$ to $\text{Gr}(A)$.

**Proof.** The proof is an adaptation of the proof of Theorem 5.1.1. Let $F = \{a_1, \ldots, a_k\} \subseteq A$ be a finite set that contains one element from each orbit of $\xi(G)$ on $A$. By Proposition 4.2.10, $\xi$ is uniformly continuous, and hence there exists a finite $F' \subseteq B$ such that

$$\text{for all } f, g \in G \text{ if } f|_{F'} = g|_{F'} \text{ then } \xi(f)|_{F} = \xi(g)|_{F}. \quad (5)$$

Choose $F'$ to be smallest possible; note that this implies that $F'$ contains at most one element from each orbit of $G$. Let $C = (F')^G$ and let $m := |C| = |F'|^k$. Let $c^1, \ldots, c^m$ be the elements of $C$, and for $j \leq k$ define $c_j = (c^1(a_j), \ldots, c^m(a_j))$. Let $S$ be the substructure of $B^m$ generated by $c_1, \ldots, c_k$; so the elements of $S$ are precisely those of the form $t^S(c_j)$ for a $\tau$-term $t$ and $j \leq k$. Define a function $\mu: S \to A$ by setting

$$\mu(t^S(c_j)) := \xi(t^B(a_j)).$$

**Claim 1.** $\mu$ is well-defined. Suppose that $t^S(c_j) = r^S(c_l)$ for $j, l \leq k$. We first show that $t^B|_{F'} = r^B|_{F'}$. Let $b \in F'$. Note that there is some $i \leq m$ such that $c^i(a_j) = b$ and $c^i(a_l) = b$. Hence,

$$t^B(b) = t^B(c^i(a_j)) = t^B(c_j)_i = r^B(c_j)_i \quad (\text{since } t^S(c_j) = r^S(c_l)$$

$$= r^B(c^i(a_l)) = r^B(b).$$

Hence, $t^B|_{F'} = r^B|_{F'}$. Therefore, using $(5)$, we obtain

$$\xi(t^B)|_{F} = \xi(r^B)|_{F}.$$

It also follows from $t^S(c_j) = r^S(c_l)$ that for all $i \leq m$ the elements $(c_j)_i$, and $(c_l)_i$, lie in the same orbit of $G$. By our assumption on $F'$ this means that $l = j$. Hence,

$$\xi(t^B)(a_j) = \xi(r^B)(a_j) = \xi(r^B)(a_l),$$

and $\mu$ is indeed well-defined.

**Claim 2.** $\mu$ is surjective. This follows immediately from the assumption that $F$ contains an element from each orbit of $\xi(G)$ on $A$.

Let $A$ be the $\tau$-structure where $g^A := \xi(g^B)$ for every $g \in \tau$. 


Claim 3. \( \mu \) is a homomorphism. Let \( f \in \tau \) and let \( s \in S \). Since \( \mathcal{S} \) is generated by \( c_1, \ldots, c_k \) there exists a \( \tau \)-term \( t \) and a \( j \leq k \) such that \( s = t(c_j) \). Then

\[
\mu(f(\mathcal{S}(s))) = \mu(f(\mathcal{S}(t(c_j)))) = \mu((f(t))\mathcal{S}(c_j)) \\
= (f(t))\Delta(a_j) \\
= f\Delta(t\Delta(a_j)) \\
= f\Delta(\mu(t\mathcal{S}(c_j))) = f\Delta(\mu(s)).
\]

It follows that \( \mathcal{A} \) is the homomorphic image of the subalgebra \( \mathcal{S} \) of \( B^n \), and so \( \mathcal{A} \in \text{HSP}^{\text{fin}}(B) \).

For the second statement we have already seen in Theorem 5.1.1 that if \( \mathcal{A} \in \text{HSP}^{\text{fin}}(B) \subseteq \text{HSP}(B) \) then the natural homomorphism \( \xi: \mathcal{G} \to \text{Gr}(\mathcal{A}) \) exists. It thus remains to show that \( \xi \) is uniformly continuous.

Let \( F \) be a finite subset of \( \mathcal{A} \). We have to find a finite subset \( F' \) of \( B \) such that for all \( k \in \mathbb{N} \) and \( f, g \in G \) if \( f|_F = g|_F \) then \( \xi(f)|_F = \xi(g)|_F \). By assumption, there exists a surjective homomorphism \( \mu \) from a subalgebra \( \mathcal{S} \) of \( B^n \), for some \( n \in \mathbb{N} \), to \( \mathcal{A} \). For each \( a \in F \) pick an \( s \in S \) such that \( \mu(s) = a \); let \( F' \subseteq B \) be the (finite) set of all entries of all the tuples in \( s \). Now let \( f, g \in \text{Gr}(B) \) be such that \( f|_F = g|_F \). Choose \( \tau \)-terms \( r, t \) such that \( r|_B = f \) and \( t|_B = g \). Clearly, \( r\mathcal{S}|_{\mathcal{S} \cap G^n} = t\mathcal{S}|_{\mathcal{S} \cap G^n} \). Since \( F \subseteq \mu(S \cap G^n) \) it follows that \( r\Delta|_F = t\Delta|_F \), which proves the statement since \( r\Delta = \xi(f) \) and \( t\Delta = \xi(g) \).

**Corollary 5.2.2** (Topological Birkhoff for permutation groups). Let \( \mathcal{G} \) be a subgroup of \( \text{Sym}(B) \) and let \( \mathcal{B} \) be a \( G \)-set with signature \( \tau \). Then there exists a continuous homomorphism \( \xi: \mathcal{G} \to \text{Sym}(\mathcal{A}) \) such that \( \xi(G) \) has finitely many orbits, if and only if there exists an \( \mathcal{A} \in \text{HSP}^{\text{fin}}(\langle B \rangle) \) such that \( \text{Gr}(\mathcal{A}) = \xi(G) \).

### 5.3. Continuous Homomorphisms and Interpretability

In this section we prove Theorem 5.0.1. we give a characterisation of topological isomorphism of automorphism groups of \( \omega \)-categorical structures in terms of bi-interpretability. We first establish the link between pseudo-varieties and full interpretations.

**Definition 5.3.1.** Let \( I \) be a \( d \)-dimensional interpretation of \( \mathcal{A} \) in \( B \). Then \( I \) is called full if a relation \( R \subseteq A^k \) is first-order definable in \( \mathcal{A} \) if and only if the relation \( h^{-1}(R) \), defined as

\[
\{(b_1', \ldots, b_d'), \ldots, b_k') \in B^{d_1 \times d_2} \mid (b_1, \ldots, b_d), \ldots, (b_k, \ldots, b_k') \in S \text{ and } (h(b_1', \ldots, b_d'), \ldots, h(b_k', \ldots, b_k')) \in R \}
\]

is first-order definable in \( B \).

Observe that any structure with an interpretation in a structure \( B \) is a reduct of a structure with a full interpretation in \( B \).

**Theorem 5.3.2.** Let \( B \) be a finite or countably infinite \( \omega \)-categorical structure and let \( B' \) be an \( \text{Aut}(B) \)-set. Then \( \mathcal{A} \) has a full interpretation in \( B \) if and only if there is an \( \mathcal{A}' \in \text{HSP}^{\text{fin}}(B') \) such that \( \text{Gr}(\mathcal{A}') = \text{Aut}(\mathcal{A}) \).

