Amenable groups
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Throughout this note, the convention is right multiplication and right convolution to be coherent with the notation in other lectures.

1 How to build amenable groups

\[ A \triangle B = (A \setminus B) \cup (B \setminus A) \]. Groups are assumed countable and discrete.

**Definition 1.1. (Følner, 1955)** A group \( \Gamma \) is amenable if and only if there exists a Følner sequence, i.e. a sequence of finite sets \( \{F_n\}_{n \geq 1} \) such that for all \( \gamma \in \Gamma \), \( \lim_{n \to \infty} \frac{|F_n \gamma \triangle F_n|}{|F_n|} = 0 \) ★

**Example 1.2.** Here are a few basic examples:

- A finite group is amenable, take \( F_n = \Gamma \) for all \( n \). Then \( F_n \gamma = F_n \) and \( \frac{|F_n \gamma \triangle F_n|}{|F_n|} = 0 \).

- A direct union of finite groups is amenable \( \Gamma = \bigoplus_{n \in \mathbb{N}} G_i \) (where \( G_i \) are finite groups). An element \( \gamma \in \Gamma \) is a sequence \( \gamma_i \in G_i \), so that for some \( n_0 \) (depending on \( \gamma \)), \( \forall n \geq n_0 \), \( \gamma_n = e_{G_i} \). Take \( F_n = \bigoplus_{i=0}^{n_0} G_i \), then for any \( \gamma \in \Gamma \), \( F_n \gamma = F_n \) for \( n \) large enough. So \( \frac{|F_n \gamma \triangle F_n|}{|F_n|} = 0 \) for \( n \) large enough.

- \( \mathbb{Z} \) is amenable. Take \( F_n = [1, n] \). Without loss of generality, one can compute only \( \gamma > 0 \). Then \( \frac{|F_n \gamma \triangle F_n|}{|F_n|} = \frac{1}{n} |[1, \gamma] \cup [n + 1, n + \gamma]| = 2|\gamma|/n \to 0 \). ♦

When the graph is finitely generated by a set \( S \), \( \partial S F_n = \{ \text{edges (in the Cayley graph of} \ \Gamma \text{ for} \ S \} \) between \( F_n \) and \( F_n^c := \Gamma \setminus F_n \). A first useful lemma that reduces some computation is:

**Lemma 1.3**

\[
\text{Assume} \ \Gamma \text{ is finitely generated. A sequence} \ \{F_n\} \text{ is Følner if and only if} \ \lim_{n \to \infty} \frac{\partial F_n}{|F_n|} = 0.
\]

The proof is left as an exercise. An isoperimetric profile for the group is an increasing function \( \overline{\mathcal{F}} : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) such that \( \exists C > 0 \) so that for any finite set \( F \), \( \overline{\mathcal{F}}(|F|) \leq |\partial F| \). Amenable groups are groups with sublinear \( \overline{\mathcal{F}} \).

In L. Saloff-Coste's lectures it was shown that growth of balls which is \( \geq C n^d \) implies \( \overline{\mathcal{F}}(x) = x^{(d-2)/d} \) works. However putting \( d = \infty \) is a risky business: there are groups which are amenable but have growth faster than any polynomial (e.g. \( BS(1, 2) \) from C.Pittet's lecture). It's an easy exercise to show \( \overline{\mathcal{F}}(x) = x^{(d-1)/d} \) is optimal for \( \mathbb{Z}^d \) (see exercises).

Actually one can easily show that groups with “slow” volume growth are amenable. Recall a group is of subexponential growth if \( \beta_{S}(n) := |B(n)| = e^{\ell(n)} \) for some (increasing) function \( f : \mathbb{N} \to \mathbb{R}_{>0} \) satisfying \( \lim_{n \to \infty} \frac{f(n)}{n} = 0 \).

1 without much preparation! These notes are in an incomplete state. Please email comments / corrections / improvements / references / insults / etc... to firstname.name@unine.ch
Proposition 1.4 (Hulanicki, 1971)²

Let $\Gamma$ be a [finitely generated] group of subexponential growth, then $\Gamma$ is amenable.

Proof: The number of edges in $\partial B_n$ is at most the number of vertices in the sphere of radius $n + 1$ times the maximal degree of a vertex, $|S|$:

$$\frac{\partial B_n}{B_n} \leq |S| \frac{B_{n+1} - B_n}{B_n} = |S|(e^{f(n+1)} - f(n)) - 1.$$

The claim is that there is some sequence of integers $\{n_k\}_{k \geq 0}$ so that $f(n_k + 1) - f(n_k)$ tends to 0. If this is the case, then $F_k = B_{n_k}$ would be a Følner sequence.

So assume there are no such sequences, or, in other words that $\exists \epsilon > 0$ such that $\forall n > 0, f(n+1) - f(n) > \epsilon$ (since $f(n)$ is increasing, absolute values are not required). This implies that

$$f(k) = f(0) + \sum_{i=1}^{k} (f(i) - f(i-1)) > k \epsilon.$$

This a contradiction with $\lim_{n \to \infty} \frac{f(n)}{n} = 0$. Hence the desired subsequence exists and $\Gamma$ is amenable.

Note that it is highly non-trivial to see whether the sequence of all balls works (and not just a subsequence). This turns out to be true (but it's difficult) in nilpotent groups. It is open in a group of intermediate growth (intermediate = superpolynomial and subexponential).

The converse of proposition 1.4 is not true: $BS(1, 2)$ is amenable but has exponential growth (see C. Pittet’s lecture).

Proposition 1.4 gives many groups without too much effort. The next theorem is useful to build groups out of known amenable groups.

Theorem 1.5 (“The closure properties”)³

Let $\Gamma$, $N$ and $\{\Gamma_i\}_{i \geq 0}$ be amenable groups.

