Special difficulties in the three-dimensional container loading problem

Jürgen Rietz
Universidade do Minho
4710-057 Braga, Portugal
juergen_rietz@gmx.de

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Problem formulation

Given a rectangular container of dimensions $L \times W \times H$, pack rectangular boxes of sizes $\ell_i \times w_i \times h_i$ into the container, such that no box contains points outside the container or penetrates another box. We assume that the boxes must not be rotated and their sides shall be parallel to the container sides. This problem is generally very difficult because of a lot of possible partial packing patterns.

A negative conjecture

If all boxes are packed one by one such that every box is moved as far as possible to the bottom left front (until it touches in all three directions another piece or the container) then probably there are instances of the packing problem, where no feasible packing of all boxes can be found in spite of its existence.

For motivation look at the figure 1. It shows, how five unit cubes and six boxes of sizes $1 \times 2 \times 4$ and $2 \times 2 \times 3$, which may be rotated, can be put into a cube of side length 5. In the corners each times three larger boxes come together and they cannot be moved to the corner, even if the unit cubes are removed. The entire packing is now mirrored several times and changed further. The result (without unit cubes) is shown in figure 2. For more flexibility choose a parameter $n \in \mathbb{N}$ with $n \geq 3$. The side length of the large cube is $n + 4$. Ten unit cubes were removed. The other 31 boxes are arranged as in the table 1 given.

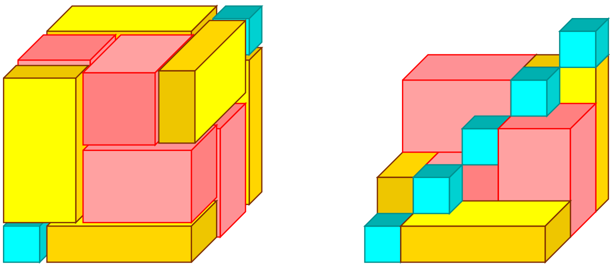

Figure 1: A three-dimensional puzzle

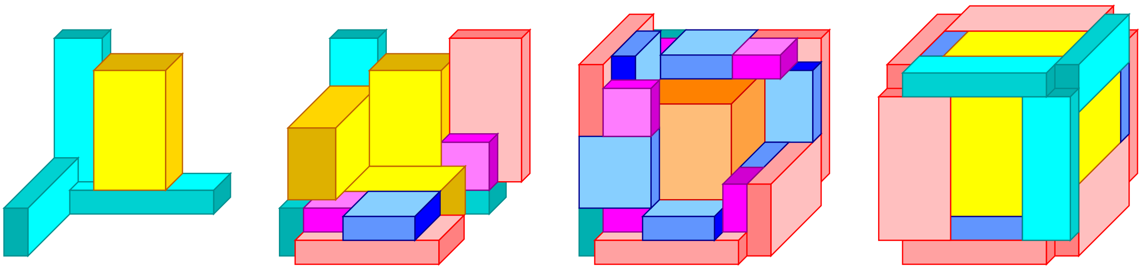

Figure 2: three-dimensional puzzle for the conjecture (here with $n = 4$)

Trying to prove the conjecture

The packing problem can be modeled as an integer linear problem. The models of the BEASLEY type have the disadvantage of a large number of binary variables, especially when the container size is increased. Another model is due to FASANO and PADBERG, see e.g. [1]. As reference point always the lower left front corner is chosen.

Each piece must fit into the container. This yields the conditions

$$0 \leq x_i \leq L - \ell_i \tag{1}$$
$$0 \leq y_i \leq W - w_i \tag{2}$$
$$0 \leq z_i \leq H - h_i \tag{3}$$

Since no piece may overlap another, for any different indices i, j at least one of six inequalities must hold. This can be expressed by binary variables $X_{ij}, X_{ji}, Y_{ij}, Y_{ji}, Z_{ij}, Z_{ji}$, which tell, which of the inequalities is valid. Hence, the following conditions arise:

$$x_i - x_j + L * X_{ij} \leq L - \ell_i \tag{4}$$
$$x_j - x_i + L * X_{ji} \leq L - \ell_j \tag{5}$$
$$y_i - y_j + W * Y_{ij} \leq W - w_i \tag{6}$$
$$y_j - y_i + W * Y_{ji} \leq W - w_j \tag{7}$$
$$z_i - z_j + H * Z_{ij} \leq H - h_i \tag{8}$$
$$z_j - z_i + H * Z_{ji} \leq H - h_j \tag{9}$$
$$X_{ij} + X_{ji} + Y_{ij} + Y_{ji} + Z_{ij} + Z_{ji} = 1 \tag{10}$$

Table 1: solution of the puzzle

type	x	y	z	ℓ_i	w_i	h_i
4	1	0	0	$n+2$	2	1
4	0	0	1	2	1	$n+2$
4	0	1	0	1	$n+2$	2
6	2	0	1	$n-1$	2	$n+1$
2	1	1	$n+1$	1	2	2
5	$n+1$	0	1	3	1	$n+2$
6	1	2	0	$n+1$	$n-1$	2
2	$n+1$	1	1	2	1	2
3	$n+2$	2	0	1	$n-1$	3
5	1	$n+1$	0	$n+2$	3	1
2	1	$n+1$	1	2	2	1
6	0	1	2	2	$n+1$	$n-1$
5	0	1	$n+1$	1	$n+2$	3
3	1	3	$n+1$	1	$n-1$	3
3	$n+1$	1	3	3	1	$n-1$
3	3	$n+1$	1	$n-1$	3	1
1	2	2	2	n	n	n
3	2	0	$n+2$	$n-1$	3	1
3	0	$n+2$	2	3	1	$n-1$
2	$n+2$	$n+1$	1	1	2	2
5	$n+3$	1	0	1	$n+2$	3
6	$n+2$	2	3	2	$n+1$	$n-1$
2	$n+1$	1	$n+2$	2	2	1
5	1	0	$n+3$	$n+2$	3	1
6	2	3	$n+2$	$n+1$	$n-1$	2
2	1	$n+2$	$n+1$	2	1	2
5	0	$n+3$	1	3	1	$n+2$
6	3	$n+2$	2	$n-1$	2	$n+1$
4	$n+2$	$n+3$	1	2	1	$n+2$
4	1	$n+2$	$n+3$	$n+2$	2	1
4	$n+3$	1	$n+2$	1	$n+2$	2

The conditions (1)–(10) yield an integer linear problem. Since 31 pieces shall be packed, 93 coordinates and $31 * 30 * 3 = 2790$ binary variables arise. The equation (10) can be dissolved, such that one of the six binary variables can be eliminated. Since setting more than one of these six binaries to 1 cannot give more freedom in the conditions (4)–(9), it is no problem that removing one of the binary variables yields an extended feasible area. Because 15 of the 31 pieces are exactly the same as 15 other pieces, we can demand $x_i \leq x_{i+15}$ if the pieces are numbered appropriately ($i = 1, \ldots, 15$). Then again 15 binary variables can be removed. But in spite of some further additional conditions, e.g. to remove some symmetry, the decision problem, if a solution exists, where one piece is put into the corner $(0,0,0)$, remains too complicated for ILOG CPLEX interactive optimizer.

References

[1] PADBERG, M.: Packing small boxes into a big box. Math. Methods of Operations Research (ZOR), 52:121, 2000.