The Connection between Stochastic Games and Constraint Satisfaction

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Mean Payoff Games

Stochastic games: Shapley 1953

A stochastic mean payoff game is given by

- a finite directed graph \((V, E)\);
- a partition \(V = V_{\text{max}} \uplus V_{\text{min}} \uplus V_{\text{stoch}}\);
- a start vertex \(s\);
- a payoff function \(w: E \rightarrow \{-W, \ldots, 0, \ldots, +W\}\);
- two players min and max (a 2\(\frac{1}{2}\)-game).

A mean payoff game (MPG) has no stochastic vertices (a 2-player game).
The rules of the game

The players move a token along edges from $E$. On nodes from $V_{\text{max}}$ it is max’ turn. On nodes from $V_{\text{min}}$ it is min’s turn. One nodes from $V_{\text{stoch}}$ an out-neighbour is chosen uniformly at random. The game continues forever. The long-run average payoff (or limiting average payoff) is defined to be

$$\liminf_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} w(u_{i-1}, u_i)$$

Goal of max (min): maximise (minimize) the long-run average. MGP as a decision problem: decide whether max has a winning strategy, i.e., a strategy that guarantees non-negative long-run average.

What is the computational complexity of mean payoff games?
Example

Stochastic Games and CSPs
Memoryless Strategies

A strategy is called memoryless (or positional) if the next move only depends on the current position (and not on the history of the previous moves).

Theorem (Ehrenfeucht+Mycielski).

Both players in a mean payoff game have optimal memoryless strategies.

Corollary: Mean payoff games is in NP.

- Guess a memoryless strategy $\sigma$ for max.
- Remove all edges not chosen by max; let $G_\sigma$ be the resulting graph.
- min is looking in $G_\sigma$ for a path leading to a negative cycle.

This can be done in time $O(|V| \cdot |E|)$ by the Bellman-Ford algorithm.
Containment in coNP

**Corollary:** Mean payoff games are in coNP.

- Guess a memoryless strategy $\sigma$ for min.
- Remove all edges not chosen by min; let $G_\sigma$ be the resulting graph.
- max is looking in $G_\sigma$ for a reachable cycle with maximum mean weight.
- This can be done in time $O(|V| \cdot |E|)$ by Karp’s algorithm for the maximum mean weight cycle and testing reachability.

**Remarks.**

- Mean payoff games not known to be in $P$!
- Problems in $\text{NP} \cap \text{coNP}$ that are not known to be in $P$ are rare.
- Already the special case of *parity games* is not known to be in $P$. Parity games are equivalent to model-checking for the $\mu$-calculus. Recent breakthrough (Calude, Jain, Khoussainov, Li, Stephan’17): quasipolynomial algorithm (in $O(n^{\log(W)+6})$).
Exercise

Play with one of your neighbours:

1. Invent on paper a mean payoff game $G$ including a start vertex $s$.
2. Give the paper to your neighbour.
3. Your neighbour decides whether (s)he wants to be player max or player min in this game.
4. Expose a memoryless strategy.
5. Your neighbour exposes a memoryless strategy.
6. Check who is the winner.
Max-Atoms

And/Or Precedence Constraints (Möhring+Stork+Skutella’04) aka Max-Atom Problem (Bezem+Nieuwenhuis+Rodriguez-Charbonell’08) equiv to Emptyness of Tropical Polyhedra (Akian+Gaubert+Guterman’11) equiv to Solvability of Tropical Linear Systems (Grigoriev+Podelskii’15):

Definition (Max-Atoms).

**Input:** A finite set of variables $V$, and a finite set of constraints

$$x \leq \max(y, z) + c$$

where $x, y, z \in V$ and $c \in \mathbb{Z}$ is represented in binary.

**Question:** Is there a mapping $V \rightarrow \mathbb{Q}$ that satisfies all constraints?

- MSS’04, BNR’08, AGG’11: in coNP.
- MSS’04, Atserias+Maneva’09, AGG’11:

Max-Atoms $\equiv_p$ Mean Payoff Games
Equivalence MPGs and Max-Atom

Max-Atoms \(\equiv_p\) Mean Payoff Games: why should we care?

