The Connection between Stochastic Games and Constraint Satisfaction

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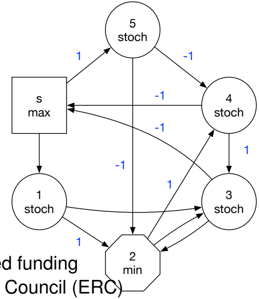

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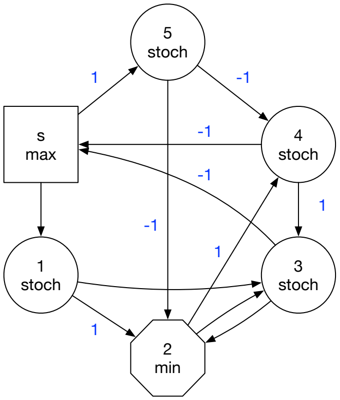
Mean Payoff Games

Stochastic games: Shapley 1953

A stochastic mean payoff game is given by

- a finite directed graph (V, E);
- a partition $V = V_{\text{max}} \uplus V_{\text{min}} \uplus V_{\text{stoch}}$;
- a start vertex s;
- a payoff function
  $w\colon E \to \{-W, \ldots, 0, \ldots, +W\}$;
- two players min and max (a $2\frac{1}{2}$-game).

A mean payoff game (MPG)
has no stochastic vertices
(a 2-player game).



The rules of the game

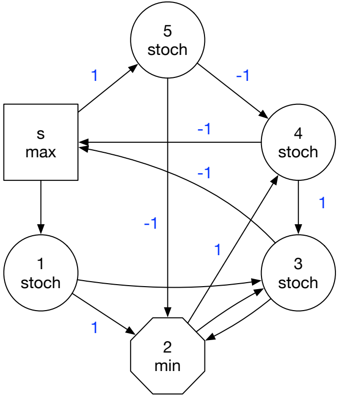
The players move a token along edges from E.
On nodes from V_{max} it is max' turn.
On nodes from V_{min} it is min's turn.
One nodes from V_{stoch} an out-neighbour is chosen uniformly at random.
The game continues forever.
The long-run average payoff (or limiting average payoff) is defined to be

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} w(u_{i-1}, u_i)$$

Goal of max (min): maximise (minimize) the long-run average.
MGP as a decision problem: decide whether max has a winning strategy, i.e., a strategy that guarantees non-negative long-run average.

What is the computational complexity of mean payoff games?

Example


5
stoch
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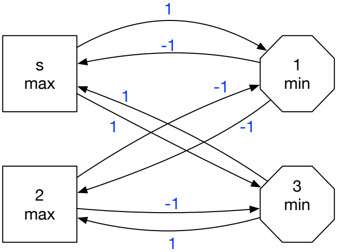
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1

Memoryless Strategies

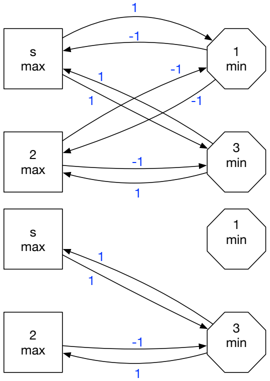
A strategy is called memoryless (or positional) if the next move only depends on the current position (and not on the history of the previous moves).

Theorem (Ehrenfeucht+Mycielski).

Both players in a mean payoff game have optimal memoryless strategies.

Corollary: Mean payoff games is in NP.

- Guess a memoryless strategy σ for max.
- Remove all edges not chosen by max; let G_σ be the resulting graph.
- min is looking in G_σ for a path leading to a negative cycle.
- This can be done in time $O(|V| \cdot |E|)$ by the Bellman-Ford algorithm.



Containment in coNP

Corollary: Mean payoff games are in coNP.

- Guess a memoryless strategy σ for min.
- Remove all edges not chosen by min;
  let G_σ be the resulting graph.
- max is looking in G_σ for a reachable cycle with maximum mean weight.
- This can be done in time $O(|V| \cdot |E|)$ by Karp's algorithm for the maximum mean weight cycle and testing reachability.

Remarks.

- Mean payoff games not known to be in P!
- Problems in NP $\cap$ coNP that are not known to be in P are rare.
- Already the special case of parity games is not known to be in P.
  Parity games are equivalent to model-checking for the μ-calculus.
  Recent breakthrough (Calude,Jain,Khoussainov,Li,Stephan'17):
  quasipolynomial algorithm (in $O(n^{\log(W)+6})$).