**Proof.** Suppose first that \( \mathcal{A} \) has a \( d \)-dimensional full interpretation \( I \) in \( B \). Since \( I^{-1}(\mathcal{A}) \) is definable in \( B \), it is preserved by all operations in \( B' \), and therefore induces a subalgebra \( \mathcal{C}' \) of \( B'^n \). Let \( K \) be the kernel of \( I \). Since \( I^{-1}(\mathcal{A}) \) is definable in \( B \), all operations of \( B' \) preserve \( K = I^{-1}(\mathcal{A}) \), so \( K \) is a congruence of \( \mathcal{C}' \). Thus, \( I \) is a surjective homomorphism from \( \mathcal{C}' \) to \( \mathcal{A}' := \mathcal{C}'/K \). We verify that \( \text{Gr}(\mathcal{A}') = \text{Aut}(\mathcal{A}) \).
Corollary 3.1.6 it suffices to show that a relation \( R \subseteq A^k \) is definable in \( A \) if and only if it is preserved by all operations of \( A' \). For every \( f \in \tau \), the relation \( R \) is preserved by \( f \mathcal{L} \mathcal{B} \) if and only if \( f \mathcal{L} \mathcal{B} \) preserves \( I^{-1}(R) \), which is the case if and only if \( I^{-1}(R) \) is definable in \( B \). This in turn is the case if and only if \( R \) is definable in \( A \) by the assumption that the interpretation \( I \) is full. Now suppose that there is an algebra \( A' \in \mathsf{HSP}^{\text{fin}}(\mathcal{B}') \) such that \( \mathsf{Gr}(A') = \mathsf{Aut}(A) \). So there exists a finite number \( d \geq 1 \), a subalgebra \( C' \) of \( \mathcal{B}'^d \), and a surjective homomorphism \( I' \) from \( C' \) to \( A' \).

We claim that \( I \) is a \( d \)-dimensional interpretation of \( A \) in \( B \). All operations of \( \mathcal{B}' \) preserve \( C \) (viewed as a \( d \)-ary relation over \( B \)) since \( C' \) is a substructure of \( \mathcal{B}'^d \). By Theorem 3.1.1, this implies that \( C \) has a definition in \( B \), which becomes the domain formula of the interpretation. Since \( I \) is an algebra homomorphism, the kernel \( K \) of \( I \) is a congruence of \( C' \). It follows that \( K \), viewed as a \( 2d \)-ary relation over \( B \), is preserved by all operations from \( \mathcal{B}' \). Theorem 3.1.1 implies that \( K \) has a definition in \( B \). This definition becomes the interpreting formula of the equality relation on \( A \).

To see that \( I \) is a full interpretation, let \( R \subseteq A^k \) be a relation of \( A \), let \( \tau \) be the signature of \( B' \), and let \( f \in \tau \) be arbitrary. By assumption, \( f \mathcal{L} \mathcal{A} \) preserves \( R \). Therefore, \( f \mathcal{L} \mathcal{B} \) preserves \( I^{-1}(R) \). Hence, all automorphisms of \( B \) preserve \( I^{-1}(R) \), and because \( B \) is \( \omega \)-categorical, the relation \( I^{-1}(R) \) has a definition in \( B \) (Theorem 3.1.1), which becomes the interpreting formula for \( R(x_1, \ldots, x_k) \). We have verified that \( I \) is an interpretation of \( A \) in \( B \). To see that \( I \) is a full interpretation, let \( R \subseteq A^k \) be a relation such that \( I^{-1}(R) \) is definable in \( B \). Then \( I^{-1}(R) \) is preserved by \( \mathsf{Gr}(\mathcal{B}') \) and \( R \) is preserved by \( \mathsf{Gr}(A') \). By assumption \( \mathsf{Gr}(\mathcal{A}') = \mathsf{Aut}(A) \), and hence \( R \) is preserved by all automorphisms of \( A \) and definable in \( A \) by Theorem 3.1.1.

The following corollary is a direct consequence of Theorem 5.3.2 and Theorem 3.1.1.

**Corollary 5.3.3.** Let \( B \) be a countable \( \omega \)-categorical and \( A \) an arbitrary structure. Then \( A \) has a full interpretation in \( B \) if and only if \( A \) is at most countable \( \omega \)-categorical and there exists a continuous homomorphism from \( \mathsf{Aut}(B) \) to \( \mathsf{Aut}(A) \) whose image is dense in \( \mathsf{Aut}(A) \).

**Proof.** Let \( B' \) be a \( \mathsf{Aut}(B) \)-set. By Theorem 5.3.2 there is a full interpretation of \( A \) in \( B \) if and only if there is an \( A' \in \mathsf{HSP}^{\text{fin}}(B') \) such that \( \mathsf{Gr}(A') = \mathsf{Aut}(A) \).

Corollary 5.2.2 shows that this is the case if and only if there exists a continuous homomorphism from \( \mathsf{Aut}(B) \) to \( \mathsf{Aut}(A) \) whose image is dense in \( \mathsf{Aut}(A) \).

In Corollary 5.3.3 we cannot in general require surjectivity of the homomorphism (instead of requiring that the image being dense) as we have seen in Example 4.2.13.

### 5.4. Topological Isomorphism and Bi-interpretability

We continue to state some consequences of the topological Birkhoff theorem.

**Theorem 5.4.1.** Let \( C, D \) be \( \omega \)-categorical structures. Then (1) \( \iff \) (2) \( \iff \) (3):

1. \( \mathsf{Aut}(C) \) and \( \mathsf{Aut}(D) \) are isomorphic as topological groups.
2. There is an \( \mathsf{Aut}(C) \)-set \( C' \) and an \( \mathsf{Aut}(D) \)-set \( D' \) such that 
   \[ \mathsf{HSP}^{\text{fin}}(C') = \mathsf{HSP}^{\text{fin}}(D') \.] 
3. \( C \) and \( D \) are bi-interpretable.

**Proof.** The equivalence between (1) and (2) follows from Theorem 5.2.1. For the implication from (2) to (3), we assume that there is a \( d_1 \geq 1 \), a substructure \( S_1 \) of \( C^{d_1} \), and a surjective homomorphism \( h_1 \) from \( S_1 \) to \( D \). Moreover, we assume that
there is a $d_2 \geq 1$, a subalgebra $S_2$ of $D^{d_2}$, and a surjective homomorphisms $h_2$ from $S_2$ to $C$. The proof of Theorem 5.3.2 shows that $h_1$ is an interpretation of $D$ in $C$ and $h_2$ is an interpretation of $C$ in $D$. Because the statement is symmetric it suffices to show that the (graph of the) function $h_1 \circ h_2: (S_2)^{d_1} \to D$ defined by

$$(y_1,1, \ldots, y_1,d_2, \ldots, y_{d_1},1, \ldots, y_{d_1},d_2) \mapsto h_1(h_2(y_1,1, \ldots, y_1,d_2), \ldots, h_2(y_{d_1},1, \ldots, y_{d_1},d_2))$$

is definable in $D$. Theorem 3.1.1 asserts that this is equivalent to showing that $h_1 \circ h_2$ is preserved by $\text{Aut}(D) = \text{Gr}(D)$. So let $b$ be an element of $(S_2)^{d_1}$. Then indeed

$$f^C((h_1 \circ h_2)(b)) = h_1(f^D(h_2(b_1)), \ldots, f^D(h_2(b_1))))$$

$$= h_1(h_2(f^C(b_1)), \ldots, h_2(f^C(b_1))))$$

$$= (h_1 \circ h_2)(f^D(b)).$$

For the implication from (2) to (1), suppose that $C$ and $D$ are bi-interpretable via an interpretation $I_1: C^{d_1} \to D$ and $I_2: D^{d_2} \to C$. Let $A'$ be an $\text{Aut}(A)$-set. As we have seen in the proof of Theorem 5.3.2, the domain $S_1$ of $I_1$ induces a structure $S_1$ in $(A')^{d_1}$ and $I_1$ is a surjective homomorphism from $S_1$ onto an $\text{Aut}(D)$-set $D'$ with the same signature $\tau$ as $C'$. Similarly, the domain $S_2$ of $I_2$ induces in $(D')^{d_2}$ a structure $S_2$ and $I_2$ is a homomorphism from $S_2$ onto an $\text{Aut}(C)$-set $A''$ with the same signature as $D'$.