(a) If $H$ is a subgroup⁴ of $\Gamma$ then $H$ is amenable “Subgroup”

(b) If $H$ is an extension $N$ by $\Gamma$ (i.e. $1 \to N \to H \to \Gamma \to 1$ is an exact sequence) then $H$ is amenable “Extension”

(c) If $N \vartriangleleft \Gamma$ then $H = \Gamma/N$ is amenable “Quotient”

(d) If $H$ is a direct limit of the $\Gamma_i$ then $H$ is amenable “Direct limit”

It’s perfectly possible to prove these properties from the current definition of amenability. It turns out to be much easier to use the most convenient of the many equivalent definition of amenability to do this.

But before moving to these considerations, it's nice to wander a bit.

²Conflicting references: Pier [see p.114 and p.117] attributes it to Hulanicki 1971. In the book by Greenleaf (1969), the proof of theorem 3.3.6 clearly contains a very similar argument. Some people told me they were told that it is in a paper in Russian... but I have no clear references, possibly Adels'son-Vel'skiĭ & Štreicher (1957 in Uspehi Mat. Nauk.)...

³(a)-(c) are in von Neumann's article [1929]. (d) is also in there for direct unions.

⁴When dealing with groups which are not with the discrete topology, one should require that the subgroups be closed. Otherwise, it is possible to have a free group as a subgroup of some compact groups.
Definition 1.6. A group is called elementarily amenable (short notation: EA) if it is obtained by (many) applications of the closure properties (a)-(d) starting from the following class of groups: finite groups and \( \mathbb{Z} \).

Day asked whether “amenable” = EA (“Day’s conjecture”). Here are two important facts about EA groups:

**Theorem 1.7 (Chou, 1980)**

If \( \Gamma \) is elementarily amenable, then \( \Gamma \) has either polynomial or exponential growth.

Since there are groups of intermediate growth [Grigorchuk, recall J. Brieussel’s talk], amenable \( \neq \) EA. In fact, it’s even known that throwing in all groups of subexponential growth in the “construction via closure properties” is not sufficient to get all amenable groups [Bartholdi-Virag].

**Theorem 1.8 (Chou, 1980)**

If \( \Gamma \) is elementarily amenable, then \( \Gamma \) can be obtained by using only (b) and (d).

There are finitely generated simple amenable groups [Juschenko-Monod, 2013]. To see that a [fin.gen.] simple amenable groups is not EA, one needs to know that the only necessary properties are (b) and (d). If \( \Gamma \) is simple, clearly the last operation was not an extension. But since \( \Gamma \) is finitely generated, the direct limits is unimportant: the generators live in a finite part of the limit. Thus one cannot get a simple group.

**Example 1.9.** Solvable groups are given by successive extensions of Abelian group, hence they are EA (and, in particular, amenable).

Mixing extensions and direct unions can be useful to create groups with “strange” properties. See “More examples” in §4.A (e.g. EA groups which are neither linear nor residually finite) or Reiter and Liouville §5.A (e.g. amenable groups with non-constant bounded harmonic functions) below.

Projective limits [a.k.a. inverse limits] of amenable groups are not amenable. I lack a convincing example at the moment.

### 2 Equivalent definitions and proof of “closure properties”

Much like hyperbolic groups, the great thing about amenable groups is that there are lots of ways to see them which turn out to be equivalent.

Although many of these equivalent definitions do not require the axiom of choice, it will be used without warning.

Some preliminary notations and terminology... A map (of a vector space into another vector space) is affine if it preserves mid-points (see §5.B), i.e. \( f(\frac{x+y}{2}) = \frac{1}{2}(f(x) + f(y)) \). Denote by \( 1_A \) the characteristic function of \( A \) and \( 1 := 1_\Gamma \). A function satisfies \( f \geq 0 \) if \( \forall x, f(x) \geq 0 \). The operator \( R_\mu \) is defined by right convolution by \( \mu \), i.e. \( R_\mu f = f \ast \mu \). The measure \( \mu^{(k)} \) is the \( k \)th iterated convolutions of \( \mu \). The function \( \delta_\gamma \) is the Dirac mass at \( \gamma \).

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5 A naive idea would be to look at the projective limits of \( \text{SL}_2(\mathbb{Z}_n) \). Indeed, since \( \mathbb{F}_2 \) embeds in \( \text{SL}_2(\mathbb{Z}) \) it will embed in there. However, the natural topology on such a projective limit makes it into a compact group (and not a discrete one)... and compact groups are always amenable.

6 Indeed, it is not possible to find an element in \( (\ell^\infty \Gamma)^* \setminus \ell^1 \Gamma \) without the axiom of choice. Of course, such an element exists only if \( \Gamma \) is infinite.
Theorem 2.1

Assume $\Gamma$ is countable (and discrete). The following are equivalent:

(i) There exists a Følner sequence, i.e. $\{F_n\}$ finite sets such that $\lim_{n \to \infty} |F_n \Delta F_1|/|F_1| = 0$.  
[Følner 1955]

(ii) If $\Gamma$ acts by affine transformations on $K$ a (non-empty) compact and convex set in a LCTVS\(^7\), then there is a fixed point.  
[Day 1961]

(iii) There is an invariant mean, i.e. a linear map $m : \ell^\infty\Gamma \to \mathbb{R}$ such that $m(1) = 1$, $m(f) \geq 0$ whenever $f \geq 0$ and $m(f * \delta_\gamma) = m(f)$.  
[von Neumann 1929]

(iv) For any $\mu$ finitely supported probability measure on $\Gamma$, $\|R_\mu\|_{\ell^1/\ell^2} = 1$, i.e. the probability of return decays subexponentially: $\mu(2^n)(e)^{1/2n} \to 1$.  
[Kesten 1959]

(v) There is a sequence $f_n \in \ell^\infty\Gamma$ of finitely supported elements so that $\|f_n\|_{\ell^2} = 1$ and, $\forall \gamma \in \Gamma$, $\|f_n * \delta_\gamma - f_n\|_{\ell^2} \to 0$.  
[Dixmier 1950?]

