A Fast Exact Algorithm for the Optimum Cooperation Problem *

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In this work, we will deal with the following optimization problem. Suppose the benefit of cooperation between two vertices, say, researchers, is represented by the weight of the edge joining them. Furthermore, there is a unit gain for each component, say, research project. We search for an optimal cooperation, i.e., want to decide which researchers should collaborate in order to maximize the total benefit (see Auriac et al. [1]).

In graph-theoretic terms, the problem can be stated as follows. Let a graph $G = (V, E)$ with edge weights $w_e \in \mathbf{R}$ for the edges $e \in E$ be given. We want to solve the problem

$$\max\{c_G(A) + w(A) : \; A \subseteq E\}, \tag{1}$$

where $w(A) = \sum_{e \in A} w_e$ and $c_G(A)$ is the number of connected components of the induced graph $G(A) = (V, A)$.

The problem was first mentioned in the literature by Cunningham [5] in the context of determining optimum attacks in networks. At this the edge weight w_e can be interpreted as a measure for the effort required by an attacker to destroy edge e. The task is to minimize the difference between the effort of destroying a set of edges and the number of newly generated components of the graph.

Another relevant application is the separation of partition inequalities, as introduced by Baïou, Barahona and Mahjoub [2]. Given a partition $\{S_1, \ldots, S_p\}$ of the node set V, we denote by $\delta(S_1, \ldots, S_p)$ the set of all edges having endnodes in different sets of the partition. Then, for given real numbers a and b, the inequality $w(\delta(S_1, \ldots, S_p)) \leq ap+b$ is called partition inequality. Partition inequalities arise as valid inequalities for a number of combinatorial problems. In order to use these inequalities inside a cutting

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plane algorithm, we have to solve the separation problem that, given edge weigths $w_e \geq 0$, returns a partition violating the inequality, if it exists. Baïou et al. [2] show that the separation problem can be solved by computing an optimal solution of (1).

An important model in statistical physics is the so-called Potts model (see Hartmann/Rieger [7]). It was introduced as a generalization of the so-called Ising model to describe several physical systems. It is a model on a graph where the vertices are assigned (spin) variables that each can take values between $\{1, \ldots, q\}$. Interactions between pairs of spins may be present. The aim is to compute the so-called partition function that completely encodes the physics of the system. It depends on the number q and the interactions between the spins. For many relevant physics systems, computing the partition function is a difficult task. However, as pointed out by Juhasz, Rieger, Iglói [8] and Anglès d'Auriac, Iglói, Preissmann and Sebö [1], for big numbers q, determining the dominant contribution in the Potts partition function amounts to solving a problem of type (1).

Several solution algorithms have been presented in the literature. As it is not hard to see that the function to be maximized is supermodular, any algorithm for submodular function minimization could be used to solve the problem. By now several polynomial algorithms [4, 6] are known to solve this task. However, the specific properties of the problem allow the usage of algorithms with better worst-case asymptotic running time.

Cunningham [5] developed the first combinatorial algorithm with polynomial running time for the solution of the optimum attack problem that is based on $|E|$ minimum cut computations in an associated network. Thereafter, the worst-case running time of the algorithm was decreased to only $|V| - 1$ minimum cut computations by Baïou, Barahona and Mahjoub, and Barahona [2, 3]. Anglès d'Auriac et al. [1] built upon the existing work and presented an algorithm that also needs $|V| - 1$ minimum cut computations but is easier to implement. They presented some experimental results for instances coming from the physics application.

In the talk we present an algorithm that is based on the one of Anglès d'Auriac et al. but has a better average running time, as by graph-theoretic considerations only a fraction of the minimum cut computations are necessary in practice. We also provide optimality conditions and theoretical results that prove the correctness of our algorithm. We discuss details of our implementation and present experimental results on instances coming from the physics applications. Furthermore, we compare our algorithm with the original method of Anglès d'Auriac et al. It turns out that the running times can be reduced considerably for many practical instances.

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