We claim that $\text{HSP}^{\text{fin}}(C') = \text{HSP}^{\text{fin}}(D')$. The inclusion `$\supseteq$' is clear since $D' \in \text{HSP}^{\text{fin}}(C')$. For the reverse inclusion it suffices to show that $C' = A''$ since $A'' \in \text{HSP}^{\text{fin}}(D')$. Let $f \in \tau$; we show that $f^{C'} = f^{A''}$. Let $a \in C$. Since $I_2 \circ I_1$ is surjective onto $C$, there are $c = (c_1,1, \ldots, c_{d_1},d_2) \in C^{d_1,d_2}$ such that $a = I_2(I_1(c))$. Then

$$f^{A''}(a) = f^{A''}(I_2 \circ I_1(c))$$

$$= I_2(f^{D'}(I_1(c_1,1, \ldots, c_{d_1},1)), \ldots, f^{D'}(I_1(c_{d_1},1, \ldots, c_{d_1},d_2))))$$

$$= I_2 \circ I_1(f^{C'}(c))$$

$$= f^{C'}(I_2 \circ I_1(c)) = f^{C'}(a)$$

where the second and third equations hold since $I_2$ and $I_1$ are algebra homomorphisms, and the fourth equation holds because $f^{C'}$ preserves $h_2 \circ h_1$, because $I_2 \circ I_1$ is homotopic to the identity.

**Example 5.4.2.** The structures

$$C := (\mathbb{N}^2; \{(u_1, u_2), (v_1, v_2) \mid u_1 = v_1\}) \quad \text{and} \quad D := (\mathbb{N}; =)$$

are mutually interpretable, but not bi-interpretable. To see this, observe that $\text{Aut}(C)$ has a proper non-trivial closed normal subgroup $N$. To specify $N$, let $P_i := \{(u_1, u_2) \mid u_1 = i\}$ for $i \in \mathbb{N}$. Then $\text{Aut}(C, P_1, P_2, \ldots)$ is a non-trivial (and $\text{Aut}(C) \cap N$ is isomorphic to $\text{Aut}(D)$) proper closed subgroup, and it can be verified that $N$ is normal (see Proposition 4.6.2), whereas $\text{Aut}(D)$, the symmetric permutation group of a countably infinite set, has no proper non-trivial closed normal subgroups (it has exactly three proper non-trivial normal subgroups), none of which is closed.

We also mention that Theorem 5.4.1 combined with Proposition 5.5.8 shows that for every $\omega$-categorical structure $B$, having essentially infinite signature only depends on the automorphism group of $B$ viewed as a topological group.

**Exercises.**

(44) Show that every finite structure has an interpretation in every structure with at least two elements.
(45) Show that the line graph of an undirected graph $G$ has a first-order interpretation in $G$.

(46) Show that the automorphism group of the line graph of the infinite clique (called the Johnson graph $J(\omega, 2)$) is topologically isomorphic to $\text{Sym}(\mathbb{N})$.

(47) Show that the automorphism group of infinitely many disjoint copies of the 2-element clique $K_2$ is not topologically isomorphic to infinitely many disjoint copies of the three-element clique $K_3$. 

CHAPTER 6

Reconstruction of Topology and Automatic Continuity

6.1. Reconstruction Notions

We study the question whether we can reconstruct the topology of closed subgroups of \(\text{Sym}(\mathbb{N})\) from the abstract group.

**Definition 6.1.1.** Let \(G\) be a closed subgroup of \(\text{Sym}(\mathbb{N})\). We say that
- \(G\) is **reconstructible** (or that \(G\) has reconstruction) iff for every other closed subgroup \(H\) of \(\text{Sym}(\mathbb{N})\), if there exists an isomorphism between \(H\) and \(G\), then there also exists a group isomorphism between \(H\) and \(G\) which is a homeomorphism;
- \(G\) has **automatic homeomorphicity (AH)** iff every group isomorphism between \(G\) and a closed subgroup of \(\text{Sym}(\mathbb{N})\) is a homeomorphism;
- \(G\) has **automatic continuity (AC)** iff every homomorphism from \(G\) to \(\text{Sym}(\mathbb{N})\) is continuous.

Obviously, automatic homeomorphicity implies reconstruction. Less obviously, we will later see in Proposition 6.3.13 that automatic continuity implies automatic homeomorphicity.

There are two dominant methods for the reconstruction of group topology. The first method is via showing the **small index property**, the second is based on Rubin’s forall-exists interpretations.

We have seen an example of a closed oligomorphic permutation group without automatic continuity in Example 4.2.10. This example has still reconstruction; this can be shown using the results of Rubin (see Remark 5.4.3 in \([82]\)). A more involved example of a closed oligomorphic permutation group without reconstruction has been found by Evans and Hewitt \([47]\).

6.2. The Small Index Property

Recall that a subgroup of \(\text{Sym}(\mathbb{N})\) is open if it contains the stabiliser \(G_a\) for some \(a \in \mathbb{N}^n, n \in \mathbb{N}\). Clearly, these groups have countable index, so all open subgroups of \(\text{Sym}(\mathbb{N})\) have countable index. A topological group \(G\) has the **small index property (SIP)** if the converse holds as well, i.e., if every subgroup of \(G\) of at most countable index is open.

**Proposition 6.2.1 (Folklore).** Let \(G\) be a closed subgroup of \(\text{Sym}(\mathbb{N})\). Then \(G\) has automatic continuity if and only if it has the small index property.

**Proof.** Let \(G\) be a topological group with automatic continuity, and let \(U\) be a subgroup of \(G\) of at most countable index. We have to show that \(U\) is open. Let \(\xi: G \to G/U\) be the action of \(G\) on the left cosets of \(U\) in \(G\) by left translation (Example 1.3.4), where \(G/U\) is equipped with the discrete topology (we do not yet know
that this equals the quotient topology; recall Proposition 4.2.11 for this topology, \( \xi \) would be always continuous 4.2.9). By automatic continuity, \( \xi \) is continuous. In particular, the pre-image of the open set \( \{ \alpha \in \text{Sym}(G/U) \mid \alpha(U) = U \} \) is open. But this pre-image is precisely \( U \), which proves the SIP.

Now suppose that \( G \) has the SIP, and let \( \xi \) be an isomorphism between \( G \) and another closed subgroup of \( \text{Sym}(N) \). We show that for every basic open set \( U := S(a,b) \cap \xi(G) \) for \( a, b \in N^n \), \( n \in N \), the set \( \xi^{-1}(U) \) is open, too. Writing \( S_a \) for the stabiliser subgroup \( \{ \alpha \in \xi(G) \mid \alpha(a) = a \} \), we have \( U = \beta S_a \) for \( \beta \in U \). Since \( \xi \) is a homomorphism, \( \xi^{-1}(U) = \xi^{-1}(\beta) \xi^{-1}(S_a) \). The subgroup \( S_a \) of \( \xi(G) \) has countable index, and therefore \( \xi^{-1}(S_a) \) is a subgroup of \( G \) of countable index, too. By assumption, \( \xi^{-1}(S_a) \) is open, and since multiplication by \( \xi^{-1}(\beta) \) is continuous, \( \xi^{-1}(\beta) \xi^{-1}(S_a) = \xi^{-1}(U) \) is open, which establishes continuity of \( \xi \). The proof that \( \xi \) is open can be found in Proposition 6.3.13.

\[
\square
\]

The small index property has been verified for the following groups:

(1) \( \text{Sym}(N) \) \[41,102,107\];
(2) the automorphism groups of countable vector spaces over finite fields \[45\];
(3) \( \text{Aut}(Q;<) \) and the automorphism group of the atomless Boolean algebra \[117\];
(4) the automorphism group of the \( \omega \)-categorical dense semi-linear order giving rise to a meet-semilattice \[44\];
(5) the automorphism group of the random graph \[59\];
(6) \( \omega \)-categorical \( \omega \)-stable structures \[59\];
(7) the automorphism group of the Henson graphs \[56\].

On the other hand, the small index property is not known for the automorphism groups of the countable universal homogeneous tournament, the countable universal poset, or the homogeneous universal permutation (see \[82\]).

### 6.3. Consistency of Automatic Continuity

In this section we prove that it is consistent with Zermelo-Fraenkel set theory that every closed subgroup of \( \text{Sym}(N) \) has automatic homeomorphicity (Theorem 6.3.12).

#### 6.3.1. Nowhere dense, meager, Baire

Let \( X \) be a topological space.

**Definition 6.3.1.** A set \( A \subseteq X \) is called **somewhere dense** if its closure contains a nonempty open set, and nowhere dense otherwise.

Note that \( A \subseteq X \) is nowhere dense if and only if for every nonempty open set \( U \subseteq X \) there is a nonempty open set \( V \subseteq U \) such that \( V \) and \( A \) are disjoint. Yet another equivalent phrasing of the definition is: \( A \subseteq X \) is nowhere dense if and only if \( X \setminus \overline{A} \) is dense.