(vi) There is a Reiter sequence, i.e. a sequence $\xi_n$ of finitely supported probability measures such that, $\forall \gamma \in \Gamma$, $\|\xi_n * \delta_\gamma - \xi_n\|_{\ell^1} \to 0$.  
[Reiter 1964~6]

Definition (i) is saying that these groups have sublinear isoperimetry. Strangely (at first), this is equivalent to a very powerful fixed point property: (ii) (the case $\Gamma = \mathbb{Z}$ is the Kakutani-[Markov] theorem)\(^8\). (iii) is the original definition and, as shown in I. Chatterji’s lecture, is an obstruction to paradoxical decompositions ([Hausdorff]-Banach-Tarski paradox). Maps $m$ as in (ii) are called invariant means. (iv) bridges (surprisingly) these properties with those of the random walk.

(v), in the original language, is to say that the right-regular representation (the “natural” action of $\Gamma$ on $\ell^2\Gamma$) has almost fixed vectors\(^9\). (vi) is a variant of (v) with 2 replaced by 1; it remains actually true for any $p$. The importance of $p = 1$ is obvious for those who like Banach algebras (make $\ell^1\Gamma$ an algebra by taking convolution for the product)\(^10\).

Funnily, the fact that the Følner condition (i) implies amenability (the existence of an average) is already done in Ahlfors\(^11\). Følner does not refer to Ahlfors, but uses a different method (and his result is stronger; see section “More equivalences; combinatorial” in §4.B below) which goes through the “Dixmier condition” (to which he also does not refer; see section “More equivalent conditions; paradoxical”). The Dixmier condition is already fairly present in von Neumann’s original paper, which leaves a possibility that Følner was simply unaware of these works.

One could think that $\mu^{(n)}$ or its “lazy version” or $\frac{1}{N} \sum_{i=0}^{N} \mu^{(i)}$ would be good candidates for $\xi_n$ in (vi). It turns out this is related to bounded harmonic functions, see “Reiter and

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\(^7\) LCTVS = Locally convex topological vector space

\(^8\) The first statement of this I could locate is by Day (1961, corrected in 1964). The statement in its full generality is in Rickert 1967. Day refers to Kakutani 1938 and Markov 1936.

\(^9\) The official formulation of this is to say that the trivial representation is weakly contained in the left-regular representation. In fact, all irreducible representations are weakly contained in the regular representation (the Hulanicki-Reiter theorem).

\(^10\) (vi) $\Rightarrow$ (v) was known to Dieudonné (1960), and (v) $\Rightarrow$ (vi) is due to Reiter (1964). The full variations for all $p$ could be due to Stegeman (1965). The equivalence with amenability seems to be only noticed in Hulanicki (1966). In any case I could not find this statement in the 1960 paper of Dixmier...

Liouville” in §5.A below.

The proof of these equivalence can be tedious. In fact, it is much more difficult to do this for the general case of locally compact groups. So let’s start slow by looking at how easy the closure properties are if one picks the right definition of amenability.

Proof of (iv) $\implies$ (a): If $H$ is a subgroup of $\Gamma$ then a measure $\mu$ on $H$ is also a measure on $\Gamma$ (irreducibility is not required!). The random walk will be identical, so there is nothing else to say.

Proof of (ii) $\implies$ (b): Recall we have a quotient map $\pi: H \to \Gamma$ and $N = \ker \pi$. We wish to show $H$ is amenable. Take any affine action on a compact convex $K \neq \emptyset$ on which $H$ acts. Then there is an action by $N$. Since $N$ is amenable, the points fixed by $N$, $K^N = \{k \in K \mid \forall n \in N, k \cdot n = k\}$, is a non-empty subset. Because the action is affine, it is still convex. It’s also clearly closed in $K$, hence compact.

There is no way in general to define an action of $\Gamma$ on $K$, but it’s possible on $K^N$. Define, for $k \in K^N$, $k \cdot \gamma := k \cdot h$ where $h$ is some element in $\pi^{-1}(\gamma)$. This is well-defined since $h = nh_i$ for a fixed $h_i$ representative of the coset and some $n$: the choice of $n$ is irrelevant since $K^N$ is fixed by $N$.

Thus $K^N$ has a non-empty fixed point set by $H$, $(K^N)^\Gamma$. This is easily seen equal to $K^H$.

Proof of (iii) $\implies$ (c): Let $\pi: \Gamma \to H$ (and $N = \ker \pi$). Let $m_\Gamma$ be the invariant mean on $\Gamma$.\(^{12}\) Given a $f \in \ell^\infty H$ we wish to compute an invariant mean. Define $\pi_* f \in \ell^\infty \Gamma$ by $\pi_* f (\gamma) = f (\pi (\gamma))$. Put $m_H(f) := m_\Gamma (\pi_* f)$. To see this defines an invariant mean on $H$, observe that $\pi_* f \in (\ell^\infty \Gamma)^N$. As in the previous proof, $\pi_* (f * \delta_h) = \pi_* f * \delta_{\pi^{-1}(h)}$ is well-defined, since it does not depend on the actual element chosen in $\pi^{-1}(h)$. Hence, $m_H$ is an invariant mean on $H$.

Proof of (i) $\implies$ (d): For simplicity, restrict to direct unions, $H = \bigoplus_i \Gamma_i$. For direct limits, you can either modify this argument to direct limits or try a different approach in the exercises.

Let $F_n^{(i)}$ be Følner sequences for $\Gamma_i$. Try the following sequence for $H$: $F_n = \bigoplus_{i=0}^n F_n^{(i)}$. Given $\gamma \in H$, let $k$ be the index starting from which it is trivial, i.e. $\gamma_i = e_{\Gamma_i}$ for $i > k$. Leaving brutally many terms out of the inclusion-exclusion principle,

$$\frac{|F_n \gamma \triangle F_n|}{|F_n|} \leq \sum_{i=1}^k \frac{|F_n^{(i)} \gamma_i \triangle F_n^{(i)}|}{|F_n^{(i)}|} \to 0$$

\(^{12}\)The proof does not require that $N$ be amenable, although this is automatic thanks to (a).

