The Feder-Vardi-Kun-Bulatov-Zhuk Dichotomy for MMSNP, via Ramsey Theory

Manuel Bodirsky

Institut für Algebra, TU Dresden
joint work with Antoine Mottet and Florent Madelaine

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Existential Second-order Logic

Let τ be a finite relational signature.
An existential second-order (ESO) τ-sentence is a sentence of the form

$$\exists R_1, \ldots, R_k. \, \phi$$

where ϕ is first-order $\tau \cup \{R_1, \ldots, R_k\}$-sentence.

Theorem (Fagin'74).

An isomorphism-closed class $\mathcal{C}$ of finite τ-structures is in NP if and only if there is an ESO sentence Φ whose finite models are $\mathcal{C}$.

Example: 3-colorability of a graph $(V; E)$ can be expressed by

$$\exists R, B, G. \, \forall x, y : \big(R(x) \vee B(x) \vee G(x)\big) \\ \wedge \big(E(x,y) \Rightarrow \neg\big(R(x) \wedge R(y) \vee B(x) \wedge B(y) \vee G(x) \wedge G(y)\big)\big)$$

Question: Which ESO sentences express problems in P?
Fact: This is undecidable.

Fragments of ESO

Feder+Vardi 1993:

Fragments of ESO with hope for complexity classification?

An SNP τ-sentence is an ESO τ-sentence of the form

$$\exists R_1, \ldots, R_k. \, \forall x_1, \ldots, x_l. \, \phi$$

where ϕ is quantifier-free over the signature $\tau \cup \{R_1, \ldots, R_k\}$.

- Monotone if all symbols from τ appear negatively.
- Monadic if $R_1, \ldots, R_k$ are unary.

Example 1. The SNP sentence

$$\exists R, B, G. \, \forall x, y : \big(R(x) \vee B(x) \vee G(x)\big)$$
$$\wedge \big(E(x,y) \Rightarrow \neg\big(R(x) \wedge R(y) \vee B(x) \wedge B(y) \vee G(x) \wedge G(y)\big)\big)$$

is monadic and monotone.

The Non-Classifiable

Theorem (Feder+Vardi'98).

Let C be a problem in NP. Then

- C is polynomial-time equivalent to a problem in monotone SNP.
- C is polynomial-time equivalent to a problem in monadic SNP.

What about monotone and monadic SNP (MMSNP)?

- Which sentences in MMSNP express NP-complete problems?
- Which sentences in MMSNP express problems in P?
- Are there other options than P and NP-complete?

Constraint Satisfaction Problems

Let B be a structure with a finite relational signature τ.
B also called the template.

Definition (CSP).
$\text{CSP}(B)$ is the class of all finite τ-structures A with a homomorphism to B.

Example: $\text{CSP}(K_3)$ is 3-Colorability Problem.

$\text{CSP}(B)$, for finite-domain B, can be expressed in MMSNP:

- use a unary predicate P_b for each $b \in B$;
- write a sentence that expresses that P_b is the pre-image of b under a homomorphism to B.

Example 2

$$\{G \mid G \text{ finite triangle-free graph}\}$$

is in monotone monadic SNP:
it is even a first-order property

$$\forall x, y, z : \neg\big(E(x,y) \wedge E(y,z) \wedge E(z,x)\big)$$

But not of the form CSP(B) for finite-domain B:
there are triangle-free graphs of arbitrarily high chromatic number (Erdős).

The Results of Feder, Vardi, et al.

MMSNP has a complexity dichotomy:

Theorem (Feder+Vardi'93, Kun'13, Bulatov'17, Zhuk'17).
Every problem in MMSNP is in P or NP-complete.

1. Feder+Vardi'98: every problem in MMSNP is equivalent to a finite-domain CSP via randomised reductions.
2. Kun'13: derandomization based on expander structures.
3. Zhuk'17, Bulatov'17: CSP(B) is in P if there exists a homomorphism $s \colon B^6 \to B$ (polymorphism) satisfying

$$\forall x, y, z : s(x, x, y, y, z, z) = s(y, z, x, z, x, y) \quad (\text{Siggers operation})$$

   or otherwise NP-complete.

Goal: Find larger fragments of ESO that have a complexity dichotomy!

Connection of MMSNP with infinite-domain CSPs

- A structure B is called homogeneous if every isomorphism between finite substructures can be extended to an automorphism of B.
- $\text{Age}(B)$:= the class of all finite structures that embed into B.
- A structure B with finite relational signature τ is finitely bounded iff there exists a finite set of finite τ-structures $\mathcal{F}$ such that $\text{Age}(B)$ equals the class of all finite $\mathcal{F}$-free structures.
- A first-order reduct of a structure B is a structure C with the same domain such that all relations of C are first-order definable over B.

