We consider Feller processes whose generators have the test functions as an operator core. In this case, the generator is a pseudo differential operator with negative definite symbol $q(x, \xi)$. If $|q(x, \xi)| < c(1 + |\xi|^2)$, the corresponding Feller process can be approximated by Markov chains whose steps are increments of Lévy processes.

This approximation can easily be used for a simulation of the sample path of a Feller process.

Further we provide conditions in terms of the symbol for the transition operators of the Markov chains to be Feller. This gives rise to a sequence of Feller processes approximating the given Feller process.

**Keywords**: Feller process; pseudo differential operator; Markov chain approximation; Euler scheme; jump process.

**AMS Subject Classification**: Primary: 60J35, Secondary: 47G30, 60J25, 60J10.

1. **Introduction and the main result**

Stochastic processes with stationary and independent increments or Lévy processes are well studied in the literature. A natural generalization of these processes are Feller processes, they include Lévy processes as special case but, in general, they are not spatially homogeneous. Although all Feller processes can be characterized in terms of their infinitesimal generators, it is still a major problem to construct Feller processes with a given generator. Many authors have studied this problem using various approaches,

- the Hille-Yosida theorem and Kolmogorov’s construction, e.g. [11, 8];
Björn Böttcher, René L. Schilling

- solving the associated evolution equation (Kolmogorov’s backwards equation), e.g. [3, 2, 14, 15];
- the well-posedness of the martingale problem, e.g. [1, 8, 20];
- solving a stochastic differential equation, e.g. [10, 13, 21].

Each of these methods also provides an approximation of the Feller process, but only the scheme of Stroock [21] can be used for simulations. Note that the conditions in [21] are quite restrictive—partly due to the fact that they also imply the existence of the Feller process—and that they are given in terms of the Lévy triplet, but not in terms of the symbol. In general it is unknown how to write these conditions in terms of the symbol, see [22] for a detailed discussion.

In this note, we will assume the existence of a Feller process \( X = (X_t)_{t \geq 0} \) with symbol \(-q(x, \xi)\) and prove the convergence of an approximation scheme under general conditions. Our approximation resembles the Euler scheme for a stochastic differential equation; but as it is in general not explicitly possible to find an SDE with solution \( X \), our proofs are necessarily quite different to those in, say, [21]. Existing papers on the Euler scheme for SDEs do either not include the general Feller case (for example Protter and Talay [17]) or have a semimartingale driving term, which of course includes Feller processes, but it is not discussed how to simulate it.

Other approximations, e.g. the powerful Markov chain approximation of Dirichlet processes by Ma, Röckner and Zhang [16], also apply to Feller processes, but they are not amenable to simulation. In contrast, our approximation scheme is easily implemented for simulations, see [4].

We will now state our main result. The necessary definitions and the proof are presented in Sections 2 and 3, respectively. In Section 4 there are some remarks on the assumptions of the theorem below and an extension which provides an approximation by a sequence of Feller processes. The paper closes with some examples.

**Theorem.** Let \( X = (X_t)_{t \geq 0} \) be a Feller process with generator \( A \). Assume that

\[
C^\infty_0 \text{ is an operator core of } A, \text{ i.e. the closure of } A|_{C^\infty_0} \text{ is } A. \quad (A1)
\]

Let \(-q(x, \xi)\) be the symbol of \( A|_{C^\infty_0} \) and assume that

\[
\exists c > 0 : |q(x, \xi)| \leq c(1 + |\xi|^2) \quad \text{for all } x \text{ and } \xi. \quad (A2)
\]

For each \( n \in \mathbb{N}_0 \) define a Markov chain \((Y^n(k))_{k \in \mathbb{N}_0}\) with \( Y^n(0) := X_0 \) and transition kernel \( \mu_{x, \frac{1}{n}}(dy) \) where

\[
\int_{\mathbb{R}^d} e^{iy \xi} \mu_{x, \frac{1}{n}}(dy) = e^{ix \xi - \frac{1}{n}q(x, \xi)}.
\]

Then

\[
Y^n([., n]) \xrightarrow{d} X \quad \text{as } n \to \infty.
\]