2.1 Some details on (iv)

For the convenience of the reader (which might have not benefited from L. Saloff-Coste’s lectures), it might be good to recall why the two conditions in (iv) are equal. A very simple (but important) remark to begin with is that $\mu^{(n)}(e)^{1/n}$ need not converge. Indeed, if $\Gamma$ admits a Cayley graph which is bipartite and $\mu$ is supported on the generators for this Cayley graph, then $\mu^{(n)}(e)$ is alternatively 0 and $>0$. The basic example is the infinite line (the classical Cayley graph for $\mathbb{Z}$). In this example, one may compute hands on that $\mu^{(2k)}(e)^{1/2k}$ tends to 1 while the limit on odd numbers is clearly 0.
Note that, for any \(f, g \in \ell^\Gamma\),
\[
\langle \delta_\gamma * f \mid g \rangle = \sum_{\gamma \in \Gamma} f(\gamma^{-1})g(\gamma) \gamma^{-\gamma} \sum_{\eta \in \Gamma} f(\eta)g(x\eta) = \langle f \mid \delta_{\equiv^{-1}} * g \rangle
\]
Similarly, \(\langle f \mid \delta_\gamma * g \rangle = \langle f \mid g \delta_{\equiv^{-1}} \rangle\). This implies that for a symmetric \(\mu\),
\[
(\text{SA}) \quad \langle \mu \mid f \rangle = \langle f \mid \mu \rangle \quad \text{and} \quad \langle f \mid \mu \rangle = \langle f \mid g \mu \rangle.
\]
The second equality is exactly saying that \(R_\mu\) is self-adjoint. Let us explore some easy consequences. Recall \(R_\mu\) is the operator defined by \(R_\mu f = f \ast \mu\).

**Lemma 2.2**

For any \(\mu \in \ell^1 \Gamma\) symmetric, one has
\[
\begin{align*}
\cdot \quad & \mu^{(2n)}(e) = \| \mu^{(n)} \|^2_{\ell^2} = \| R_\mu^{n} \delta_\equiv \|^2_{\ell^2}; \\
\cdot \quad & \sup_{x \in \Gamma} \mu^{(2n)}(x) = \mu^{(2n)}(e); \\
\cdot \quad & \text{and } n \mapsto \mu^{(2n)}(e) \text{ is non-increasing.}
\end{align*}
\]

**Proof:** For the first one, just note that
\[
\mu^{(2n)}(e) = \langle \mu^{(2n)} \mid \delta_\equiv \rangle = \langle \mu^{(n)} \mid \mu^{(n)} \rangle = \| \mu^{(n)} \|^2_{\ell^2}.
\]
Then remark that
\[
\| R_\mu^{n} \delta_\equiv \|^2_{\ell^2} = \langle \delta_\equiv * \mu^{(n)} \mid \delta_\equiv * \mu^{(n)} \rangle = \langle \delta_{\equiv^{-1}} * \delta_\equiv * \mu^{(n)} \mid \mu^{(n)} \rangle = \langle \mu^{(n)} \mid \mu^{(n)} \rangle.
\]
The second comes from a slight variation:
\[
\mu^{(2n)}(x) = \langle \mu^{(2n)} \mid \delta_\equiv \rangle = \langle \mu^{(n)} \mid \delta_\equiv * \mu^{(n)} \rangle = \langle \mu^{(n)} \mid \mu^{(n)} \rangle \leq \| \mu^{(n)} \|_{\ell^2} \| \delta_\equiv * \mu^{(n)} \|_{\ell^2} = \| \mu^{(n)} \|^2_{\ell^2} = \mu^{(2n)}(e).
\]
The last one can be obtained as a corollary:
\[
\mu^{(2n+2)}(e) = \sum_x \mu^{(2n)}(x) \mu^{(2)}(x^{-1}) \leq \mu^{(2n)}(e) \sum_x \mu^{(2)}(x^{-1}) = \mu^{(2n)}(e). \quad \blacksquare
\]

**Lemma 2.3**

For any \(\mu \in \ell^1 \Gamma\) symmetric, the operator norm of \(R_\mu : \ell^2 \to \ell^2\) is \(\leq 1\).

**Proof:** Indeed, by invoking Young’s inequality\(^{13}\): \(\| R_\mu f \|_{\ell^2} = \| f \ast \mu \|_{\ell^2} \leq \| f \|_{\ell^2} \| \mu \|_{\ell^2}\). If you don’t like Young’s inequality, a direct computation (where C-S stands for the Cauchy-Schwarz inequality) also works. First, note that
\[
\| R_\mu f \|_{\ell^2}^2 = \sum_s |f(xs)\mu(s)|^2 \leq \left( \sum_s |f(xs)|^2 \mu(s) \right) \left( \sum_s \mu(s) \right) = \sum_s |f(xs)|^2 \mu(s).
\]
Then by rearranging:
\[
\| R_\mu f \|_{\ell^2}^2 \leq \sum_s \sum_s |f(xs)|^2 \mu(s) = \sum_y |f(y)|^2 \sum_t \mu(t^{-1}) = \sum_y |f(y)|^2 = \| f \|_{\ell^2}^2. \quad \blacksquare
\]

\(^{13}\)In fact, this shows that \(R_\mu\) has norm \(\leq 1\) as an operator from \(\ell^p\) to \(\ell^p\). To avoid this inequality for \(p \neq 2\), use Hölder instead of Cauchy-Schwarz.
Another easy consequence of (SA) is that, for any $f \in \ell^2$, the map $n \mapsto \frac{\|R_n^{n+1}f\|_{\ell^2}}{\|R_n^n f\|_{\ell^2}}$ is increasing:

$$\|R_n^n f\|_{\ell^2} = \langle f * \mu^{(n)} | f * \mu^{(n)} \rangle = \langle f * \mu^{(n-1)} | f * \mu^{(n+1)} \rangle \quad \text{for } n \geq 0,$$

Hence the limit $\rho_f := \lim_{n \to \infty} \frac{\|R_n^{n+1}f\|_{\ell^2}}{\|R_n^n f\|_{\ell^2}}$ exists (these ratios form an increasing sequence bounded by the operator norm of $R_\mu$, which is 1).