**Definition 6.3.2.** Let \( X \) be a topological space. A set \( A \subseteq X \) is meager if it is a countable union of nowhere dense sets. The complement of a meager set is called comeager.

Clearly, every subset of a meager set is meager, and every countable union of meager sets is meager. For example, \( \mathbb{Q} \) is meager in \( \mathbb{R} \). A set is comeager if and only if it contains a countable intersection of dense open sets. Recall the following definition from Theorem 1.1.26.

**Definition 6.3.3.** A topological space \( X \) is called **Baire** if every countable intersection of open dense sets is dense.

**Proposition 6.3.4.** Let \( X \) be a topological space. Then the following statements are equivalent:

1. \( X \) is Baire.
2. Every closed subgroup of \( \text{Sym}(N) \) has automatic homeomorphicity.
3. Every closed subgroup of \( \text{Sym}(N) \) has automatic continuity.
4. Every closed subgroup of \( \text{Sym}(N) \) is automatically continuous.
5. Every closed subgroup of \( \text{Sym}(N) \) is automatically homeomorphic.

**Theorem 6.3.12.** It is consistent with Zermelo-Fraenkel set theory that every closed subgroup of \( \text{Sym}(N) \) has automatic homeomorphicity.
(1) \( X \) is Baire.
(2) Every comeager set in \( X \) is dense.
(3) Every nonempty open set in \( X \) is non-meager.

Proof. (1) implies (2). Let \( A \) be comeager, so it is the countable intersection of complements of nowhere dense sets \((A_i)_{i \in \mathbb{N}}\). Since \( X \setminus A_i \) is dense for each \( i \in \mathbb{N} \) the set \( A \) is a countable intersection of dense sets, and since \( X \) is Baire \( A \) must be dense.

(2) implies (3). If \( U \) is meager, then \( X \setminus U \) is comeager and by (2) it is dense. Hence, \( U \) is either empty or not open.

(3) implies (1). Let \( A = \bigcap_{i \in \mathbb{N}} A_i \). If \( A \) is not dense, it contains a nonempty open set \( U \). By (3), \( U \) is non-meager. If all of the \( A_i \) would be dense open, then \( A \) and hence \( U \) would be meager.

\( \square \)

Exercises.

(48) We have already mentioned that \( \mathbb{R} \setminus \mathbb{Q} \) is comeager. Show that it is not meager.

Definition 6.3.5. A set \( A \) in a Baire space \( X \) is said to have the Baire property (with respect to \( X \)) if for some open set \( U \subseteq X \) the symmetric difference \( U \Delta A \) between \( A \) and \( U \) is meager.

Example 6.3.6. Using the Axiom of Choice (AC), we construct a subset of \( \mathbb{R} \) which does not have the Baire property: TODO. Using the existence of non-principal ultrafilters, we can construct subsets of \( \mathbb{N} \) which do not have the Baire property: TODO.

Proposition 6.3.7 (Levi [79]). Let \( X \) be Polish, \( Y \) a metric space, and \( f \) continuous. Then the image of every open set under \( f \) has the Baire property in \( Y \).

The axiom of dependent choice (DC) is a weak form of the axiom of choice (AC) that is still sufficient to develop most of real analysis. It says that for every non-empty set \( X \) and every binary relation \( R \subseteq X^2 \) such that for every \( x \in X \) there exists a \( y \in X \) such that \((x, y) \in R\) there exists a sequence \((x_n)_{n \in \mathbb{N}}\) such that \((x_n, x_{n+1}) \in R\) for all \( n \in \mathbb{N} \). Over Zermelo-Fraenkel set theory (ZF), the axiom of dependent choice is equivalent to the version of Theorem 4.1.26 where we only require that \( S \) is completely metrisable (instead of Polish).

Theorem 6.3.8 (Shelah). It is consistent with ZF+DC that every subset of \( \mathbb{R} \) has the Baire property.

Recall that the irrational numbers are homeomorphic to the Baire space (Theorem 4.1.14). Hence, we obtain the following.

Corollary 6.3.9. It is consistent with ZF+DC that every subset of \( \mathbb{N}^\mathbb{N} \) has the Baire property.

6.3.2. The Baire property and permutation groups. We now discuss the concepts from the previous section (nowhere dense, meager, Baire) in the context of permutation groups.

Lemma 6.3.10. Let \( G \) be a Polish group. If \( U \) is a subgroup of \( G \) of countable index, then \( U \) is not meager.

Proof. If \( U \) is meager, then all cosets of \( U \) are meager, too. \( G \) is the union of all the cosets of \( U \), but is not meager by Proposition 6.3.4 (3). Since a countable union of a meager set is meager, \( U \) must have uncountable index.

A topological group is called Baire if it is Baire as a topological space.
**Lemma 6.3.11** (Lemma 2.6 in [77]). Let $G$ be a closed subgroup of $\text{Sym}(\mathbb{N})$, and let $H$ be a subgroup of $G$ which has the Baire property. Then $H$ is either meager or open.

**Proof.** It follows from Theorem 4.1.26 that $G$ is Baire. Let $U \subseteq \text{Sym}(\mathbb{N})$ be open such that $H \Delta U$ is meager. If $U$ is empty then $H$ is meager. Otherwise, we have to show that $H$ is open. Since $U$ is nonempty open it contains $fG_a$ for some $f \in G$, $a \in \mathbb{N}^n$, and $n \in \mathbb{N}$.

**Claim.** The subgroup $H \cap G_a$ of $G_a$ is comeager in $G_a$. The set $fG_a$ is non-meager by Proposition 6.3.4 (3). If $fG_a \subseteq U \setminus H$, then $U \setminus H$ and therefore $H \Delta A$ would not be meager. Hence, there exists an $h \in H \cap fG_a$. Since $fG_a \setminus H$ is meager, we have that $h^{-1}fG_a \setminus h^{-1}H$ is meager. But $h^{-1}H = H$ and $h^{-1}fG_a = G_a$ and hence $G_a \setminus H$ is meager, which proves the claim.

The claim implies that all cosets of $H \cap G_a$ in $G_a$ are comeager. If there would be more than one coset this would imply that all the cosets are also meager, in contradiction to $G_a$ being not meager (Proposition 6.3.4 (3)). Therefore, $H = G_a$ and $H$ is open.

**Theorem 6.3.12** (Lascar [77]). It is consistent with ZF+DC that every closed subgroup $G$ of $\text{Sym}(\mathbb{N})$ has automatic continuity.

**Proof.** Assume that every subset of $G$ has the Baire property; this is consistent with ZF+DC by Corollary 6.3.9. Let $U$ be a subgroup of $G$ of countable index. Then $U$ cannot be meager by Lemma 6.3.10, so it must be open by Lemma 6.3.11. Hence, $G$ has automatic continuity by Proposition 6.2.1.

The following statement shows that the machinery of this section can also be used to prove some absolute statements that do not rely on some consistency assumptions.

**Proposition 6.3.13** (Corollary 2.8 in [77]). Let $G$ be a closed subgroup of $\text{Sym}(\mathbb{N})$. Then every continuous isomorphism $\phi: G \to \text{Sym}(\mathbb{N})$ is a homeomorphism.

**Proof.** Let $U$ be an open subgroup of $G$. We have to show that $\phi(U)$ is open in $H$. Proposition 6.3.7 asserts that $\phi(U)$ has the Baire property. Recall that $U$ has countable index in $G$, so $\phi(U)$ has countable index in $H$. Hence, it cannot be meager according to Lemma 6.3.10, and hence must be open because of Lemma 6.3.11.

Proposition 6.3.13 shows that automatic continuity of closed subgroups of $\text{Sym}(\mathbb{N})$ is a property of the abstract group in the sense that if two closed subgroups of $\text{Sym}(\mathbb{N})$ are isomorphic as abstract groups, and one has automatic continuity, then so has the other.