Here \([x] := \max\{k \in \mathbb{Z} \mid k \leq x\}\) and \(\xrightarrow{d}\) denotes the convergence in the space of right continuous functions with left limits equipped with the Skorohod topology.
2. Preliminaries

Throughout, $d$ denotes the dimension. We write $C = C(\mathbb{R}^d, \mathbb{C})$ for the continuous functions $f : \mathbb{R}^d \to \mathbb{C}$ and $B_b$ for the Borel measurable and bounded functions. By $C_b$, $C_{\infty}$ or $C_c$ we denote the continuous functions which are bounded, vanish at infinity or have compact support, respectively; the superscript ‘$\infty$’ indicates that the functions are arbitrarily often differentiable. All random variables and processes will be defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and the corresponding expectation is written as $\mathbb{E}$.

A stochastic process $X = (X_t)_{t \geq 0}$ with values in $\mathbb{R}^d$ is called Feller process if the family of operators

$$T_t u(x) = \mathbb{E}(u(X_t) \mid X_0 = x), \quad u \in C_{\infty},$$

is a Feller semigroup. This means that $(T_t)_{t \geq 0}$ is a semigroup of linear operators on the space of continuous functions vanishing at infinity, $C_{\infty}$, equipped with the supremum norm $\| \cdot \|_{\infty}$, which is

- strongly continuous: $\lim_{t \to 0} \| T_t u - u \|_{\infty} = 0$;
- contractive: $\| T_t u \|_{\infty} \leq \| u \|_{\infty}$, and
- positivity preserving: $T_t u \geq 0$ whenever $u \geq 0$.

As usual, one defines the infinitesimal generator $(A, \mathcal{D}(A))$ by

$$A u := \lim_{t \to 0} \frac{T_t u - u}{t} \text{ on } \mathcal{D}(A) := \left\{ u \in C_{\infty} : \lim_{t \to 0} \frac{T_t u - u}{t} \text{ exists strongly} \right\}.$$

If the test functions $C_{c\infty}^\circ$ are contained in the domain of the generator $A$ of the semigroup, a result due to Courrège [6] shows that $A|_{C_{c\infty}^\circ}$ is a pseudo differential operator:

$$A u(x) = -q(x, D) u(x) = -\int_{\mathbb{R}^d} q(x, \xi) e^{i\xi \cdot \hat{u}(\xi)} \, d\xi, \quad u \in C_{c}^\infty, \quad (2.1)$$

where $\hat{u}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} u(x) \, dx$ is the Fourier transform of $u$. The function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is called the symbol of the pseudo differential operator. It is measurable and locally bounded in $(x, \xi)$, and continuous and negative definite as a function of $\xi$, that is $\xi \mapsto q(x, \xi)$ has for each $x$ the following Lévy-Khinchine representation:

$$q(x, \xi) = i\ell(x) \cdot \xi + \frac{1}{2} \xi \cdot Q(x) \xi + \int_{\mathbb{R}^d \setminus \{0\}} \left( 1 - e^{iy \cdot \xi} + \frac{iy \cdot \xi}{1 + |y|^2} \right) \nu(x, dy) \quad (2.2)$$

where $\ell(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d \times d}$ is positive semidefinite and $\nu(x, dy)$ is a kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that $\int_{\mathbb{R}^d \setminus \{0\}} \min(|y|^2, 1) \nu(x, dy) < \infty$ for all $x$. The (Lévy-)triplet $((\ell(x), \frac{1}{2} Q(x), \nu(x, dy)))$ corresponds to the ‘coefficients’ of the pseudo differential operator $-q(x, D)$.

Note that the real part $\text{Re} q(x, \xi)$ is always positive.
A Lévy process is a stochastic process with càdlàg (right-continuous with finite left-hand limits) paths and stationary and independent increments. Among the Feller processes, the Lévy processes play a special role: they are characterized by the fact that their symbol does not depend on \( x \) and we will write it as \( \psi(\xi) \). This is equivalent to saying that the corresponding semigroup has the form

\[
T_t u(x) = \int_{\mathbb{R}^d} u(x + y) \mu_t(dy)
\]

where \( (\mu_t)_{t \geq 0} \) is a convolution semigroup of probability measures on \( \mathbb{R}^d \) whose characteristic function is given by

\[
\int_{\mathbb{R}^d} e^{ix\xi} \mu_t(d\xi) = e^{-t\psi(\xi)}.
\]

This means that Lévy processes are exactly those Feller processes whose generators have constant 'coefficients' and do not depend on \( x \). In contrast to the situation of Lévy processes it is a difficult problem to show that a given \( x \)-dependent symbol \( q(x,\xi) \) will yield a stochastic process. For a comprehensive treatment we refer to the monographs by Jacob [11] and the paper [12].