A bound on all the $\rho_f$ would be a bound on the operator norm: as the ratio is increasing,

$$\|R_\mu^1 f\|_{\ell^2} \leq \|f\|_{\ell^2} \leq \frac{\|R_\mu^2 f\|_{\ell^2}}{\|R_\mu^1 f\|_{\ell^2}} \leq \cdots \leq \frac{\|R_\mu^{n+1} f\|_{\ell^2}}{\|R_\mu^n f\|_{\ell^2}} \leq \|f\|_{\ell^2} \rho_f.$$

Remember that if a ratio test converges, then the root test also converges to the same value (but it is possible that the root test converges and not the ratio test). Hence,

$$\rho_f = \lim_{n \to \infty} \frac{\|R_\mu^n f\|_{\ell^2}}{\|f\|_{\ell^2}} = \rho_f^{1/n}.$$

Note that the ratio is easy to compute for any Dirac mass:

$$\|R_\mu^n \delta_e\|_{\ell^2} = \frac{\|\mu^{(n+1)}\|_{\ell^2}}{\|\mu^{(n)}\|_{\ell^2}} = \left(\frac{\mu^{(2n+2)}(e)}{\mu^{(2n)}(e)}\right) = \left(\frac{\mu^{(2n+2)}(e)}{\mu^{(2n)}(e)}\right).$$

Here is a corollary:

**Lemma 2.4**

The function $n \mapsto \frac{\mu^{(2n+2)}(e)}{\mu^{(2n)}(e)}$ is increasing; consequently, so is $n \mapsto \mu^{(2n)}(e)^{1/2n}$.

Turning back to our problem, $\rho_{\delta_e}$ is the same as $\mu^{(2n)}(e)^{1/2n}$.

For $f = \sum a_\gamma \delta_e$ of finite support, note that

$$\|R_\mu^n f\|_{\ell^2}^{1/n, \text{T.I.}} \leq \left(\sum |a_\gamma| \cdot \|R_\mu^n \delta_e\|_{\ell^2}\right)^{1/n} = \|f\|_{\ell^1, \text{T.I.}} \mu^{(2n)}(e)^{1/2n},$$

where “T.I.” stands for the triangle inequality. So that $\rho_f \leq \lim_{n \to \infty} \mu^{(2n)}(e)^{1/2n}$. From this, one actually sees

**Lemma 2.5**

$$\|R_\mu^n f\|_{\ell^2} \to \|f\|_{\ell^1} = \mu^{(2n)}(e)^{1/2}.$$

**Proof:** The inequality $\leq$ was just proved (and extends to $\ell^1$ by density of finitely supported functions) and the inequality $\geq$ is obtained by looking at a Dirac mass (or in fact, any $f = \sum a_\gamma \delta_e$ with $a_\gamma \geq 0$).

**Lemma 2.6**

$$\lim_{n \to \infty} \mu^{(2n)}(e)^{1/2n} = \mu^{(2n)}(e)^{1/2n}.$$

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If you want to see directly in this case that the root test converges, note that $\mu^{(m+n)}(e) \geq \mu^{(m)}(e)\mu^{(n)}(e)$. This means that $F(n) = -\ln \mu^{(2n)}(e)$ is a sub-additive function. Invoke Fekete’s lemma to conclude.
Proof: Let \( \bar{\rho} = \lim_{n \to \infty} \mu(2n)\epsilon^{1/2n} \).

Until now we have shown that, for any function of finite support, one has \( \rho_f \leq \bar{\rho} \). Using (R), one also gets that for such functions \( \|R_f \|_{\ell^2} \leq \bar{\rho}\|f\|_{\ell^2} \). The result then follows by density of finitely supported functions. Let us nevertheless put it in full details as approximating elements of the spectrum by finitely supported functions will come up again later.

Suppose there is a \( f \in \ell^2 K \) with \( \|R_f \|_{\ell^2} = \rho\|f\|_{\ell^2} \) for some \( \rho > 0 \) and some \( \epsilon > 0 \). WLOG, \( \|f\|_{\ell^2} = 1 \). Then, for any \( \epsilon \in [0,1] \), one may write \( f = h + g \) where \( g \) and \( h \) have disjoint support, \( h \) is finitely supported, \( ||g||_{\ell^2} \leq \epsilon \) and \( 1 - \epsilon \leq ||h||_{\ell^2} \leq 1 \).

On one hand,
\[
\|R_\mu f\|_{\ell^2} = \|R_\mu(h + g)\|_{\ell^2} \leq \|R_\mu h\|_{\ell^2} + \|R_\mu g\|_{\ell^2} \leq \|R_\mu h\|_{\ell^2} + ||g||_{\ell^2} \leq \|R_\mu h\|_{\ell^2} + \epsilon
\]
while, on the other,
\[
\|R_\mu f\|_{\ell^2} = \rho\|f\|_{\ell^2} = \rho\sqrt{\|h\|_{\ell^2}^2 + ||g||_{\ell^2}^2} \geq \rho\|h\|_{\ell^2}
\]
Putting this together one has
\[
\|R_\mu h\|_{\ell^2} \geq \rho\|h\|_{\ell^2} - \epsilon \geq (\rho - \frac{\epsilon}{1 - \epsilon})||h||_{\ell^2}.
\]
Since one has \( \|R_\mu h\|_{\ell^2} \leq \bar{\rho}\|h\|_{\ell^2} \), for any finitely supported \( h \), one gets that \( \rho - \frac{\epsilon}{1 - \epsilon} \leq \bar{\rho} \).

Since \( \epsilon \) is arbitrarily small, this shows \( \|R_\mu f\|_{\ell^2} \leq \bar{\rho}\|f\|_{\ell^2} \).