Theorem (MB+Dalmau'07).

The finite models of an MMSNP sentence Φ are equal to $\text{CSP}(B_1) \cup \cdots \cup \text{CSP}(B_n)$ for countably infinite structures $B_1, \ldots, B_n$. The B_i can be chosen to be first-order reducts of finitely bounded homogeneous structures.

Easy fact:

MMSNP has a dichotomy if and only if CSPs in MMSNP have a dichotomy.

Infinite-Domain Tractability Conjecture

Conjecture (MB+Pinsker'11, Barto+Pinsker'16).

Let B be first-order reduct of a finitely bounded homogeneous structure, and suppose that B is model-complete core. Then

- B has a pseudo-Siggers polymorphism s, i.e., s satisfies

$$\forall x, y, z\colon e_1(s(x,x,y,y,z,z)) = e_2(s(y,z,x,z,x,y))$$

  for some unary polymorphisms e_1, e_2 of B, and $\mathrm{CSP}(B)$ is in P, or
- $\mathrm{CSP}(B)$ is NP-hard.

Conjecture true for:

- finite structures B (Bulatov'17, Zhuk'17)
- first-order reducts of $(\mathbb{Q}; <)$ (MB+Kara'07)
- first-order reducts of the Random Graph (MB+Pinsker'11)
- and many more

How about CSPs in MMSNP?

Main Result

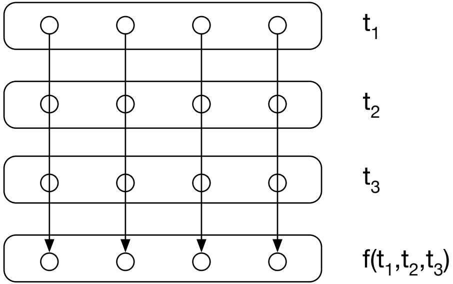
Theorem (Mottet+MB'18).
The Infinite-Domain Tractability Conjecture is true for CSPs in MMSNP.

1. Use a powerful Ramsey-type theorem of Hubička+Nešetřil'2016
2. Use a reduction to finite-domain CSPs from (MB+Mottet'2016)
3. Don't use expanders of Kun

Canonical Functions

Let B be a homogeneous structure.

Definition. A function $f \colon B^n \to B$ is called canonical (wrt. B) iff
for all $k \in \mathbb{N}$, $t_1, \dots, t_n \in B^k$
the orbit of $f(t_1, \dots, t_n)$ wrt. $\mathrm{Aut}(B)$
only depends on
the orbits of $t_1, \dots, t_n$ wrt. $\mathrm{Aut}(B)$.



Examples:

- The map $x \mapsto -x$ is canonical with respect to $(\mathbb{Q}; <)$.
- The map $(x, y) \mapsto \max(x, y)$ is not canonical with respect to $(\mathbb{Q}; <)$.

Infinite-to-finite Reduction

Theorem (B+Mottet'16).

Let A be a first-order reduct of a finitely bounded homogeneous structure B. If A has a pseudo-Siggers polymorphism which is canonical wrt B, then $\mathrm{CSP}(A)$ is in P.

Uses Zhuk'17/Bulatov'17 as black-box.

How do we get canonical functions?

Lemma (Canonisation lemma; MB+Pinsker+Tsankov'11).

Suppose B is homogeneous with finite signature, $\mathrm{Age}(B)$ is a Ramsey class. Then for any $f\colon B^\ell \to B$, the set

$$\overline{\{a_0(f(a_1,\dots,a_\ell)) \mid a_0,a_1,\dots,a_\ell \in \mathrm{Aut}(B)\}}$$

contains a function g that is canonical wrt B. (g is 'canonisation' of f)

The Theorem of Hubička+Nešetřil

Theorem (consequence of Hubička+Nešetřil'16).

For every CSP described by a MMSNP sentence Φ
there exists an infinite structure HN and a linear order $<$ on HN so that

- $(\text{HN}, <)$ is first-order reduct of a finitely bounded homogeneous structure;
- $A \to \text{HN}$ if and only if $A \models \Phi$ for all finite structures A;
- $\text{Age}(\text{HN}, <)$ is a Ramsey class.

Uses so-called partite method from structural Ramsey theory.

Open Problems

1. Which finitely bounded homogeneous structures have an expansion which is Ramsey and still finite bounded homogeneous?
2. Find more expressive logics with a complexity dichotomy.

   Candidate: MMSNP_2 (quantification over subsets of edges)