Exercise

Play with one of your neighbours:

1. Invent on paper a mean payoff game G including a start vertex s.
2. Give the paper to your neighbour.
3. Your neighbour decides whether (s)he wants to be player max or player min in this game.
4. Expose a memoryless strategy.
5. Your neighbour exposes a memoryless strategy.
6. Check who is the winner.

Max-Atoms

And/Or Precedence Constraints (Möhring+Stork+Skutella'04)
aka Max-Atom Problem (Bezem+Nieuwenhuis+Rodriguez-Charbonell'08)
equiv to Emptyness of Tropical Polyhedra (Akian+Gaubert+Guterman'11)
equiv to Solvability of Tropical Linear Systems (Grigoriev+Podelskii'15) :

Definition (Max-Atoms).

Input: A finite set of variables V, and a finite set of constraints

$$x \leq \max(y, z) + c$$

where $x, y, z \in V$ and $c \in \mathbb{Z}$ is represented in binary.

Question: Is there a mapping $V \to \mathbb{Q}$ that satisfies all constraints?

- MSS'04, BNR'08, AGG'11: in coNP.
- MSS'04, Atserias+Maneva'09, AGG'11:

Max-Atoms $\equiv_P$ Mean Payoff Games

Equivalence MPGs and Max-Atom

Max-Atoms $\equiv_P$ Mean Payoff Games: why should we care?

- Both reductions are natural, elegant, easy
- The correctness proof of the reductions is surprisingly complicated
- Side-product: new proof for the existence of memoryless strategies!
- The correspondence is the starting point for important generalisations of both MPGs and Max-Atom.
- Personal claim: CSP techniques will lead to a proof that MGPs are in P.

Massaging MPGs

Work with computationally equivalent variant MPG*:
decide whether max has a winning strategy for every starting vertex.

- Reduction from MPG* to MPGs:
  introduce start vertex $s \in V_{\text{min}}$ with edges to each $v \in V_{\text{max}}$.
- Reduction from MPGs to MPG*:
  add an edge of weight 0 from every $v \in V_{\text{max}}$ to start vertex s.

Massaging Max-Atoms

Observations.

- $x \leq \max(y_1, y_2, y_3)$ has the primitive positive definition

$$\exists z\big(x \leq \max(y_1, z) \wedge z \leq \max(y_2, y_3)\big).$$

- $x \leq \max(y_1 + c_1, y_2 + c_2)$ has the primitive positive definition

$$\exists z\big(x \leq \max(y_1, z) + c_1 \wedge \underbrace{z \leq \max(y_2, y_2) + (c_2 - c_1)}_{i.e.,\ z \leq y_2 + c_2 - c_1}\big).$$

- $x \leq \min(y_1 + c_1, \ldots, y_k + c_k)$ has the primitive positive definition

$$x \leq \max(y_1, y_1) + c_1 \wedge \cdots \wedge x \leq \max(y_k, y_k) + c_k.$$

Max-Atoms: Polymorphisms

Let ϕ be a max-atoms instance.

Lemma.

Suppose that $s \colon V \to \mathbb{Q}$ is a solution to ϕ. Then for any $d \in \mathbb{Q}$

$$x \mapsto s(x) + d \qquad \text{(\textit{Translation})}$$

is a solution to ϕ, too. Suppose that $s_1, s_2 \colon V \to \mathbb{Q}$ are solutions to ϕ. Then

$$x \mapsto \max(s_1(x), s_2(x))$$

is a solution to ϕ, too.

$$R_c^{\max} := \{(x, y, z) \mid x \leq \max(y, z) + c\}$$

The structure $(\mathbb{Q}; (R_c^{\max})_{c \in \mathbb{Q}})$ has the polymorphisms

$$(x, y) \mapsto \max(x, y)$$

and $\quad x \mapsto x + d \quad$ for all $d \in \mathbb{Q}$.

Max-Atoms: Important Properties

A max-atom instance ϕ is called left-distinct if for any two conjuncts of ϕ

$$x_1 \leq \max(y_1, z_1) + c_1$$
$$\text{and } x_2 \leq \max(y_2, z_2) + c_2$$

the head variables x_1 and x_2 are distinct.