For the reverse inequality, simply recall that \( \|R_\mu \mu^{(n)}\|_{\ell^2} \) tends to this limit. \( \square \)

Remark 2.7. The above applies more generally to bounded self-adjoint operators \( R \) on \( \ell^2 K \). Namely, the operator norm of \( R \) is equal to \( \sup_{x \in K} ||R^\delta x||_{\ell^2}^{1/n} \).
\( \spadesuit \)

3 Proof of equivalences

Most of these proofs are significantly more technical in the general locally compact case. Here is the pattern that will be followed here:

\[
(i) \iff (iv) \implies (i) \implies (ii) \implies (v) \implies (vi) \iff (iii)
\]

Proof of (i) \( \implies \) (ii): This proof is very intuitive. We are given a Følner sequence \( F_n \) for \( \Gamma \) and an action \( \Gamma K \). The aim is to find a fixed point in \( K \).

Pick any \( v \in K \) and define \( v_n = \frac{1}{|F_n|} \sum_{x \in F_n} v \cdot x \). This sequence is infinite so it has accumulation points in \( K \) (by compactness). It remains to check these accumulation points are fixed by \( \Gamma \). Notice that \( |F_n \gamma \setminus F_n| = |F_n \setminus F_n \gamma| = \frac{1}{2} |F_n \gamma \triangle F_n| \)

Compute
\[
v_n \cdot \gamma - v_n = \frac{1}{|F_n|} \left( \sum_{x \in F_n \gamma \setminus F_n} v \cdot x - \sum_{x \in F_n \setminus F_n \gamma} v \cdot x \right)
= \frac{1}{2|F_n|} \left( |F_n \gamma \setminus F_n| \sum_{x \in F_n \gamma \setminus F_n} v \cdot x - \frac{1}{|F_n \setminus F_n \gamma|} \sum_{x \in F_n \gamma \setminus F_n} v \cdot x \right)
\]

\( ^{15}g \) is just \( f \) restricted to the complement of some large enough finite set.
This seems nasty but notice that the two sums in parenthesis are actually convex combinations. Since $K$ is convex,

$$v_n \cdot \gamma - v_n \in \frac{|F_n \gamma \Delta F_n|}{2|F_n|} (K - K).$$

Since $K$ is compact, it's quite intuitive this tends to 0. To see it, here are the required facts about LCTVS: 1- the topology is given by a family of semi-norms; 2- a sequence tends to 0 if and only if it tends in to 0 in any semi-norm compatible with the topology. Since $K$ is compact, it is bounded in any semi-norm. Thus $v_n \cdot \gamma - v_n$ tends to 0 in any semi-norm (it's a scalar tending to 0 times a bounded quantity).

This insures invariance of accumulation points and shows the existence of a fixed point in $K$. □

Before moving on, some background on means is required.

**Definition 3.1.** A mean is a linear map $\mu : \ell^\infty \Gamma \to \mathbb{R}$ such that $\mu(1) = 1$ and $\mu(f) \geq 0$ whenever $f \geq 0$.

Denote by $M$ the set of means. Let $\mathcal{P}_r(\Gamma)$ be the set of finitely supported probability measures on $\Gamma$. Then $\mathcal{P}_r(\Gamma) \subset M$. Here is a small lemma which will be useful

**Lemma 3.2**

Means are elements of $(\ell^\infty \Gamma)^*$ of norm 1. Hence $M$ is a convex set which is bounded in norm (hence compact in the weak* topology). Furthermore, $\mathcal{P}_r(\Gamma)$ is weak* dense in $M$.

**Proof of the Lemma:** To show a mean belongs to $(\ell^\infty \Gamma)^*$ it suffices to show it is bounded. But, $\forall f \in \ell^\infty \Gamma$,

$$-\|f\|_{\ell^\infty} \leq f \leq \|f\|_{\ell^\infty}.$$

Using positivity and linearity, $m(\|f\|_{\ell^\infty} + f) = \|f\|_{\ell^\infty} + m(f) \geq 0$. Similarly, $m(f) - \|f\|_{\ell^\infty} \geq 0$. Thus $|m(f)| \leq \|f\|_{\ell^\infty}$. This implies $\|m\|_{\ell^\infty \rightarrow \mathbb{R}} \leq 1$. Since $m(1) = 1$, we actually have equality.

For convexity, it's quite easy to check that the conditions are stable under convex combinations.

Assume there is an $m \in M \setminus \mathcal{P}_r(\Gamma)^*$. By Hahn-Banach, this means $m$ can be separated from $\mathcal{P}_r(\Gamma)$ by a careful evaluation at some $f : \exists f \in \ell^\infty \Gamma$ such that $m(f) \neq 0$ but $\mu(f) = 0$ for all $\mu \in \mathcal{P}_r(\Gamma)$. But if $m(f) \neq 0$ then there must be a $x \in \Gamma$ for which $f(x) \neq 0$ (otherwise $\|f\|_{\ell^\infty} = 0$, and by continuity $m(f) = 0$). Pick $\delta_x \in \mathcal{P}_r(\Gamma)$, then $\delta_x(f) \neq 0$. This shows that $\mathcal{P}_r(\Gamma)$ is weak* dense in $M$. □

**Proof of (ii) $\implies$ (iii):** By the lemma $M$ is a convex set which is compact in the weak* topology. It sits in $(\ell^\infty \Gamma)^*$ a LCTVS. There is a natural action of $\Gamma$ on $M$: $(m \cdot \gamma)(f) = m(f \ast \delta_\gamma)$.16 This action is the action dual to the action on $\ell^\infty \Gamma$.

To conclude, observe that a fixed point by this action is exactly an invariant mean. Since the action is affine17, the conclusion follows. ■

---

16 This is just a "change of variable" in the integration. If $\mu \in \mathcal{P}_r(\Gamma)$, $\mu(f) = \int f(x) \phi(x)$. Then act as usual on $\mathcal{P}_r(\Gamma) \subset \ell^1 \Gamma$ by $\mu \cdot \gamma = \mu * \delta_\gamma$. A change of variable gives $(\mu * \delta_\gamma)(f) = \mu(f \ast \delta_\gamma^{-1})$.