- Both reductions are natural, elegant, easy
- The correctness proof of the reductions is surprisingly complicated
- Side-product: new proof for the existence of memoryless strategies!
- The correspondence is the starting point for important generalisations of both MPGs and Max-Atom.
- Personal claim: CSP techniques will lead to a proof that MGPs are in P.
Massaging MPGs

Work with computationally equivalent variant MPG*: decide whether max has a winning strategy for every starting vertex.

- Reduction from MPG* to MPGs:
  introduce start vertex $s \in V_{\text{min}}$ with edges to each $v \in V_{\text{max}}$.

- Reduction from MPGs to MPG*:
  add an edge of weight 0 from every $v \in V_{\text{max}}$ to start vertex $s$. 
Massaging Max-Atoms

Observations.

- $x \leq \max(y_1, y_2, y_3)$ has the primitive positive definition
  $$\exists z (x \leq \max(y_1, z) \land z \leq \max(y_2, y_3)).$$

- $x \leq \max(y_1 + c_1, y_2 + c_2)$ has the primitive positive definition
  $$\exists z (x \leq \max(y_1, z) + c_1 \land z \leq \max(y_2, y_2) + (c_2 - c_1)) .$$
  i.e., $z \leq y_2 + c_2 - c_1$

- $x \leq \min(y_1 + c_1, \ldots, y_k + c_k)$ has the primitive positive definition
  $$x \leq \max(y_1, y_1) + c_1 \land \cdots \land x \leq \max(y_k, y_k) + c_k.$$
Let $\phi$ be a max-atoms instance.

**Lemma.**

Suppose that $s: V \rightarrow \mathbb{Q}$ is a solution to $\phi$. Then for any $d \in \mathbb{Q}$

$$x \mapsto s(x) + d$$

is a solution to $\phi$, too. Suppose that $s_1, s_2: V \rightarrow \mathbb{Q}$ are solutions to $\phi$. Then

$$x \mapsto \max(s_1(x), s_2(x))$$

is a solution to $\phi$, too.

$$R_c^{\text{max}} := \{(x, y, z) \mid x \leq \max(y, z) + c\}$$

The structure $(\mathbb{Q}; (R_c^{\text{max}})_{c \in \mathbb{Q}})$ has the polymorphisms

$(x, y) \mapsto \max(x, y)$

and $x \mapsto x + d$ for all $d \in \mathbb{Q}$. 
Max-Atoms: Important Properties

A max-atom instance $\phi$ is called **left-distinct** if for any two conjuncts of $\phi$

$$x_1 \leq \max(y_1, z_1) + c_1$$

and

$$x_2 \leq \max(y_2, z_2) + c_2$$

the head variables $x_1$ and $x_2$ are distinct.

**Lemma (Bezem+Nieuvenhuis+Rodriguez-Charbonell’08).**

$\phi$ is unsatisfiable $\iff$ $\phi$ contains a left-distinct unsatisfiable subset.

**Example.**

$$x \leq \max(y, z)$$

$$y \leq \max(x, z)$$

$$z \leq \max(x, x) + 2$$

$$z \leq \max(y, y) + 1$$
Left-distinct unsatisfiability certificates

Let $\phi$ be a max-atoms instance and $x$ a variable of $\phi$. Let $\psi_1, \ldots, \psi_k$ be all conjuncts of $\phi$ with head $x$.