A Feller process has a version whose sample paths are in the space of right continuous functions with left limits. Convergence of processes on this space is described using the Skorohod topology, for details see for example [7].

3. Proof of the main result

Let \( X \) be an \( \mathbb{R}^d \)-valued Feller process with generator \( A \). Our assumption (A1) that the test functions \( C_\infty^\infty \) are an operator core for \( A \) entails that \( C_\infty^\infty \subset D(A) \). Therefore, cf. Jacob [11], vol. 1, Sections 3.7 and 4.5 or Courrègue [6], the operator \( A \) is on \( C_\infty^\infty \) a pseudo differential operator with negative definite symbol \( -q(x,\xi) \), i.e. \( A \) has for test functions the representation (2.1).

Thus following our discussion in Section 2, we see that \( e^{-tq(x,\cdot)} \) is for frozen \( x \) the characteristic function of some Lévy process at time \( t \). Thus, for each \( x \in \mathbb{R}^d \) there exists a family of probability measures \( (\mu_{x,\frac{1}{n}})_{n \in \mathbb{N}} \) such that the characteristic function or inverse Fourier transform \( F^{-1} \) is of the form

\[
\int_{\mathbb{R}^d} e^{iy\xi} \mu_{x,\frac{1}{n}}(dy) = F^{-1} \left[ \mu_{x,\frac{1}{n}} \right](\xi) = e^{ix\xi - \frac{1}{n}q(x,\xi)}.
\]

For each \( n \in \mathbb{N} \) define a Markov chain \( (Y^n(k))_{k \in \mathbb{N}_0} \) with \( Y^n(0) := X_0 \) and transition kernel \( \mu_{x,\frac{1}{n}}(dy) \). The corresponding transition operator on \( u \in C_b \) is given by

\[
W_{\frac{1}{n}} u(x) := \int_{\mathbb{R}^d} u(y) \mu_{x,\frac{1}{n}}(dy).
\]

On \( C_\infty^\infty \) the operator \( W_{\frac{1}{n}} \) has a representation as pseudo differential operator with symbol \( e^{-\frac{1}{n}q(x,\xi)} \). This follows from Plancherel’s theorem:

\[
W_{\frac{1}{n}} u(x) = \int_{\mathbb{R}^d} F^{-1} \left[ \hat{u} \right](y) \mu_{x,\frac{1}{n}}(dy)
= \int_{\mathbb{R}^d} \hat{u}(\xi) \cdot F^{-1} \left[ \mu_{x,\frac{1}{n}} \right](\xi) d\xi = \int_{\mathbb{R}^d} \hat{u}(\xi) e^{ix\xi} e^{-\frac{1}{n}q(x,\xi)} d\xi.
\]
From (3.2) we see that $W^n_\frac{1}{n}$ is positivity preserving and that for $u \in B_b$

$$|W^n_\frac{1}{n} u(x)| \leq \int_{\mathbb{R}^d} |u(y)| \mu_{x, \frac{1}{n}}(dy) \leq \|u\|_\infty \mu_{x, \frac{1}{n}}(\mathbb{R}^d) = \|u\|_\infty;$$

thus, $W^n_\frac{1}{n}$ is also a contraction.

To proceed further, we need the following result which can be found, in a much more general form, in the monograph by Ethier and Kurtz [7], Theorem 6.5, Chapter 1.