17 One does not need to write the action down for this. Indeed the Mazur-Ulam theorem insures an isometric action is affine.
Proof of (iii) $\Rightarrow$ (vi): We are given an invariant mean $m$. By the lemma, $\mathcal{P}_c(\Gamma)$ is weak* dense in $M$. This means that there is a sequence $\mu_i \in \mathcal{P}_c(\Gamma)$ with $\mu_i \xrightarrow{w^*} m$. It remains to show they are Reiter. For any $f \in \ell^\infty \Gamma$ and any $\gamma$,

$$\mu_i * \delta_{\gamma^{-1}}(f) - \mu_i(f) = \mu_i(f * \delta_{\gamma}) - \mu_i(f) \to m(f * \delta_{\gamma}) - m(f) = 0.$$ 

So at least $\mu_i \to m \xrightarrow{w^*} 0$.

To improve this to a norm convergence, fix some finite set $F \subset \Gamma$ and consider $X = \oplus_{g \in F} \ell^1 \Gamma$ with the norm $\|\{(\xi_g)\} \|_X = \max_{g \in F} \|\xi_g\|_{\ell^1}$. The set

$$K = \{\oplus_{g \in F} (\xi * \delta_g - \xi) \mid \xi \in \mathcal{P}_c(\Gamma)\}$$

is (again) a convex set. By the previous argument, it contains $0$ in its weak* closure. In terms of separation by evaluation (Hahn-Banach), we actually get $0$ in the weak convergence (since we are looking at elements in $\ell^1$ and $0 \in \ell^1$). A wonderful corollary of Hahn-Banach\footnote{This consequence is sometimes referred to as Mazur's theorem or the Mazur trick.} comes to the rescue: the closure of a convex sets in the weak topology is the same as the closure in the norm topology\footnote{Actually, in $\ell^1$ weak and norm convergence coincide, see Lemma 5.1. But this argument passes more easily to non-discrete groups.}. This yields a sequence $\xi_{i,F}$ so that $\max_{g \in F} \|\xi_{i,F} * \delta_g - \xi_{i,F}\|_{\ell^1} \to 0$. Taking a well-chosen subsequence $\xi_{i_k,F_k}$ will give the desired Reiter sequence.

Proof of (vi) $\Rightarrow$ (i): The idea is to find Følner sets in the level sets of $\xi_n$. Define for $\mu \in \mathcal{P}_c(\Gamma)$, $F(\mu,t) = \{x \in \Gamma \mid \mu(x) > t\}$. Then for a fixed $x$

$$|\xi(x) - \mu(x)| = \int_0^1 |1_{F(\mu,t)}(x) - 1_{F(\mu,t)}(x)|d\xi.$$

Summing over $x$ one gets,

$$|\xi_n * \delta_\gamma - \xi_n|_{\ell^1} = \int_0^1 |F(\xi_n * \delta_\gamma, t) \Delta F(\xi_n, t)|d\xi.$$

Assume, there is no way to find a Følner sequence in $F(\xi_n, t)$ (as $t$ and $n$ vary). This implies that there is some $\epsilon > 0$ such that $|F(\xi_n * \delta_\gamma, t) \Delta F(\xi_n, t)| \geq \epsilon |F(\xi_n, t)|$. But then

$$|\xi_n * \delta_\gamma - \xi_n|_{\ell^1} \geq \epsilon \int_0^1 |F(\xi_n, t)|d\xi = \epsilon |\xi_n|_{\ell^1} = \epsilon.$$

This contradicts the fact that $\xi_n$ is a Reiter sequence.
Proof of (i) \(\implies\) (iv): Take \(f_n = 1_{F_n}/\sqrt{|F_n|}\) (the \(\ell^2\)-normalised characteristic function of \(F_n\)). Suppose the measure \(\mu\) is supported on \(U \subset \Gamma\) (finite). Then a direct computation gives
\[
\|R_\mu f_n\|_2 \geq \frac{|\delta^{-U} F_n|}{|F_n|} \to 1,
\]
where \(\delta^{-U} F_n = \{\gamma \in F_n \mid \gamma U \subset F_n\}\). (To see why this should converge to 1, prove lemma 1.3) By definition, of the operator norm, one has \(\|R_\mu\| = 1\).

Note that one can also prove (i) \(\implies\) (v) and (i) \(\implies\) (vi) by the same argument. In fact, one can get elements of \(\ell^p\) which are almost invariant in \(\ell^p\)-norm by taking the \(\ell^p\)-normalised characteristic functions of the \(F_n\).

Proof of (iv) \(\implies\) (v): Recall that \(R_\mu\) is contracting, i.e. its operator norm is \(\leq 1\) (see §2.1). Next, \(R_\mu\) has no eigenvector of eigenvalue 1, i.e. no \(f \in \ell^2\Gamma\) such that \(f \ast \mu = f\). Indeed, such a function would be harmonic and, hence, satisfy a maximum principle. But since \(f \in \ell^2\Gamma\) it must decrease to 0 at infinity. This implies \(f\) must be 0 everywhere.

If you are not familiar with harmonic functions, just think of it as follows. Pick some element \(x\). The equation \(f = f \ast \mu\) tells you you can express the value at \(x\) in terms of a convex combination of values taken on the set \(S = \text{Supp} \mu\). Now repeat this for each element of \(S\). This might make \(f(x)\) pop-up again but you can cancel it using the first step. Repeating this process over and over again, you will have an expression of \(f(x)\) in terms of a convex combination of values of \(f\) which are as far away as you like. However, these need to tend to 0 because \(f \in \ell^2\Gamma\). Hence, at any point the value is 0.