**Claim.** If $\phi$ is unsatisfiable, then $\phi \setminus \psi_i$ is for some $i \leq k$ unsatisfiable.

**Proof:** Assume that for every $i \leq k$ the formula $\phi \setminus \psi_i$ has solution $s_i$.

Produce solution to $\phi$ using translations and max:

$$x \leq \max(y, z) + 1 \land x \leq \max(u, v) + 2$$

$s_1$:

$$x = 1, y = 2, z = 3, u = ?, v = ?$$

$s_2$:

$$x = 2, y = ?, z = ?, u = 8, v = 3$$

Translate $s_2$:

$s_3$:

$$x = 1, y = ?, z = ?, u = 7, v = 2$$

Max $s_1, s_3$:

$s_4$:

$$x = 1, y \geq 2, z \geq 3, u \geq 8, v \geq 3$$

Lemma follows by induction.\[\square\]
Max-Atoms: Pseudo-Polynomial Algorithm

Suppose that all conjuncts of max-atom instance $\phi$ are of the form

$$x \leq \max(y, z) + 1.$$ 

There is a polynomial-time algorithm to decide satisfiability of $\phi$.

**Theorem (Bodirsky, Martin, Mottet’18).**

Let $\Gamma = (\mathbb{Z}; R_1, R_2, \ldots, R_m)$ be a structure with a fo-definition over $(\mathbb{Z}; <)$. If $\Gamma$ has max or min or $(\ldots)$ as polymorphism, then $\text{CSP}(\Gamma)$ is in P. Otherwise, $\text{CSP}(\Gamma)$ is NP-complete.

**Algorithm** (in case that $\Gamma$ has polymorphism max):

- Let $S$ be a **sufficiently big** finite substructure of $\Gamma$.
- Note: $S$ has max as polymorphism, too!
- Solve $\phi$ as an instance of $\text{CSP}(S)$.
- For that, use **Arc-Consistency Algorithm**
  - **Fact**: answers correctly since $S$ has max as polymorphism.
  - has polynomial running time in both $\phi$ and $S$. 

Stochastic Games and CSPs
From Mean Payoff Games To Max-Atoms

Let $G = (V, V_{\text{max}}, V_{\text{min}}, w, s)$ be an MPG.

We create a max-atoms instance $\phi_G$ with variables $V$:

1. For each $v \in V_{\text{max}}$ with out-neighbours $u_1, \ldots, u_k$, add a constraint
   \[ v \leq \max(u_1 + w(v, u_1), \ldots, u_k + w(v, u_k)) \]

2. For each $v \in V_{\text{min}}$ with out-neighbours $u_1, \ldots, u_k$, add a constraint
   \[ v \leq \min(u_1 + w(v, u_1), \ldots, u_k + w(v, u_k)) \]

(These constraints can be expressed by Max-Atom constraints.)
From Max-Atoms to Mean Payoff Games

Let $\phi$ be a max-atoms instance with variables $V$ and inequalities $C$.

Create an MPG $G_\phi$ such that $V_{\text{max}} := V$ and $V_{\text{min}} := C$, and

- If $e \in C$ is of the form $x \leq \max(y, z) + k$
  add edges $(e, y), (e, z)$ of weight $k$;
- For every $x \in V$ and every $e \in C$
  of the form $x \leq \max(y, z) + c$
  add an edge $(x, e)$ of weight 0.
Max-atom are polynomial-time equivalent to system with $n$ variables $x_1, \ldots, x_n$ and $n$ constraints $e_1, \ldots, e_n$ where $e_i$ is of the form

\[ x_i \leq \max(x_{i_1} + c_1, \ldots, x_{i_k} + c_k) \]

or

\[ x_i \leq \min(x_{i_1} + c_1, \ldots, x_{i_k} + c_k) \]

‘Max-min offset operator system’
For those systems, the correspondence is particularly nice:

\[ x_s \leq \max(x_1 + 1, x_3 + 1) \]
\[ x_2 \leq \max(x_1 - 1, x_3 - 1) \]
\[ x_1 \leq \min(x_s - 1, x_2 - 1) \]
\[ x_3 \leq \min(x_s + 1, x_2 + 1) \]
Have to prove:

1. max wins on $G$ iff $\phi_G$ satisfiable.
2. $\phi$ satisfiable iff max wins on $G_\phi$.

Idea: suffices to prove correctness of 2!