### 6.4. Ample Generics

An element of $G$ is called *generic* if it lies in a comeager orbit with respect to the action of $G$ on $G$ by conjugation (Example 1.3.5). The **diagonal conjugacy action** of $G$ on $G^n$ is the action given by

$$g \cdot (g_1, \ldots, g_n) := (gg_1g^{-1}, \ldots, gg_ng^{-1}).$$

A Polish group $G$ has *ample generics* if for each $n \in \mathbb{N}$ this action of $G$ on $G^n$ has a generic element. We will see many examples of structures with ample generics later.

From Macpherson survey: “For example, Hodkinson [personal communication] showed that for $(\mathbb{Q}, <)$, the automorphism group $G$ does not have a comeagre orbit in its diagonal action by conjugation on $G^2$.”
Lemma 6.4.1 (Lemma 6.7 in [69]). Let $G$ be a Polish group with ample generics acting continuously on a Polish space $X$. For every $n \in \mathbb{N}$ let $A_n, B_n \subseteq X$ be such that $A_n$ is not meager and $B_n$ is not meager in any non-empty open subset of $X$. Then there is a continuous map $h: 2^\mathbb{N} \to G$ such that if $a, b \in 2^\mathbb{N}$ are such that $a|_{\{1, \ldots, n\}} = b|_{\{1, \ldots, n\}}$, $a(n) = 0$, and $b(n) = 1$, then $h(a)A_n \cap h(b)B_n \neq \emptyset$.

Proof. TODO. □

Theorem 6.4.2 (Theorem 6.9 in [69]). Let $G$ be a Polish group with ample generics. Then $G$ has the small index property.

Proof. TODO. □

6.4.1. Turbulence. Let $G$ be a Polish group acting continuously on a Polish space $X$. Let $x \in X$, let $U$ be an open subset of $X$ that contains $x$, and let $V$ be an open subset of $X$ that contains $1$ such that $V^{-1} = V$. Then the local orbit $O(x, U, V)$ is defined as

$$\{ y \in X : \exists k \in \mathbb{N}, \beta_0, \beta_1, \ldots, \beta_k \in V \forall i \in \{0, \ldots, k\}. \beta_i, \beta_{i-1} \ldots \beta_0 \cdot x \in U$$

and

$$\beta_k \beta_{k-1} \ldots \beta_0 \cdot x = y \}$$

Then $x \in X$ is called turbulent if for every open $U \subseteq X$ that contains $x$ and every open $V \subseteq X$ that contains $1$ such that $V^{-1} = V$, the local orbit $O(x, U, V)$ is somewhere dense.

Proposition 6.4.3 (Proposition 3.2 in [69]). Let $G \leq \text{Sym}(\mathbb{N})$ be closed and $\xi: G \to X$ be a continuous action of $G$ on a Polish space $X$. Then $x \in X$ is turbulent if and only if the orbit of $x$ is non-meager.

Proof. TODO. □

6.4.2. Dense conjugacy classes. We have seen in the previous section that if $G \leq \text{Sym}(\mathbb{N})$ has ample generics, then it has the small index property: but how do we prove that $G$ has ample generics? To this end, we present a powerful technique from [69] which itself is based on ideas from [59]. Let $\mathcal{K}$ be an amalgamation class. Then $\mathcal{K}_p$ denotes the class of all tuples

$$(A, B, C, e: B \to C)$$

such that

- $A, B, C \in \mathcal{K}$,
- $B$ and $C$ are substructures of $A$,
- $e$ is an isomorphism between $B$ and $C$.

An embedding of $(A, B, C, e: B \to C) \in \mathcal{K}_p$ into $(A', B', C', e': B' \to C') \in \mathcal{K}_p$ is an embedding $f: A \hookrightarrow A'$ such that $f(B) \subseteq B'$, $f(C) \subseteq C'$, and $f \circ e = e' \circ (f|_B)$. Note that the definition of the joint embedding property (JEP) and the amalgamation property (AP) were purely categorical in the sense that their definition only requires a notion of embedding; so JEP and AP are naturally defined not only for structures and embeddings, but also for classes of the form $\mathcal{K}_p$ as introduced above.

Theorem 6.4.4 (Theorem 2.1 in [69]). Let $\mathcal{K}$ be an amalgamation class and let $\mathcal{L}$ be its Fraïssé limit. Then the following are equivalent.

- $\mathcal{G} := \text{Aut}(\mathcal{L})$ has a dense conjugacy class.
- $\mathcal{K}_p$ has the JEP.
Proof. Fix an element $\alpha \in G$ having a dense conjugacy class in $G$. To show that $K_p$ satisfies the JEP, let $(A_i, B_i, C, e_i : B_i \to C_i) \in K_p$ for $i \in \{1,2\}$; we assume that $A_i$ is a substructure of $L$. By the homogeneity of $L$, the embedding $e_i$ has an extension in $G$, so by the density of the conjugacy class of $f$ there is a $\beta_i \in G$ such that $e = \beta_i^{-1}\alpha\beta_i|_B$. Let

$$
A := L[\beta_1^{-1}A_1 \cup \beta_2^{-1}A_2]
$$

$$
B := L[\beta_1^{-1}B_1 \cup \beta_2^{-1}B_2]
$$

$$
C := L[\beta_1^{-1}C_1 \cup \beta_2^{-1}C_2]
$$

$$
e := \alpha|_B.
$$

Then $h_i := \beta_i^{-1}|_{A_i}$ is an embedding of $(A_i, B_i, C_i, e_i)$ into $(A, B, C, e)$.

Conversely, suppose that $K_p$ has the JEP. We find an $f \in G$ with a dense conjugacy class by a Baire category argument. Let $(A, B, C, e) \in K_p$. Recall that the sets of the form $D(e) := \{\alpha \in G \mid e = \alpha|_B\}$ form a basis of the topology of $G$. Then

$$
\{\alpha \in G \mid \exists \beta \in G, \beta e \beta^{-1} = \alpha|_B\}
$$

is open as the union of open sets. So let $D(e)$ be also dense in $G$. So let $D(e')$ be another basic open set, and choose $A', B', C'$ such that $D' = (A', B', C', e') \in K_p$. By the JEP of $K_p$ there exists $D'' = (A'', B'', C'', e'') \in K_p$ and an embedding $f : D' \to D''$. By the homogeneity of $L$ there exists a $\beta \in G$ that extends $e''$, and $\beta$ witnesses that $D(e) \cap D(e') \neq \emptyset$. Since there are countably many sets of the form $D(e)$, the Baire category theorem (Theorem 4.1.26) implies that the intersection over all those sets is still dense, and in particular non-empty, concluding the proof. \hfill \Box

Corollary 6.4.5. $\text{Sym}(\mathbb{N}; =)$, the automorphism group of the random graph, and more generally the automorphism group of Fraïssé-limits with free amalgamation have a dense conjugacy class.

Proof. Let $K$ be the class of all finite structures with the empty signature so that the automorphism group of the Fraïssé-limit of $K$ is isomorphic (as a permutation group) to $\text{Sym}(\mathbb{N})$. By Theorem 6.4.4 it suffices to verify that $K_p$ has the JEP. So let $(A_i, B_i, C_i, e_i : B_i \to C_i)$ for $i \in \{1,2\}$. We may assume that $A_1 \cap A_2 = \emptyset$ and define $A := A_1 \cup A_2$, $B := B_1 \cup B_2$, $C := C_1 \cup C_2$, and $e : B \to C$ as the common extension of both $e_1$ and $e_2$. Then the identity map $f_1 : A_1 \to A$ is an embedding of $(A_1, B_1, C_1, e_1)$ into $(A, B, C, e)$ showing JEP. The same proof works for the other groups in the statement. \hfill \Box

6.4.3. The Weak Amalgamation Property. For the characterisation of the existence of ample generics we need the following variant of the amalgamation property, called the weak amalgamation property, which is of independent interest in model theory.

Definition 6.4.6. A class $K$ of finite structures satisfies the weak amalgamation property (WAP) if every $A \in K$ has an extension $A' \in K$ such that for all $B_1, B_2 \in K$ and embeddings $e_i : A' \to B_i$ for $i \in \{1,2\}$ there exists $C \in K$ and embeddings $f_i : B_i \to C$ such that $f_1 \circ e_1 = f_2 \circ e_2$.