For $n \in \mathbb{N}$ let $W^n_\frac{1}{n}$ be a linear contraction on $B_b$. Let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on $C_\infty$ with generator $A$ such that $C_\infty$ is a core for $A$. Then

$$\lim_{n \to \infty} \sup_{t \in [a, b]} \left\| W^n_\frac{1}{n} u - T_t u \right\|_\infty = 0 \text{ for all } u \in C_\infty \text{ and } [a, b] \subset [0, \infty) \quad (3.4)$$

is equivalent to

$$\lim_{n \to \infty} \left\| W^n_\frac{1}{n} u - u - A u \right\|_\infty = 0 \text{ for all } u \in C_\infty^c. \quad (3.5)$$

Our next step is to verify (3.5) for the situation at hand. For this we need assumption (A2), i.e. $|q(x, \xi)| \leq c(1 + |\xi|^2)$ for all $x$ and $\xi$. Using the mean value theorem twice with suitable intermediate values $h, s \in (0, \frac{1}{n})$ it follows that

$$\frac{e^{-\frac{1}{n}q(x, \xi)} - 1}{\frac{1}{n}} + q(x, \xi) = q(x, \xi) \left( e^{-sq(x, \xi)} - 1 \right) = sq(x, \xi)^2 e^{-hq(x, \xi)}.$$

Thus for $u \in C_\infty^c$ and with the representations (2.1) and (3.3) of $A$ and $W^n_\frac{1}{n}$, respectively, we obtain

$$\left| W^n_\frac{1}{n} u(x) - u(x) - A u(x) \right| = \left| \int e^{ix\xi} \left( e^{-\frac{1}{n}q(x, \xi)} - 1 \right) + q(x, \xi) \hat{u}(\xi) \, d\xi \right|$$

$$\leq \int |sq(x, \xi)^2 e^{-hq(x, \xi)}||\hat{u}(\xi)|| \, d\xi$$

$$\leq s \int |q(x, \xi)^2||\hat{u}(\xi)|| \, d\xi$$

$$\leq \frac{c^2}{n} \int (1 + |\xi|^2)^2|\hat{u}(\xi)|| \, d\xi.$$

The right hand side is independent of $x$ and finite since $u \in C_\infty^c$.

Thus (3.5) holds and (3.4) follows under the assumption (A1). This shows that the finite dimensional distributions of $Y^n = (Y^n(\lfloor tn \rfloor))_{t \geq 0}$ converge uniformly on compact time intervals. Since $(Y^n(k))_{k \in \mathbb{N}_0}$ is a Markov chain, this implies that the laws of the $Y^n$ are tight and we get convergence in Skorohod’s space $D_{\mathbb{R}^d}[0, \infty)$. Indeed, an application of Theorem 2.12 of Chapter 4 in [7] with $E_n = \mathbb{R}^d$, $\eta_n = \pi_n = id$ and $\varepsilon_n = \frac{1}{n}$ shows

$$Y^n \overset{d}{\longrightarrow} X \quad \text{as} \quad n \to \infty.$$
4. Remarks and Extensions

Let \( X \) be a Feller process with generator \( A \) such that \( C^\infty_c \subset D(A) \); then \( A|_{C^\infty_c} \) is a pseudo differential operator and we denote its symbol by \( -q(x, \xi) \).

\((A1):\) The assumption that \( C^\infty_c \) is an operator core of \( A \) is equivalent to

\[(A, D(A)) \text{ is the only extension of } (A|_{C^\infty_c}, C^\infty_c) \text{ generating a Feller semigroup.}\]

This is usually required in applications.

We do not know any non-trivial Feller process on the whole space \( \mathbb{R}^d \) whose generator is defined on the test functions \( C^\infty_c \) but where \( C^\infty_c \) is not an operator core. This is quite different on domains. Consider on the open unit ball \( B_1(0) \subset \mathbb{R}^d \) the Laplacian with Dirichlet \(-\Delta_D\) resp. Neumann \(-\Delta_N\) boundary conditions. The corresponding process is killed resp. reflecting Brownian motion. In both cases \( C^\infty_c = C^\infty_c(B_1(0)) \) is in the domain of \(-\Delta\) but \( -\Delta|C^\infty_c \subset -\Delta_D \).\(\]

\((A2):\) This is a more restrictive assumption

\[\exists c > 0 : |q(x, \xi)| \leq c(1 + |\xi|^2) \text{ for all } x \text{ and } \xi. \quad (A2)\]