Lemma (Bezem+Nieuvenhuis+Rodriguez-Charbonell'08).

ϕ is unsatisfiable $\Leftrightarrow$ ϕ contains a left-distinct unsatisfiable subset.

Example.

$$x \leq \max(y, z)$$
$$y \leq max(x, z)$$
$$z \leq \max(x, x) + 2$$
$$z \leq \max(y, y) + 1$$

Left-distinct unsatisfiability certificates

Let ϕ be a max-atoms instance and x a variable of ϕ.
Let $\psi_1, \dots, \psi_k$ be all conjuncts of ϕ with head x.

Claim. If ϕ is unsatisfiable, then $\phi \setminus \psi_i$ is for some $i \leq k$ unsatisfiable.

Proof: Assume that for every $i \leq k$ the formula $\phi \setminus \psi_i$ has solution s_i.

Produce solution to ϕ using translations and max:

$$x \leq \max(y, z) + 1 \wedge x \leq \max(u, v) + 2$$

$$s_1 : \quad x = 1, y = 2, z = 3, u = ?, v = ?$$

$$s_2 : \quad x = 2, y = ?, z = ?, u = 8, v = 3$$

Translate s_2: $\quad s_3 : \quad x = 1, y = ?, z = ?, u = 7, v = 2$

Max s_1, s_3: $\quad s_4 : \quad x = 1, y \geq 2, z \geq 3, u \geq 8, v \geq 3$

□

Lemma follows by induction.

Max-Atoms: Pseudo-Polynomial Algorithm

Suppose that all conjuncts of max-atom instance ϕ are of the form

$$x \leq \max(y, z) + 1.$$

There is a polynomial-time algorithm to decide satisfiability of ϕ.

Theorem (Bodirsky,Martin,Mottet'18).

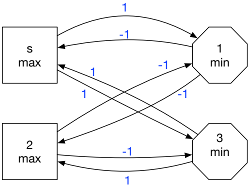
Let $\Gamma = (\mathbb{Z}; R_1, R_2, \ldots, R_m)$ be a structure with a fo-definition over $(\mathbb{Z}; <)$.
If Γ has max or min or (...) as polymorphism, then CSP(Γ) is in P.
Otherwise, CSP(Γ) is NP-complete.

Algorithm (in case that Γ has polymorphism max):

- Let S be a sufficiently big finite substructure of Γ.
- Note: S has max as polymorphism, too!
- Solve ϕ as an instance of CSP(S).
- For that, use Arc-Consistency Algorithm
  - **Fact:** answers correctly since S has max as polymorphism.
  - has polynomial running time in both ϕ and S.

From Mean Payoff Games To Max-Atoms

Let $G = (V, V_{\max}, V_{\min}, w, s)$ be an MPG.



We create a max-atoms instance ϕ_G with variables V:

1. For each $v \in V_{\max}$ with out-neighbours $u_1, \ldots, u_k$, add a constraint

$$v \leq \max(u_1 + w(v, u_1), \ldots, u_k + w(v, u_k))$$

2. For each $v \in V_{\min}$ with out-neighbours $u_1, \ldots, u_k$, add a constraint

$$v \leq \min(u_1 + w(v, u_1), \ldots, u_k + w(v, u_k))$$

(These constraints can be expressed by Max-Atom constraints.)

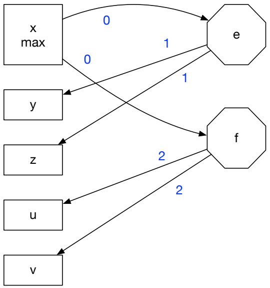
From Max-Atoms to Mean Payoff Games

Let ϕ be a max-atoms instance with variables V and inequalities C.

$$\underbrace{x \leq \max(y, z) + 1}_{=:e}, \underbrace{x \leq \max(u, v) + 2}_{=:f}$$

Create an MPG G_ϕ such that $V_{\max} := V$ and $V_{\min} := C$, and

- If $e \in C$ is of the form $x \leq \max(y, z) + k$ add edges $(e, y), (e, z)$ of weight k;
- For every $x \in V$ and every $e \in C$ of the form $x \leq \max(y, z) + c$ add an edge (x, e) of weight 0.