Also the WAP only depends on the notion of embeddings, and hence makes also sense for the class $K_p$ from the previous section. A good example of a class of structures with JEP and WAP, but not the AP is the age of $(\mathbb{Z}; \{(x,y) \mid x = y + 1\})$. An $\omega$-categorical structure whose age has the WAP but not the AP is the structure $(\mathbb{Q}^\omega; <)$.

Exercises.
(49) Prove that the age of \((\mathbb{Z}; \{(x, y) \mid x = y + 1\})\) does not have the AP, but the WAP.

(50) Prove that the age of \((\mathbb{Q}_+^*; <)\) does not have the AP, but the WAP.

Also for classes of the form \(K_p\) from Section 6.4.2 we can study the question whether they have the AP or the WAP.

**Example 6.4.7.** Let \(K\) be the class of finite structures with the empty signature. Then \(K_p\) does not have the AP: let \((A, B, C, e; B \to C) \in K_p\) be such that \(B = \emptyset\), and let \((A, B, C, e_i; B \to C) \in K_p\) for \(i \in \{1, 2\}\) be such that \(B_1 = B_2\) is non-empty, but \(e_1 \neq e_2\). Then \((A, B, C, e; B \to C)\) embeds into \((A, B_1, C_1, e_1; B \to C)\) and into \((A, B_2, C_2, e_2; B \to C)\), but \(e_1\) and \(e_2\) cannot have a common extension, so amalgamation fails.

To verify that \(K_p\) has the WAP, let

\[ T = (A, B, C, e; B \to C). \]

Let \(a\) be an extension of \(e\) to a permutation of \(A\). Clearly, \(T\) embeds into \(T' := (A, B, C, a; B \to C)\). Let

\[ T_1 = (A_1, B_1, C_1, e_1; B_1 \to C_1) \in K_p \quad \text{and} \quad T_2 = (A_2, B_2, C_2, e_2; B_2 \to C_2) \in K_p \]

be such that there exist embeddings \(f_i: T' \to T_i\); we can assume without loss of generality that \(A_1 \cap A_2 = A\) that \(e_1\) and \(e_2\) preserve \(A\) Let \(A'\) be the free amalgam of \(A^1\) and \(A^2\), and for \(i \in \{1, \ldots, n\}\) let \(e'_i\) be the union of \(e_1\) and \(e_2\); this is well-defined by the stipulation that \(a\) and that \(e'_1|_A = e'_2|_A\) for all \(i \in \{1, \ldots, n\}\). Then \(A'\) is the amalgam of \(A^1\) and \(A^2\) via the identity mappings from \(A^1\) and \(A^2\) into \(A'\), proving the WAP.

**Theorem 6.4.8 (Theorem 3.4 in [69]).** Let \(K\) be an amalgamation class and let \(L\) be its Fraïssé limit. Then the following are equivalent.

- \(G := \text{Aut}(L)\) has a generic element.
- \(K_p\) has the JEP and the WAP.

**Proof.** Suppose that \(G\) has a generic element \(\alpha\), i.e., the orbit of \(\alpha\) with respect to the action of \(G\) on \(G\) by conjugation is comeager. Then the orbit is in particular dense (since \(G\) is Polish and by Proposition 6.3.4). Theorem 6.4.4 then shows that \(K_p\) satisfies the JEP.

The orbit of \(\alpha\) is also non-meager (if it would be meager then \(G\) would be meager, in contradiction to Proposition 6.3.4). Hence, \(\alpha\) is turbulent. To show that \(K_p\) satisfies the WAP, let \((A, B, C, e) \in K_p\) be given, and suppose that \(A\) is a substructure of \(F\). Since \(\alpha\) is in a dense conjugacy class we may assume that \(\alpha\) is an extension of \(e\). Let \(V := \{\beta \in G \mid |_A = \text{id}_A\}\). By the turbulence of \(\alpha\) we know that \(\{\beta^{-1}\alpha \beta \mid \beta \in V\}\) is dense in some open subset of \(G\) that contains \(f\).

Conversely, suppose that \(K_p\) has the JEP and the WAP. TODO.

**Example 6.4.9.** Discuss \((\mathbb{Q}; <)\).

Similarly, if \(L\) is the Fraïssé-limit of \(K\) we want to characterise when \(\text{Aut}(L)\) has ample generics. For this we introduce the class \(K_p^n\) for \(n \geq 1\), which consists of tuples \((A, e_1: B_1 \to C_1, \ldots, e_n: B_n \to C_n)\) where \(A, B_1, \ldots, B_n, C_1, \ldots, C_n \in K\) with \(B_i, C_i\) substructures of \(A\) and \(e_i\) an isomorphism between \(B_i\) and \(C_i\). Embeddings between elements of \(K_p^n\) are defined analogously as embeddings between elements of \(K_p\), and again properties like the JEP and the AP make sense.
Theorem 6.4.10 (Theorem 6.2 in [51]). Let \( \mathcal{K} \) be an amalgamation class and let \( L \) be its Fraïssé limit. Then the following are equivalent.

- \( \text{Aut}(L) \) has ample generics.
- for every \( n \), the class \( \mathcal{K}_n^p \) has the JEP and the WAP.

Lemma 6.4.11. \( \text{Sym}(\mathbb{N}) \) has ample generics.

Proof. Let \( \mathcal{K} \) be the class of all finite structures over the empty signature. We have already seen in Corollary 6.4.5 that \( \mathcal{K}_p \) has the JEP; the proof that \( \mathcal{K}_n^p \) has the JEP is analogous. Moreover, we have already seen in Example 6.4.7 that \( \mathcal{K}_p \) has the WAP; the proof that \( \mathcal{K}_n^p \) has the WAP is analogous. Now the statement follows from Theorem 6.4.10. \( \square \)

Corollary 6.4.12. \( \text{Sym}(\mathbb{N}) \) has the small index property and automatic homeomorphism.

Proof. We have just seen that \( \text{Sym}(\mathbb{N}) \) has ample generics, so it follows from Theorem 6.4.2 that \( \text{Sym}(\mathbb{N}) \) has the small index property. Automatic homeomorphism then follows from Proposition 6.2.1. \( \square \)

6.5. The Hrushovski Property

Theorem 6.5.1 (of [51]). For every finite graph \( G \) there exists a finite graph \( H \) that contains \( G \) as an induced subgraph such that every partial isomorphism of \( G \) extends to an automorphism of \( H \).

Proof. TODO: give a better proof than the original proof. \( \square \)

Corollary 6.5.2. The automorphism group of the random graph has the small index property and automatic continuity.

Proof sketch. Let \( \mathcal{K} \) be the class of all finite graphs so that the Fraïssé-limit \( L \) of \( \mathcal{K} \) is the random graph. Theorem 6.5.1 can be used to show that \( \mathcal{K}_n^p \) has the WAP. Theorem 6.4.10 therefore implies that \( G = \text{Aut}(L) \) has ample generics, and the small index property follows from Theorem 6.4.2. Finally, automatic continuity follows from Proposition 6.2.1 and the fact that the automorphism group of the random graph is simple. \( \square \)

We say that a class \( C \) of finite relational \( \tau \)-structures has the Hrushovski property if every structure \( A \in C \) has an extension \( B \in C \) such that every partial isomorphism of \( A \) extends to an automorphism of \( B \). Hence, Theorem 6.5.1 states that the class of all finite undirected graphs has the Hrushovski property. Clearly, the classes of all finite linear orders or the class of all finite partial orders do not have the Hrushovski property.

Herwig [55] showed the Hrushovski property for the class of all \( \tau \)-structures, for any finite relational signature \( \tau \), and Herwig and Lascar [56] for the class of all \( K_n \)-free graphs. Hence, it follows that the automorphism group of the countable universal homogeneous \( K_n \)-free graph has automatic homeomorphism. More generally, they showed for all classes that are described by homomorphically forbidding finitely many structures, i.e., classes \( C \) such that there exists a finite class of structures \( \mathcal{F} \) such that \( A \in C \) if no structure in \( \mathcal{F} \) admits a homomorphism to \( A \). They left as an open problem whether the class of all tournaments has the Hrushovski property [56].
6.6. The Strong Small Index Property

A permutation group $G \leq \text{Sym}(\mathbb{N})$ has the strong small index property (SSIP) if every countable index subgroup of $G$ lies between the pointwise and the setwise stabilizer of a finite subset of $\mathbb{N}$.