Since the symbol \( q(x, \xi) \) always satisfies

\[|q(x, \xi)| \leq cx(1 + |\xi|^2) \text{ for all } x \text{ and } \xi,\]

\((A2)\) is a boundedness condition for the ‘coefficients’ of the generator. In fact, it is shown in [18] that sup\(x \in \mathbb{R}^d c_x < \infty\) if, and only if, for the ‘coefficients’ \( \ell(x), Q(x) \) and \( \nu(x, \cdot) \) from (2.2) the expression

\[\sup_{x \in \mathbb{R}^d} ||\ell(x)|| + \sup_{x \in \mathbb{R}^d} ||Q(x)|| + \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \min(|y|^2, 1) \nu(x, dy) < \infty\]

is finite.

It is interesting to note that—with the exception of Chapter 9 in Hoh [8]—all known constructions of processes starting with pseudo differential operators require the assumption (A2).

If we subordinate the Markov chain \((Y^n(k))_{k \in \mathbb{N}_0}\) with transition operator \(W^n_1\) by an independent Poisson process with rate \(n\) we get the following semigroup:

\[S_t^{(n)}u := e^{-\tn(id-W^n_1)}u = e^{-nt} \sum_{k=0}^{\infty} \frac{(tn)^k W^n_1 u}{k!}. \quad (4.1)\]

Obviously, the generator of \(S_t^{(n)}\) is \(-n(id-W^n_1)\) and its symbol is \(-q_n(x, \xi) = n(1 - e^{-1/2}q(x, \xi))\). By Corollary 3.6.10 and Theorem 3.6.11 (Schoenberg’s theorem) of [11] (vol. 1), \(\xi \mapsto q_n(x, \xi)\) is itself continuous negative definite and it is a natural question whether \(S_t^{(n)} = e^{-tq_n(x, \cdot)}\) is itself a Feller semigroup.
Lemma. We use the notation from above. If the assumptions of the Theorem hold and if, in addition,

\[
\limsup_{\xi \to 0} |q(x, \xi)| = 0, \tag{A3}
\]

\[
x \mapsto q(x, \xi) \text{ is a continuous function for all } \xi, \tag{A4}
\]

then \( W_\frac{1}{n} : C_\infty \to C_\infty \) and the \((S_t^{(n)})_{c \geq 0}\) are Feller semigroups approximating the semigroup \( T_1 \).

Proof. If \( u \in C_\infty \), we find for every \( \epsilon > 0 \) some \( R = R_\epsilon \) such that \(|u(y)| \leq \epsilon \) whenever \(|y| \geq R\). Thus,

\[
|W_\frac{1}{n} u(x)| = \left| \int u(y) \mu_{x, \frac{1}{n}}(dy) \right| \leq \epsilon + \|u\|_{\infty} \mu_{x, \frac{1}{n}}(B_R(0)).
\]

If \(|x| > 2R\) we have \( B_R(0) \subset B_R(x) \). Since

\[
1_{B_R(x)}(y) \leq 2 \frac{1}{1 + \frac{|x-y|^2}{R^2}} = \int_{\mathbb{R}^d} \left[ 1 - \cos \frac{|x-y|}{R} \right] g(\eta) d\eta
\]

with some exponentially decaying continuous density \( g(\eta) \geq 0 \), see e.g. (2.5), (2.6) in [18], we deduce with Tonelli’s theorem

\[
|W_\frac{1}{n} u(x)| \leq \epsilon + \|u\|_{\infty} \int \int \left[ 1 - \cos \frac{|x-y|}{R} \right] \mu_{x, \frac{1}{n}}(dy) g(\eta) d\eta
\]

\[
= \epsilon + \|u\|_{\infty} \text{Re} \int \left[ 1 - e^{-\frac{1}{2} q(x, \eta)} \right] g(\eta) d\eta
\]

\[
\leq \epsilon + \|u\|_{\infty} \int \sup_x |g(x, \eta)| g(\eta) d\eta.
\]

Here we used that \( \cos(x-y) \frac{R}{R} = \text{Re} e^{i(x-y) \frac{R}{R}} \) and the definition (3.1) of \( \mu_{x, \frac{1}{n}} \). By dominated convergence we see that (A3) implies \( \lim_{|x| \to \infty} W_\frac{1}{n} u(x) = 0 \).