Ideas of Correctness Proofs, 1

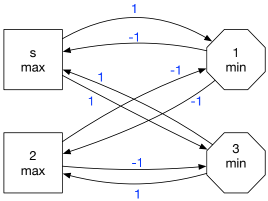
Max-atom are polynomial-time equivalent to system with n variables $x_1, \ldots, x_n$ and n constraints $e_1, \ldots, e_n$ where e_i is of the form

$$x_i \leq \max(x_{i_1} + c_1, \ldots, x_{i_k} + c_k)$$
$$\text{or } x_i \leq \min(x_{i_1} + c_1, \ldots, x_{i_k} + c_k).$$

'Max-min offset operator system'

For those systems, the correspondence is particularly nice:

$$x_s \leq \max(x_1 + 1, x_3 + 1)$$
$$x_2 \leq \max(x_1 - 1, x_3 - 1)$$
$$x_1 \leq \min(x_s - 1, x_2 - 1)$$
$$x_3 \leq \min(x_s + 1, x_2 + 1)$$



Ideas of Correctness Proofs, 2

Have to prove:

1. max wins on G iff ϕ_G satisfiable.
2. ϕ satisfiable iff max wins on G_ϕ.

Idea: suffices to prove correctness of 2!

Reason: reduction is such that $G_{\phi_G} = G$.

max wins on G iff max wins on G_{ϕ_G}
iff ϕ_G is satisfiable (by 2)

Ideas of Correctness Proof, 3

Want to prove: A max-atoms instance ϕ is satisfiable iff max wins on G_ϕ.

Idea: To prove '$\Rightarrow$', let $s \colon V \to \mathbb{Q}$ be a solution to ϕ.
For each constraint $x \leq \max(y_1 + c_1, \dots, y_k + c_k)$
let $i \leq k$ be such that $y_i + c_i = \max(y_1 + c_1, \dots, y_k + c_k)$.

Claim: the (memoryless!) strategy for max
that always moves from x to y_i wins against any strategy of min.

Idea: To prove '$\Leftarrow$', suppose that ϕ is unsatisfiable.
Then ϕ contains an unsatisfiable left-distinct subset ψ.

Claim. ψ corresponds to a (memoryless) strategy for min.

Extensions?

Questions:

- which extensions of the max-atom problem remain in NP $\cap$ coNP?
- which other relations over $\mathbb{Q}$ are preserved by max and translations?
- which other relations over $\mathbb{Q}$ are preserved by max?

Semilinear Relations

Definition .

$R \subseteq \mathbb{Q}^k$ is semilinear if R has a first-order definition in $(\mathbb{Q}; +, 1, \leq)$.

Ferrante and Rackoff'75: A relation is semilinear if and only if it is a finite intersection of finite unions of (open or closed) linear half spaces.

- CSP$(\mathbb{Q}; \leq, R_+, R_{=1})$ where $R_+ := \{(x, y, z) \mid x = y + z\}$ and $R_{=1} := \{1\}$ is essentially linear program feasibility.
- All CSPs with semi-linear constraints are in NP: guess one half space per union and verify satisfiability of resulting system of linear inequalities.
- Every finite-domain CSP falls into this class.

Big challenge:
Classify the complexity of CSP$(\mathbb{Q}; R_1, \ldots, R_n)$ for all semilinear relations $R_1, \ldots, R_n$.

Tropically Convex Sets

Definition (Develin-Sturmfels'04).

A subset R of $\mathbb{Q}^n$ is called tropically convex if

- R is preserved by max, and
- R is preserved by all *translations* $x \mapsto x + c$ for all $c \in \mathbb{Q}$.

- For every $c \in \mathbb{Q}$ the max-atom relation $R_c^{\max}$ is tropically convex
- $x \leq (y + z)/2$ is tropically convex
- $x \leq y + z$ is preserved by max, but not tropically convex.

Theorem (B+Mamino'15).

Let $S_1, \dots, S_n$ be tropically convex semilinear relations.
Then $\text{CSP}(\mathbb{Q}; S_1, \dots, S_n)$ is in NP $\cap$ coNP.

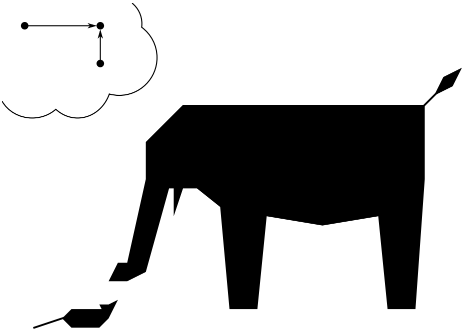
Step 1: Syntax

Can we find a syntactic description of the relations preserved by max?