Again discuss $\text{Sym}(\mathbb{N})$ and the automorphism group of the random graph. Present the results from [96].

Example 6.6.1. An example of an oligomorphic permutation group which has the small index property, but not the strong small index property is the automorphism group of an equivalence relation $E$ on a set $V$ with infinitely many infinite classes: it has the open subgroup $H$ which fixes one equivalence class of $E$. Then $H$ has countable index, but is not contained in the set-stabiliser of $\text{Aut}(V; E)$ of some finite subset of $V$. TODO: reference for small index property.

Corollary 6.6.2. Let $\xi: \text{Sym}(\mathbb{N}) \to \text{Sym}(\mathbb{N})$ be a homomorphism such that $\xi(\text{Sym}(\mathbb{N}))$ is a primitive permutation group $G$. Then there exists an $n \in \mathbb{N}$ such that $\xi(\text{Sym}(\mathbb{N}))$ is isomorphic (as a permutation group) to the setwise action of $\text{Sym}(\mathbb{N})$ on $\binom{\mathbb{N}}{n}$.

Proof. Recall from Corollary 1.4.6 that the primitivity of $G$ implies that for any $a \in \mathbb{N}$ the point stabiliser $H := G_a$ is a maximal subgroup of $G$. Hence, the strong small index property of $\text{Sym}(\mathbb{N})$ implies that $H$ is contained in the set-wise stabiliser $\text{Sym}(\mathbb{N})_F$ for some finite $F \subseteq \mathbb{N}$. By the maximality of $H$, this means that $H$ equals $\text{Sym}(\mathbb{N})_F$. Let $n := |F|$. Let $i$ be the map from $\binom{\mathbb{N}}{n} \to X$ that maps for each $\alpha \in G$ the set $\alpha(F) \in \binom{\mathbb{N}}{n}$ to $\alpha(a) \in X$. Note that $i$ is well-defined because if $\alpha(F) = \beta(F)$, then $\alpha^{-1}\beta \in G_F = H$, and hence $\alpha^{-1}\beta(a) = a$, and hence $\alpha(a) = \beta(a)$. Moreover, it is straightforward to verify that $i$ is an isomorphism between the permutation groups $G$ and setwise action of $\text{Sym}(\mathbb{N})$ on $\binom{\mathbb{N}}{n}$. □
CHAPTER 7

Ramsey Classes and Topological Dynamics

7.1. The Kechris-Pestov-Todorcevic Connection

By the Kechris-Pestov-Todorcevic correspondence \([66]\), a structure \(B\) is Ramsey (with respect to colorings of embeddings) if and only if its automorphism group \(\text{Aut}(B)\) is extremly amenable, meaning that every continuous action of it on a compact Hausdorff space has a fixed point.


Example 7.1.1. Consider the natural continuous actions of \((\mathbb{R}^2, +)\) on the sphere and on the torus. What is the universal minimal flow?

7.2. Canonical Functions

If \(f : \mathbb{Q} \to \mathbb{Q}\) is any function from the order of the rational numbers to itself, then there are arbitrarily large finite subsets of \(\mathbb{Q}\) on which \(f\) “behaves regularly”; that is, it is either strictly increasing, strictly decreasing, or constant. A direct (although arguably unnecessarily elaborate) way to see this is by applying Ramsey’s theorem (see Section 2.1.3): two-element subsets of \(\mathbb{Q}\) are coloured with three colours according to the local behaviour of \(f\) on them. In particular, it follows that

\[ \{ \beta f \alpha \mid \alpha, \beta \in \text{Aut}(\mathbb{Q}; <) \} \subseteq \mathbb{Q}^\mathbb{Q} \]

equipped with the pointwise convergence topology, contains a function which behaves regularly everywhere. This function of regular behavior is called canonical.

More generally, a function \(f : A \to B\) between two structures \(A, B\) is called canonical when it behaves regularly in an analogous way, that is, when it sends orbits of \(n\)-tuples of \(\text{Aut}(A)\) to orbits of \(n\)-tuples of \(\text{Aut}(B)\). Similarly as in the example above, canonical functions can be obtained from \(f\), in the fashion stated above, when \(A\) has sufficient Ramsey-theoretic properties (for example, the Ramsey property) and when \(\text{Aut}(B)\) is sufficiently rich (for example \(\omega\)-categorical) \([18, 19, 25]\).

The concept of canonical functions has turned out useful in numerous applications: for classifying first-order reducts they are used in \([1, 2, 13, 24, 80, 94, 98]\), for complexity classification for constraint satisfaction problems (CSPs) in \([14, 16, 20, 26, 72]\), for decidability of meta-problems in the context of the CSPs in \([25]\), for lifting algorithmic results from finite-domain CSPs to CSPs over infinite domains in \([17]\), for lifting algorithmic results from finite-domain CSPs to homomorphism problems from definable infinite structures to finite structures \([70]\), and for decidability questions in computations with atoms in \([71]\).

As indicated above, the technique is available for a function \(f : A \to B\) whenever \(A\) is a Ramsey structure and \(B\) is \(\omega\)-categorical, and the existence of canonical functions in the set \(\{ \beta f \alpha \mid \alpha \in \text{Aut}(A), \beta \in \text{Aut}(B) \} \subseteq B^A\) was originally shown under these conditions by a combinatorial argument \([18, 19, 25]\). It is natural to ask for
a perhaps more elegant proof of the existence of canonical functions via topological
dynamics, reminiscent of the numerous proofs of combinatorial statements obtained
in this fashion (cf. the survey [11] for ergodic Ramsey theory; [68] mentions some
applications of extreme amenability). In this section we present such a proof, taken
from [22].

7.2.1. Canonicity. Let \( G \leq \text{Sym}(X) \) and \( H \leq \text{Sym}(Y) \). A function \( f: X \to Y \)
is called canonical with respect to \( G \) and \( H \) if for every \( k \geq 1, t \in X^k \), and \( \alpha \in G \)there exists \( \beta \in H \) such that \( f\alpha(t) = \beta f(t) \). Hence, functions that are canonical
with respect to \( G \) and \( H \) induce for each integer \( k \geq 1 \) a function from the orbits of the
componentwise action of \( G \) of \( X^k \) to the orbits of the componentwise action of \( H \)
on \( Y^k \).

In order to formulate properties equivalent to canonicity we require some topo-
logical notions. We consider the set \( Y^X \) of all functions from \( X \) to \( Y \) as a topological
space equipped with the topology of pointwise convergence, i.e., the product topology
where \( Y \) is taken to be discrete. When \( S \subseteq Y^X \), then we write \( \overline{S} \) for the closure of
\( S \) in this space. In particular, when \( G \acts X \) is a permutation group, then \( \overline{G} \) is the
closure of \( G \) in \( X^X \). Note that \( \overline{G} \) might no longer be a group, but it is still a monoid
with respect to composition of functions. For example, in the case of the full symmet-
ric group \( G = \text{Sym}(X) \) consisting of all permutations of \( X \), \( \overline{G} \) is the transformation
monoid of all injections in \( X^X \).

For oligomorphic permutation groups we have the following equivalent character-
isations of canonicity.

**Proposition 7.2.1.** Let \( G \acts X \) and \( H \acts Y \) be permutation groups, where \( H \acts \)
\( Y \) is oligomorphic. Then for any function \( f: X \to Y \) the following are equivalent.

1. \( f \) is canonical with respect to \( G \) and \( H \);
2. for all \( \alpha \in G \) we have \( f\alpha \in \overline{H}f \coloneqq \{ \beta f \mid \beta \in H \} \);
3. for all \( \alpha \in G \) there are \( e_1, e_2 \in H \) such that \( e_1 f\alpha = e_2 f \).

A stronger condition is to require that for all \( \alpha \in G \) there is an \( e \in \overline{H} \) such that
\( f\alpha = ef \). To illustrate that this is strictly stronger, already when \( G = H \), we give an
explicit example.