Continuity follows from (A4). Since \( \text{Re} q(x, \xi) \geq 0 \) is positive, we have \( |e^{-tq(x, \xi)}| = e^{-t \text{Re} q(x, \xi)} \leq 1 \). Recall that the Fourier transform maps the test functions \( C_c^\infty \) into the rapidly decreasing functions \( S \). If we use dominated convergence in the representation (3.3), we conclude that \( x \mapsto W_\frac{1}{n} u(x) \) is continuous and bounded for all \( u \in C_c^\infty \). Further, since \( W_\frac{1}{n} \) is contractive and since \( C_c^\infty \) is dense in \( C_\infty \), a standard \( 3-e \)-argument proves that \( W_\frac{1}{n} u \) is continuous for all \( u \in C_\infty \).

The Feller property of \((S_t^{(n)})_{t \geq 0}\) follows from the mapping properties of the contractions \( W_\frac{1}{n} \) and the strong convergence of the exponential series (4.1).
As seen before (A1) and (A2) imply (3.5), i.e. the right hand side converges. Thus (3.4) and the strong continuity of $(S_t^{(n)})_{t \geq 0}$ imply that $S_t^{(n)}$ approximates $T_t$. □

**(A3):** This assumption is only needed to show that $W_{\frac{1}{n}} u(x)$, hence $S_t^{(n)} u(x)$, vanish at infinity for $u \in C_\infty$. Together with (A4) this yields the Feller property. If we content ourselves with the $C_c$-Feller property, i.e. $W_{\frac{1}{n}}, S_t^{(n)} : C_c \rightarrow C_b$, we can argue as in the proof of Theorem 3.2 (ii)⇒(iii) of [19] and do without (A3).

**(A4):** This is a very natural assumption. To our knowledge there is no example of a Feller generator $A$ with $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$ such that its symbol is not continuous as a function of $x$. (A4) is automatically satisfied in the following two cases:

a) for diffusions $X_t$. In this case the generator is a differential operator and (A4) follows immediately from the local property of the generator.

b) for Feller processes with uniformly bounded jumps, i.e. where the Lévy kernel satisfies $\nu(x, \mathbb{R}^d \setminus B_r(0)) = 0$ for some $r > 0$. Without loss of generality we may assume that in (2.2) $\ell(\cdot) = 0$ and $Q(\cdot) \equiv 0$. If we insert (2.2) into (2.1) and use Fourier inversion, we get the following alternative form of $A$:

$$Au(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x + y) - u(x) - \frac{y \nabla u(x)}{1 + |y|^2} \right) \nu(x, dy), \quad u \in C_c^\infty(\mathbb{R}^d).$$

Pick some $\chi = \chi_R \in C_c^\infty(\mathbb{R}^d)$, $R > r$, such that $1_{B_{2R}(0)} \leq \chi \leq 1$, and set $u_\xi(x) := e^{ix \xi} \chi(x)$. Clearly $u_\xi \in C_c^\infty(\mathbb{R}^d)$ and $x \mapsto Au_\xi(x)$ is continuous. Since $\nu(x, dy)$ is supported in $B_r(0)$, we find for all $x \in B_R(0)$

$$Au_\xi(x) = \int_{B_r(0) \setminus \{0\}} \left( e^{i\xi(x+y)} \chi(x+y) - e^{i\xi x} \chi(x) - \frac{y \nabla_x (e^{i\xi x} \chi(x))}{1 + |y|^2} \right) \nu(x, dy)$$

$$= \int_{B_r(0) \setminus \{0\}} \left( e^{i\xi(x+y)} - e^{i\xi x} - \frac{y \nabla_x e^{i\xi x}}{1 + |y|^2} \right) \nu(x, dy)$$

i.e. $-q(x, \xi) = e^{-ix \xi} A e^{ix} \xi(x)$ is continuous on any ball $B_R(0)$, hence on $\mathbb{R}^d$.