Reason: reduction is such that $G_{\phi_G} = G$.

\[
\text{max wins on } G \text{ iff max wins on } G_{\phi_G} \text{ iff } \phi_G \text{ is satisfiable (by 2)}
\]
Want to prove: A max-atoms instance $\phi$ is satisfiable iff max wins on $G_\phi$.

**Idea:** To prove $\Rightarrow$, let $s: V \to \mathbb{Q}$ be a solution to $\phi$.
For each constraint $x \leq \max(y_1 + c_1, \ldots, y_k + c_k)$
let $i \leq k$ be such that $y_i + c_i = \max(y_1 + c_1, \ldots, y_k + c_k)$.

**Claim:** the (memoryless!) strategy for max that always moves from $x$ to $y_i$ wins against any strategy of min.

**Idea:** To prove $\Leftarrow$, suppose that $\phi$ is unsatisfiable. Then $\phi$ contains an unsatisfiable left-distinct subset $\psi$.

**Claim.** $\psi$ corresponds to a (memoryless) strategy for min.
Extensions?

Questions:
- which extensions of the max-atom problem remain in $\text{NP} \cap \text{coNP}$?
- which other relations over $\mathbb{Q}$ are preserved by max and translations?
- which other relations over $\mathbb{Q}$ are preserved by max?
Semilinear Relations

Definition.

$R \subseteq \mathbb{Q}^k$ is semilinear if $R$ has a first-order definition in $(\mathbb{Q}; +, 1, \leq)$.

Ferrante and Rackoff’75: A relation is semilinear if and only if it is a finite intersection of finite unions of (open or closed) linear half spaces.

- CSP$(\mathbb{Q}; \leq, R_+, R_{=1})$ where $R_+ := \{(x, y, z) \mid x = y + z\}$ and $R_{=1} := \{1\}$ is essentially linear program feasibility.
- All CSPs with semi-linear constraints are in NP: guess one half space per union and verify satisfiability of resulting system of linear inequalities.
- Every finite-domain CSP falls into this class.

Big challenge:

Classify the complexity of CSP$(\mathbb{Q}; R_1, \ldots, R_n)$ for all semilinear relations $R_1, \ldots, R_n$. 
Tropically Convex Sets

Definition (Develin-Sturmfels’04).

A subset $R$ of $\mathbb{Q}^n$ is called tropically convex if

- $R$ is preserved by max, and
- $R$ is preserved by all translations $x \mapsto x + c$ for all $c \in \mathbb{Q}$.

- For every $c \in \mathbb{Q}$ the max-atom relation $R_c^{\text{max}}$ is tropically convex
- $x \leq (y + z)/2$ is tropically convex
- $x \leq y + z$ is preserved by max, but not tropically convex.

Theorem (B+Mamino’15).

Let $S_1, \ldots, S_n$ be tropically convex semilinear relations.
Then $\text{CSP}(\mathbb{Q}; S_1, \ldots, S_n)$ is in $\text{NP} \cap \text{coNP}$. 
Step 1: Syntax

Can we find a syntactic description of the relations preserved by max?

**Fact:** a relation $R \subseteq \{0, 1\}^n$ is preserved by max if and only if the complement of $R$ has a Horn definition.

\[ \text{e.g. } '(-x \lor -y \lor z) \land (-x \lor -u)' \]

For subsets of $\mathbb{Q}^2$, max-closed sets can be more complicated:
Step 1: Syntax

Theorem (B-Mamino’15).