Fact: a relation $R \subseteq \{0, 1\}^n$ is preserved by max if and only if the complement of R has a Horn definition.

e.g. '$(\neg x \vee \neg y \vee z) \wedge (\neg x \vee \neg u)$'

For subsets of $\mathbb{Q}^2$, max-closed sets can be more complicated:



Step 1: Syntax

Theorem (B-Mamino'15).

Let $R \subseteq \mathbb{Q}^n$ be semilinear and topologically closed. The following are equivalent:

- R is preserved by max
- R can be defined by a conjunction of expressions of the form

  $$x \leq \max(a_1\overline{x} + b_1, \ldots, a_m\overline{x} + b_m)$$

  where $m \in \mathbb{N}$, each of $a_1, \ldots, a_m$ is a vector from $\mathbb{Q}^n_{\geq 0}$, and $b_1, \ldots, b_n \in \mathbb{Q}$;
- R has a primitive positive definition over $(\mathbb{Q}; <, \{1\}, \{-1\}, S_1, S_2, S_3)$ where

  $$S_1 := \{(x, y) : 2x \leq y\}$$
  $$S_2 := \{(x, y, z) : x \leq y + z\}$$
  $$S_3 := \{(x, y, z) : x \leq \max(y, z)\}$$

Theorem can be generalised to non-closed case.

A syntactic description of tropical convexity

Theorem (B-Mamino'15).

Let $R \subseteq \mathbb{Q}^n$ be semilinear and topologically closed. The following are equivalent:

- R is tropically convex.
- R can be defined by a conjunction of expressions of the form

$$x \leq \max(a_1\overline{x} + b_1, \ldots, a_m\overline{x} + b_m)$$

  where $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{Q}^n_{\geq 0}$ are $\sum_{i \leq n} a_{j,i} = 1$ for all $j \leq n$, and $b_1, \ldots, b_n \in \mathbb{Q}$;
- R has a primitive positive definition in $(\mathbb{Q}; S_3, T_1, T_{-1}, S_4)$ where

$$T_{\pm 1} := \{(x, y) : x \leq y \pm 1\}$$
$$S_4 := \{(x, y, z) : x \leq (y + z)/2\}.$$

Step 2: Duality for Max-Plus-Average Ineqs

Tropically convex CSPs can be reduced to:
P: a system of n strict inequalities on n variables,
each having one the following three forms

$$x_i < \max(x_{j_1} + k_1, \ldots, x_{j_m} + k_m) \tag{1}$$

$$x_i < \min(x_{j_1} + k_1, \ldots, x_{j_m} + k_m) \tag{2}$$

$$x_i < (\alpha_1 x_{j_1} + \cdots + \alpha_m x_{j_m})/(\alpha_1 + \cdots + \alpha_m) + k \tag{3}$$

where $\alpha_1, \ldots, \alpha_m > 0$.
The dual D of P: replace $<$ by $\geq$.

Theorem (B+Mamino'15).

P has a solution in $\mathbb{Q}^n$ if and only if D has no solution in $(\mathbb{Q} \cup \{+\infty\})^n \setminus \{+\infty\}^n$.

Consequence: satisfiability of P is in NP $\cap$ coNP.
Proof: Connection to stochastic MPGs.

Link between CSPs and stochastic MPGs

Let P be the system of n constraints on n variables, and D its dual.
P corresponds to game:

- max inequalities $\leftrightarrow$ max vertices.
- min inequalities $\leftrightarrow$ min vertices.
- inequalities $x_i < (\alpha_1 x_{j_1} + \cdots \alpha_m x_{j_m})/(\alpha_1 + \cdots + \alpha_m) + c$
  $\leftrightarrow$ stochastic vertices with probabilities $\alpha_1, \ldots, \alpha_m$.

$v(x)$: expected limiting average payoff of game G, starting in x.

Theorem (B+Mamino'15).

- D is satisfiable if and only if $v(x_i) \leq 0$ for some vertex x_i of G.
- P is satisfiable if and only if $v(x_i) > 0$ for all vertices x_i of G.

This implies the duality theorem

Overview State of the Art