The following example shows what can go wrong if we have infinite jump heights and (interior) boundaries. Consider a Markov process which is defined on $(0, \infty)$ by

$$P(X_t = x \mid X_0 = x) = e^{-t \cosh t}, \quad P(X_t = \frac{1}{2} \mid X_0 = x) = e^{-t \sinh t}.$$ 

The semigroup of transition operators is then

$$T_t u(x) = e^{-t} \left( \cosh t \ u(x) + \sinh t \ u\left(\frac{1}{2}\right) \right), \quad x \in (0, \infty)$$

which is a Feller semigroup on $C_\infty((0, \infty)) := \{u|_{(0,\infty)} : u \in C_\infty(\mathbb{R}), \ u(0) = 0\}$. Its generator is given by

$$Au(x) = \int_{(0,\infty)} (u(x+y) - u(x)) \nu(x, dy)$$
with Lévy kernel \( \nu(x,dy) = \delta_{1/2-x}(dy) \chi_{\{x \neq 1\}}(x) \). Its symbol is given by
\[
-q(x,\xi) = e^{-ix\xi}A e^{i\xi}(x) = \left[ \exp \left( i\xi \frac{1}{2} - \xi \right) \right] - 1 \chi_{\{x \neq 1\}}(x).
\]

As soon as we add the state \( x = 0 \) and the coffin state \( x = \infty \), e.g.
\[
\mathbb{P}(X_t = 0 | X_0 = 0) = e^{-t} \cosh t, \quad \mathbb{P}(X_t = \infty | X_0 = 0) = e^{-t} \sinh t; \quad \mathbb{P}(X_t = \infty | X_0 = \infty) = 1, \quad \mathbb{P}(X_t = 0 | X_0 = \infty) = 0.
\]

Thus,
\[
T_t u(0) = e^{-t}(\cosh t u(0) + \sinh t u(\infty)), \quad T_t u(\infty) = u(\infty)
\]
and the infinitesimal generator changes to
\[
\tilde{A} u(x) = (-1_{\{0\}}(x) + A)u(x) \text{ on } C_\infty([0,\infty)) = \{u|_{[0,\infty)} : u \in C_\infty(\mathbb{R})\}
\]
and it is not hard to see that this is a Feller generator if, and only if we restrict ourselves to \( u(0) = 0 \) or \( u \in C_\infty((0,\infty)) \). Still, it has a symbol, given by
\[
-q(x,\xi) = -1_{\{0\}}(x) - q(x,\xi) 1_{(0,\infty)}(x)
\]
which is clearly discontinuous at \( x = 0 \).

5. Examples

We give three examples of Feller processes which satisfy the assumptions (A1), (A2), (A3) and (A4).

The Feller process corresponding to the symbol
\[
q(x,\xi) := |\xi|^{\alpha(x)}
\]
is called stable-like process and it exists if \( x \mapsto \alpha(x) \) is Dini continuous and bounded away from 0 and 2, cf. [1].

The Meixner-type process corresponds to the symbol
\[
q(x,\xi) := -im(x)\xi + 2s(x) \left[ \ln \cosh \left( \frac{a(x)\xi - ib(x)}{2} \right) - \ln \cosh \left( \frac{b(x)}{2} \right) \right]
\]
where \( a, b, m, s \in C_\infty^0 \) and \( 0 < a_0 \leq a(x), -\pi < b_- \leq b(x) \leq b_+ < \pi \) and \( 0 < s_0 \leq s(x) \) for some constants \( a_0, b_-, b_+, s_0 \), cf. [5].

A for practitioners especially interesting Feller process is the symmetric generalized hyperbolic-type process mentioned by Hoh [9]. It corresponds to the symbol
\[
q(x,\xi) := -\ln \left( \frac{(\delta(x)\alpha(x))^\lambda K_\lambda(\delta(x)\sqrt{\alpha(x)^2 + \xi^2})}{K_\lambda(\delta(x)\alpha(x)) (\delta(x)\sqrt{\alpha(x)^2 + \xi^2})^\lambda} \right)
\]
where \( K_\lambda \) is a modified Bessel function of third kind, \( \lambda \in \mathbb{R}, \alpha, \delta \in C_\infty^0 \) and \( 0 < a_0 \leq a(x), 0 < \delta_0 \leq \delta(x) \) for some constants \( a_0, \delta_0 \).

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References


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