Complexity of Constraint Satisfaction in 2019

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Constraint Satisfaction Problems

Definition (CSP\( (B) \)).
Input: A conjunction of atomic \(\tau\)-formulas \(\phi\).
Question: Is \(\phi\) satisfiable in \(B\)?

Example: CSP\( (K_3)\) is 3-Colorability Problem, and NP-hard.

Central research question: for which structures \(B\) is CSP\( (B)\) in P, and when is it NP-hard?
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Definition (CSP($B$)).

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for which structures $B$ is CSP($B$) in P, and when is it NP-hard?
Conjecture (Feder-Vardi 1998).

If the domain of $B$ is finite, then CSP($B$) is in P or NP-complete.

If $B$ admits a pp-construction of all finite structures, then CSP($B$) is NP-hard.

2017: proof of dichotomy conjecture (Bulatov, Dmitry Zhuk)

Both prove the tractability conjecture: if $B$ does not admit a pp-construction of all finite structures, then CSP($B$) is in P.
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- Both prove the tractability conjecture:
  if $B$ does not admit a pp-construction of all finite structures, then $\text{CSP}(B)$ is in P.
A has a pp-construction in B, B ≤ A, if there exists an n ∈ \mathbb{N} such that A is homomorphically equivalent to a structure with domain B^n all of whose relations have a primitive positive definition in B.

If B ≤ A, there is a polynomial-time reduction from CSP(A) to CSP(B).

Hence: if B ≤ K_3, then CSP(B) is NP-hard.

Basic fact: K_3 pp-constructs every finite structure.

Research question: study class of finite structures partially ordered by pp-constructability.

Understanding complexity of CSP(B) within P (e.g., membership in L, NL, etc).

Independently of interest in universal algebra.
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The Universal-Algebraic Approach

$\text{Pol}(B)$: set of all homomorphisms from $B^k$ to $B$, for some $k \in \mathbb{N}$. 

Theorem (Libor Barto–Oprˇsal–Pinsker).

$B \leq A$ if and only if there is a minor-preserving map from $\text{Pol}(B)$ to $\text{Pol}(A)$.

Example.

Let $C_k$ be the directed cycle with $k$ vertices.

Then $C_3$ does not have a pp-construction in $C_2$: the majority operation $m$ is a polymorphism of $C_2$ and satisfies $m(x_1, x_2, x_3) = m(x_2, x_3, x_1)$ but $C_3$ cannot have such a polymorphism.
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The poset has

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2-Element Structures

(B+Albert Vucaj)
Graphs

Up to $\leq$ there are only three undirected graphs (Hell-Nešetřil, Bulatov):

- $K_3 \leq K_2 \leq K_1$

Digraphs: seem to be difficult already for orientations of trees.

Smooth digraphs: digraphs without sources and sinks.

Niven-Barto-Kozik: up to pp-constructability, a smooth digraph is either equivalent to $K_3$, or to a union of directed cycles.

B, Florian Starke, Vucaj: complete description of $\leq$ for unions of directed cycles.

Is a completely distributive lattice.

Whether $B$ pp-constructs $A$ only depends on the prime cyclic loop conditions that are satisfied by $\text{Pol}(B)$ and $\text{Pol}(A)$.
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Three levels of finiteness for countable structures

1. $B$ is $\omega$-categorical.
   
   Equivalent: $\text{Aut}(B)$ has for each $n$ only finitely many orbits of $n$-tuples.

   Examples: $(\mathbb{Q}; <)$, the Rado graph, the random poset, the Henson digraphs, the atomless Boolean algebra.

   Consequence: $R \in \text{Inv}(\text{Pol}(B))$ if and only if $R$ has pp-definition in $B$.

2. $B$ is homogeneous in a finite relational language.
   
   Examples: $(\mathbb{Q}; <)$, Rado graph, random poset, Henson digraphs.

3. $B$ is homogeneous and finitely bounded, i.e., its finite substructures are described by finitely many forbidden finite substructures.
   
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   Consequence: $\text{CSP}(B)$ is in NP.

4. $\text{CSP}(B)$ in NP and $B$ is $\omega$-categorical also if $B$ is a first-order reduct of a finitely bounded homogeneous structure $A$, i.e., if $A$ and $B$ have the same domain and all relations of $B$ are first-order definable in $A$. 

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Infinite-Domain Constraint Satisfaction

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3. $B$ is homogeneous and finitely bounded, i.e., its finite substructures are described by finitely many forbidden finite substructures.
   
   **Examples:** $(\mathbb{Q};<)$, Rado graph, random poset.
   
   **Consequence:** $\text{CSP}(B)$ is in NP.
Three levels of finiteness for countable structures \( B \):

1. \( B \) is \( \omega \)-categorical.
   