**Example 7.2.2** (Trung Van Pham). Let \( G := \text{Aut}(\mathbb{Q}; <) \). Note that \( (\mathbb{Q}; <) \) and
\( (\mathbb{Q} \setminus \{0\}; <) \) are isomorphic, and let \( f \) be such an isomorphism. Then \( f \), viewed as a
function from \( \mathbb{Q} \to \mathbb{Q} \), is clearly canonical with respect to \( G \) and \( G \). But \( f \) does not
satisfy the stronger condition above: there is no \( e \in \overline{G} \) such that \( f\alpha = ef \). To see
this, choose \( b, c \in \mathbb{Q} \) such that \( f(b) < 0 < f(c) \). By transitivity there exists an \( \alpha \in G \)
such that \( \alpha(b) = c \). Note that \( 0 < f\alpha(b) < f\alpha(c) \). Moreover, the image of \( f\alpha \) equals
the image of \( f \), and hence any \( e \in \overline{G} \) such that \( f\alpha = ef \) must fix 0. Since \( e \) must also
preserve \(<\), it cannot map \( f(b) < 0 \) to \( f\alpha(b) > 0 \). Hence, there is no \( e \in \overline{G} \) such that
\( f\alpha = ef \).

In Proposition 7.2.1 we lift the implications from (1) to (2) and from (3) to (1) follow
straightforwardly from the definitions. For the implication from (2) to (3) we need
a lift lemma, which is in essence from [23]. This lemma has been applied frequently
lately \([8,14,17]\) in various slightly different forms. We need a yet different formulation
here; since the lemma is a consequence of the compactness of a certain space which we
need in case for the canonisation theorem in Section 7.2.2 we present its proof.

Let \( H \acts Y \) be a permutation group, and let \( f, g \in Y^X \), for some \( X \). We say that
\( f = g \) holds locally modulo \( H \) if for all finite \( F \subseteq X \) there exist \( \beta_1, \beta_2 \in H \) such that
\( \beta_1 f|_F = \beta_2 g|_F \). We say that \( f = g \) holds globally modulo \( H \) (modulo \( \overline{H} \)) if there exist
\( e_1, e_2 \in H \) (\( e_1, e_2 \in \overline{H} \), respectively) such that \( e_1 f = e_2 g \).
Of course, if \( f = g \) holds globally modulo \( \overline{H} \), then it holds locally modulo \( H \). On the other hand, if \( f = g \) holds locally modulo \( H \), then it need not hold globally modulo \( H \). To see this, let \( f(x, y) : \omega^2 \to \omega \) be an injection, set \( g := f(y, x) \), and let \( H \) be the group of all permutations of \( \omega \). Then \( f = g \) holds locally modulo \( H \), but not globally. However, there exist injections \( e_1, e_2 \in \omega^\omega \) such that \( e_1 f = e_2 g \), so \( f = g \) holds globally modulo \( \overline{H} \). This is true in general, as we see in the following lift lemma.

**Lemma 7.2.3.** Let \( H \cong Y \) be an oligomorphic permutation group, let \( I \) be an index set, and let \( X_i \) be a set for every \( i \in I \). Let \( f_i, g_i \) be functions in \( Y^{X_i} \) such that \( f_i = g_i \) holds locally modulo \( H \) for all \( i \in I \). Then there exist \( e, e_i \in \overline{H} \) such that \( e f_i = e_i g_i \) holds globally for all \( i \in I \).

To prove Lemma 7.2.3, it is convenient to work with a certain compact Hausdorff space that we also use for the canonisation theorem in Section 7.2.2. Let \( H \cong Y \) be a permutation group, and \( X \) be a set. On \( Y^X \), define an equivalence relation \( \sim \) by setting \( f \sim g \) if \( f \in g \overline{g} \), i.e., if \( f = g \) holds locally modulo \( H \); here, transitivity and symmetry follow from the fact that \( H \) is a group. The following has essentially been shown in [21] (though for the finer equivalence relation of global equality modulo \( H \)), but we give an argument for the convenience of the reader since it is used so often (cf. for example [6, 7, 15, 23]).

**Lemma 7.2.4.** If \( H \cong Y \) is oligomorphic, then the space \( Y^X / \sim \) is a compact Hausdorff space.

**Proof.** We represent the space in such a way that this becomes obvious. Extend the definition of the equivalence relation \( \sim \) to all spaces \( Y^F \), where \( F \subseteq X \). When \( F \) is finite, then \( Y^F / \sim \) is finite and discrete, because \( H \) is oligomorphic. Hence, the space

\[
\prod_{F \subseteq X, |F| < \omega} Y^F / \sim
\]

is compact. The mapping from \( Y^X / \sim \) into this space defined by

\[
[g] \sim \mapsto ([g]_F \sim \mid F \subseteq [X]^{<\omega})
\]

is well-defined. In fact, it is a homeomorphism onto a closed subspace thereof, since the topology on \( Y^X / \sim \) is precisely given by the behavior of functions on finite sets, modulo the equivalence \( \sim \). Hence, \( Y^X / \sim \) is indeed a compact Hausdorff space.

**Proof of Lemma 7.2.3.** For simplicity of notation, assume that the \( X_i \) are countable; then \( Y^{X_i} \) is a metric space (otherwise, we would have to work with more general topological notions than sequences). We have \( f_i \in H g_i \); so let \( (\beta_i, g_i)_{i \in \omega} \) be a sequence converging to \( f_i \) for all \( i \in I \). Setting \( X := Y \) we see that \( X^X / \sim \) is compact by Lemma 7.2.4. Therefore, the set

\[
\{(\delta _{\sim}, (\delta \beta _i)_i)_{i \in I} \mid j \in \omega, \delta \in H \}
\]

is a subset of a compact space, \( (X^X / \sim) \times (X^X / \sim)^I \). Hence, it has an accumulation point \( [(e)_{\sim},([e]_{\sim})_{i \in I}] \). Clearly, \( e, e_i \in \overline{H} \) for all \( i \in I \), and the functions \( e_i \) prove the lemma.

The implication from (2) to (3) in Proposition 7.2.1 now is a direct consequence of Lemma 7.2.3.
7.2.2. Canonisation. The following is the canonisation theorem, first proved combinatorially in [25] in a slightly more specialized context.

**Theorem 7.2.5.** Let $G \acts X$, $H \acts Y$ be permutation groups, where $G$ is extremely amenable and $H$ is oligomorphic, and let $f : X \to Y$. Then

$$HfG := \{ \beta f \alpha \mid \alpha \in G, \beta \in H \}$$

contains a canonical function with respect to $G$ and $H$.

**Proof.** The space $HfG/\sim$ is a closed subspace of the compact Hausdorff space $Y^X/\sim$ from Lemma 7.2.4 and hence is a compact Hausdorff space as well. We define a continuous action of $G$ on this space by

$$(\alpha, [g]_\sim) \mapsto [g \alpha^{-1}]_\sim.$$  

Clearly, this assignment is a function, it is a group action, and it is continuous. Since $G$ is extremely amenable, the action has a fixed point $[g]_\sim$. Any member $g$ of this fixed point is canonical: whenever $\alpha \in G$, then $[g \alpha]_\sim = [g]_\sim$, which is the definition of canonicity.

**7.2.3. Open Problems.** Is there a converse of Theorem 7.2.5 in the sense that extreme amenability of $G$ is equivalent to some form of the statement of the canonisation theorem? To be more specific, we ask the following question.

**Question 7.1.** Let $G$ be the automorphism group of a countably infinite linearly ordered structure with domain $X$. Is it true that $G$ is extremely amenable if and only if for all oligomorphic permutation groups $H \acts Y$ and every $f : X \to Y$ we have that $HfG$ contains a function that is canonical with respect to $H$ and $G$?
CHAPTER 8

Closed Supergroups of Oligomorphic Permutation Groups

\[ 2, 10, 27, 28, 31, 64, 80, 94, 98, 99, 114, 115 \]
CHAPTER 9

Restricted Orbit Growth

32 35 48 83 86 100 101
Exercise. Up to isomorphism, there is only one countable homogeneous linear order.

Theorem 10.0.1 (Woodrow). Up to isomorphism, there are only two countable homogeneous tournaments that do not embed the 4-element tournament $D$ from Exercise 25.
Bibliography