Let \( R \subseteq \mathbb{Q}^n \) be semilinear and topologically closed. The following are equivalent:

- \( R \) is preserved by \( \max \)
- \( R \) can be defined by a conjunction of expressions of the form
  \[
  x \leq \max(a_1 \bar{x} + b_1, \ldots, a_m \bar{x} + b_m)
  \]
  where \( m \in \mathbb{N} \), each of \( a_1, \ldots, a_m \) is a vector from \( \mathbb{Q}^n_{\geq 0} \), and \( b_1, \ldots, b_n \in \mathbb{Q} \);
- \( R \) has a primitive positive definition over \( (\mathbb{Q}; <, \{1\}, \{-1\}, S_1, S_2, S_3) \) where
  \[
  S_1 := \{(x, y) : 2x \leq y\}
  \]
  \[
  S_2 := \{(x, y, z) : x \leq y + z\}
  \]
  \[
  S_3 := \{(x, y, z) : x \leq \max(y, z)\}
  \]

Theorem can be generalised to non-closed case.
Theorem (B-Mamino’15).

Let $R \subseteq \mathbb{Q}^n$ be semilinear and topologically closed. The following are equivalent:

- $R$ is tropically convex.
- $R$ can be defined by a conjunction of expressions of the form
  
  \[ x \leq \max(a_1 \bar{x} + b_1, \ldots, a_m \bar{x} + b_m) \]

  where $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{Q}_{\geq 0}^n$ are $\sum_{i \leq n} a_{j,i} = 1$ for all $j \leq n$, and $b_1, \ldots, b_n \in \mathbb{Q}$;

- $R$ has a primitive positive definition in $(\mathbb{Q}; S_3, T_{1}, T_{-1}, S_4)$ where
  
  \[ T_{\pm 1} := \{(x, y) : x \leq y \pm 1\} \]
  \[ S_4 := \{(x, y, z) : x \leq (y + z)/2\}. \]
Step 2: Duality for Max-Plus-Average Ineqs

Tropically convex CSPs can be reduced to:

\( P: \) a system of \( n \) strict inequalities on \( n \) variables, each having one the following three forms

\[
\begin{align*}
  x_i &< \max(x_{j_1} + k_1, \ldots, x_{j_m} + k_m) \\
  x_i &< \min(x_{j_1} + k_1, \ldots, x_{j_m} + k_m) \\
  x_i &< (\alpha_1 x_{j_1} + \cdots + \alpha_m x_{j_m}) / (\alpha_1 + \cdots + \alpha_m) + k
\end{align*}
\]

where \( \alpha_1, \ldots, \alpha_m > 0. \)

The dual \( D \) of \( P \): replace \( < \) by \( \geq \).

**Theorem (B+Mamino’15).**

\( P \) has a solution in \( \mathbb{Q}^n \) if and only if \( D \) has no solution in \( (\mathbb{Q} \cup \{+\infty\})^n \setminus \{+\infty\}^n \).

**Consequence:** satisfiability of \( P \) is in \( \text{NP} \cap \text{coNP} \).

**Proof:** Connection to stochastic MPGs.
Let $P$ be the system of $n$ constraints on $n$ variables, and $D$ its dual. $P$ corresponds to game:

- max inequalities $\leftrightarrow$ max vertices.
- min inequalities $\leftrightarrow$ min vertices.
- inequalities $x_i < (\alpha_1 x_j + \cdots + \alpha_m x_{j_m}) / (\alpha_1 + \cdots + \alpha_m) + c$
  $\leftrightarrow$ stochastic vertices with probabilities $\alpha_1, \ldots, \alpha_m$.

$v(x)$: expected limiting average payoff of game $G$, starting in $x$.

**Theorem (B+Mamino’15).**

- $D$ is satisfiable if and only if $v(x_i) \leq 0$ for some vertex $x_i$ of $G$.
- $P$ is satisfiable if and only if $v(x_i) > 0$ for all vertices $x_i$ of $G$.

This implies the duality theorem.
Overview State of the Art

- **Semilinear CSPs**
  - **Max-closed semilinear CSPs**
    - **Tropically convex CSPs**
      - **Max-Atom Problem**
  - **Stochastic mean payoff games**
  - **Mean payoff Games**
  - **Parity games**
  - **Propositional $\mu$-calculus**

- **Quasi-polynomial algorithms**

- **NP-hard**
  - **?**
  - **$\mathsf{NP} \cap \mathsf{co-NP}$**
  - **Pseudo-polynomial**