   **Equivalent:** \( \text{Aut}(B) \) has for each \( n \) only finitely many orbits of \( n \)-tuples.
   
   **Examples:** \( (\mathbb{Q};<) \), the Rado graph, the random poset, the Henson digraphs, the atomless Boolean algebra.

   **Consequence:** \( R \in \text{Inv}(\text{Pol}(B)) \) if and only if \( R \) has pp-definition in \( B \).

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4. \( \text{CSP}(B) \) in NP and \( B \) is \( \omega \)-categorical also if \( B \) is a first-order reduct of a finitely bounded homogeneous structure \( A \), i.e., if \( A \) and \( B \) have the same domain and all relations of \( B \) are first-order definable in \( A \).
The First Conjecture

**Fact.** Every $\omega$-categorical structure $B$ has the same CSP as an $\omega$-categorical structure $C$ which is a model-complete core, i.e., every first-order formula is over $C$ equivalent to an existential positive formula.

**Proposition (B+Pinsker’11).** Let $C$ be an $\omega$-categorical model-complete core. If (* there are constants $c_1, \ldots, c_n$ such that $\text{Pol}(C, c_1, \ldots, c_n)$ has a continuous clone homomorphism to $\text{Pol}(K_3)$ then CSP $(C)$ is NP-hard.

**First conjecture:** Otherwise, if $C$ is a first-order reduct of a finitely bounded homogeneous structure, then CSP $(C)$ is in P.

**Theorem (Libor Barto, Pinsker’15).** Let $C$ be an $\omega$-categorical model-complete core. Then either (*) or $C$ has a pseudo-Siggers polymorphism, i.e., there are $e_1, e_2, s \in \text{Pol}(C)$ such that $e_1(s(x, x, y, y, z, z)) \approx e_2(s(y, z, x, z, x, y))$.
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The Second Conjecture

Proposition (Libor Barto, Opršal, Pinsker'14). Let \( C \) be \( \omega \)-categorical. If \( \text{Pol}(C) \) has a uniformly continuous minor-preserving map to \( \text{Pol}(K_3) \) then \( \text{CSP}(C) \) is NP-hard.

Second conjecture: Otherwise, if \( C \) is a first-order reduct of a finitely bounded homogeneous structure, then \( \text{CSP}(C) \) is in P.

Libor Barto, Michael Kompatscher, Olšak, Van Pham, Pinsker'17: For \( \omega \)-categorical structures (even model-complete cores), the two conjectures are not equivalent.

\( A \): the countable atomless Boolean algebra. \((A, \neq)\) has pseudo-Siggers polymorphism and uniformly continuous minor-preserving map to \( \text{Pol}(K_3) \).

For first-order reducts of homogeneous structures with finite relational signature, the two conjectures are equivalent.

Question (B., Quinn-Gregson): What about other \( \omega \)-categorical algebras?
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$\omega$-categorical Algebras
ω-categorical Algebras

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- If CSP($A;\neq$) is in P, then so is the validity problem and the entailment problem over $A$.
- ($A;\neq$) is always a core.
Let $A$ be an $\omega$-categorical algebra. What is the complexity of CSP$(A; \neq)$? (Satisfiability of system of equalities and inequalities over $A$.)

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ω-categorical Algebras

Let $A$ be an $\omega$-categorical algebra. What is the complexity of $\text{CSP}(A; \neq)$? (Satisfiability of system of equalities and inequalities over $A$.)

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- In the first case, verify the first conjecture,
ω-categorical Algebras

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  - semi-lattices.
- In the first case, verify the first conjecture, in the second case the second conjecture!
Fixed-Point Logics and Datalog
$B$: finite structure.
Atserias, Bulatov, Dawar’09: either

1. $\text{CSP}(B)$ is in Datalog,
Fixed-Point Logics and Datalog

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1. $\text{CSP}(B)$ is in Datalog, or
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Theorem (Libor Barto, Kozik’09). Let $B$ be a finite structure. Then either $B$ pp-contructs a finite Abelian group, or $\text{Pol}(B)$ contains $f, g$ satisfying

\[
f(x, x, x, y) = f(x, x, y, x) = f(x, y, x, x) = f(y, x, x, x) = g(y, x, x) = g(x, y, x) = g(x, x, y)\]

Question: which first-order reducts of finitely bounded homogeneous structures are in Datalog? in LFP? in LFP+counting?
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Temporal CSPs in FP

Proposition: there is no set of polymorphism identities characterising expressibility of Datalog (B., Jakub Rydval).

Already not for first-order reducts of \((Q; <)\) ('temporal CSPs').

Theorem (B+Jakub Rydval).

Let \(B\) be a first-order reduct of \((Q; <)\).

Either \(\text{Pol}(B)\) contains \(f, g_1, \ldots, g_4\) satisfying

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g_1(y, x, x) & \approx f(x, y, x, x), \\
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Complexity of Constraint Satisfaction

Manuel Bodirsky
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Relation Algebras

Definitions from the early 90s:
A representation of \( A \) is a structure \( B \) with (binary) signature \( A \) s.t. (…)

An \( A \)-network: pair \( (V; f) \) where \( V \) is finite set and \( f: V \times V \to A \).

The network satisfaction problem (NSP) for \( A \): given an \( A \)-network, is there a representation \( B \) of \( A \) and \( s: V \to B \) such that for all \( u, v \in V \) \( (s(u), s(v)) \in f(u, v) \).

Hirsch's RBCP (94): classify the complexity of the NSP for all finite \( A \).
Relation Algebras

A: finite relation algebra. Example:
Relation Algebras

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\[
\begin{array}{c}
\circ & = & < & > \\
= & = & < & > \\
< & < & < & 1 \\
> & > & 1 & > \\
\end{array}
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\leq \prec = \preceq \\
\succ \not\preceq \not\prec
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> \\
0 := \emptyset
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  (s(u), s(v)) \in f(u, v)^B.
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Relation Algebras

A: finite relation algebra. Example:

Definitions from the early 90s:

- A representation of $A$ is a structure $B$ with (binary) signature $A$ s.t. (...
- An $A$-network: pair $(V; f)$ where $V$ is finite set and $f: V^2 \rightarrow A$.
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- Hirsch’s RBCP (94): classify the complexity of the NSP for all finite $A$. 

Complexity of Constraint Satisfaction

Manuel Bodirsky
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\[ \text{An atom } R \in A \text{ is called flexible if for all atoms } S, T \text{ of } A \text{ that are not contained in } R \subseteq S \circ T. \]

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Method: Ramsey theory (Hubicka-Ne ˇsetˇril)

If CSP(B) is in P there exists a canonical pseudo-Siggers polymorphism.
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- If $CSP(B)$ is in P there exists a **canonical** pseudo-Siggers polymorphism.
Pseudo-Jonsson Polymorphisms

Operations $j_1, \ldots, j_{2n-1}: D^3 \to D$ are called quasi Jónsson if

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j_1(x, y, z) \approx j_1(x, x, x)
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for all $i \in \{1, \ldots, 2n - 1\}$

for all even $i \in \{1, \ldots, 2n - 2\}$

for all odd $i \in \{1, \ldots, 2n - 3\}$

Fact (Libor Barto, Kozik): If a finite structure $B$ with finite signature has quasi Jonsson polymorphisms, then it has a quasi near unanimity polymorphism (and CSP($B$) is in Datalog).

Question: Statement remains true for first-order reducts $B$ of finitely bounded homogeneous structures?

Proposition (B+Scheck). If $B$ is a first-order reduct of $(\mathbb{N}; =)$ with finite signature and has pseudo-Jonsson polymorphisms, then it also has quasi near unanimity polymorphisms.
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Valued CSPs

CSPs: framework for modelling decision problems
VCSPs: framework for modelling optimisation problems

Instead of fixing the allowed constraint relations, we fix a set $\Gamma$ of allowed (partial) cost functions $f : D^n \rightarrow Q$.

**Definition (VCSP($\Gamma$))**

Given:
- finite set of variables $V$
- finite set $\phi$ of cost functions from $\Gamma$ applied to variables from $V$
- threshold $u \in Q$.

Question: is there an assignment $s : V \rightarrow D$ such that $\sum \phi(s) \leq u$?

Previous research: finite $D$ (complete complexity classification).

Research question (B+Mamino+Caterina Viola): complexity of VCSPs if $D = Q$.

Example: Least-correlation-clustering with partial information.

Given:
- graph with red and blue edges.

Task:
- find vertex partition such that sum of red edges between different parts and sum of blue edges within a part is minimised.
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- No hope for complexity classification when allowing arbitrary cost functions over \( \mathbb{Q} \).
- Restrict to cost functions that are piecewise linear homogeneous.

**Theorem** (B+Mamino+Caterina Viola):

Let \( \Gamma \) be a finite set of PLH cost functions. If \( \Gamma \) has fully symmetric fractional polymorphisms of all arities then VCSP(\( \Gamma \)) can be solved in polynomial time.

What remains open:

- Full VCSP classification for PLH cost functions?
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Finite structures modulo pp-constructability:
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- 2-Element case (Albert Vucaj)
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