Repeating the same argument with \(\frac{1}{2}(\text{Id} + \mu)\) instead of \(\mu\), one sees that 1 is in the spectrum\(^{20}\) (since \(\|R_\mu\|_{\ell^2 \to \ell^2} = 1\)) but there are no such eigenvectors. This means there are almost eigenvectors: for any \(\epsilon > 0\) there is a \(f\) of norm 1 which decomposition \(^{21}\) is made of eigenvectors of eigenvalue \(\geq 1 - \epsilon\), i.e. \(\langle R_\mu f_n | f_n \rangle \geq 1 - \epsilon\). One may also assume this \(f\) is finitely supported by cutting off very far away.

Thus
\[
\sum_x (1 - \langle f \ast \delta_x | f \rangle) \mu(x) = 1 - \langle R_\mu f | f \rangle \leq \epsilon
\]
But for any \(x\), \(\|f \ast \delta_x\|_{\ell^2} = \|f\|_{\ell^2} = 1\) so \((1 - \langle f \ast \delta_x | f \rangle) \geq 0\). This implies that, for any \(x\) in the support of \(\mu\), \(1 - \langle f \ast \delta_x | f \rangle \leq \epsilon\). Hence, for all \(x\) in the support of \(\mu\),
\[
\frac{1}{2} \|f \ast \delta_x - f\|_{\ell^2}^2 = 1 - \langle f \ast \delta_x | f \rangle \leq \epsilon
\]
To conclude, let \(F_k\) be an increasing sequence of finite sets so that \(\bigcup F_k = \Gamma\). Define \(f_k\) by looking at \(\mu_k\) with uniform support on \(F_k\) and then take \(\epsilon = 1/|F_k|k\). This gives a sequence \(f_k\) which satisfies \(\|f \ast \delta_x - f\|_{\ell^2} \leq 2/k\) for any \(x \in F_k\) and concludes the proof.

Proof of (v) \(\implies\) (vi): Take \(\xi_n = f_n^2\). Then
\[
\|\xi_n \ast \delta_\gamma - \xi_n\|_{\ell^1} = \sum_{x \in \Gamma} |f_n(x\gamma)^2 - f_n(x)^2| = \sum_{x \in \Gamma} |f_n(x\gamma) - f_n(x)| \cdot |f_n(x\gamma) + f_n(x)| \leq \sum_{x \in \Gamma} |f_n(x\gamma) - f_n(x)| \cdot (|f_n(x\gamma)| + |f_n(x)|).
\]

\(^{20}\)With more efforts, one can show contractions on \(\ell^2\) always have vectors which are “almost” of eigenvalue 1.

\(^{21}\)This can, of course, be said more rigorously with spectral decomposition of self-adjoint operators.
Split in two sums and use Cauchy-Schwarz to get
\[ \| \xi_n * \delta - \xi_n \| \leq \| f_n * \delta - f_n \| (\| f_n \| + \| f_n * \delta \| ) = 2 \| f_n * \delta - f_n \|. \]
This tends to 0 as \( n \to \infty \).

4 More...

This section gathers up many things which would have been nice to say if I would have had the time and energy to do it.

4.A ... examples

Here is a classical example:

Example 4.1 (Lamplighter group). \( \mathbb{Z} \) and \( \mathbb{Z}_2 \) are amenable. So is \( L = \oplus_{g \in \mathbb{Z}} \mathbb{Z}_2 \). Elements of \( L \) can be seen as finitely supported functions from \( \mathbb{Z} \) to \( \mathbb{Z}_2 \). Consequently there is a "natural" action by \( \mathbb{Z} \) on these functions by shifting. This turns out to be an automorphism of \( L \). Let \( \Gamma = L \rtimes \mathbb{Z} \) be the semi-direct product obtained from this action:
\[(\{a_g\}_{g \in \mathbb{Z}}, k) \cdot (\{b_g\}_{g \in \mathbb{Z}}, \ell) = (\{a_g + b_{g+k}\}_{g \in \mathbb{Z}}, k + \ell)\]
It's easier to describe this in words. The \( \mathbb{Z} \) coordinate describes where the lamplighter is. The first coordinate describes which lamp are on or off. If one multiplies (on the right) by \( 1 \in \mathbb{Z} \), the lamplighter moves. If one multiplies by \( \delta_0 \) (the function taking value \( 0 \in \mathbb{Z}_2 \) everywhere but at \( 0 \in \mathbb{Z} \) where it takes value \( 1 \in \mathbb{Z}_2 \)), then the lamp at the current position of the lamplighter is turned on or off.

Through this description, one sees that the group is actually finitely generated, even though it is made of an infinitely generated one (and a finitely generated one). It's not too hard to check this group has exponential growth as well. A Følner sequence can be explicitly given by taking elements \( (\xi, k) \) so that \( \xi \) (seen as a function from \( \mathbb{Z} \to \mathbb{Z}_2 \)) is supported on \( [1, n] \) and \( k \) belongs to \( [1, n] \).

It's an exercise to show these groups are of exponential growth.

If one starts from finitely generated Abelian groups and uses only extensions, the class obtained is called polycyclic groups. These groups are much more nicely behaved than a "generic" solvable group.

There are many other groups of this type, if one changes \( \mathbb{Z}_2 \) by some group \( H \) and \( \mathbb{Z} \) by \( G \), the notation for the resulting group is \( H \wr G \) (this notation is not unanimous, it's sometimes \( G \wr H \)). The following nice example was mentioned to me by Maxime Gheysens and Adrien Le Boudec. It's an EA group which is not virtually solvable and not linear.

Example 4.2. Take \( A_5 \) (the alternating group on 5 elements). Let \( \Gamma = A_5 \wr \mathbb{Z} \). \( \Gamma \) is EA but it is not linear (i.e. is not isomorphic to a subgroup of some matrix group). To see this the Tits alternative is required: a linear group either contains a free group or is virtually solvable (i.e. has a solvable subgroup of finite index).

Assume \( \Gamma \) is linear. Since it is amenable it cannot contain a free group. Thus it should be virtually solvable. It takes some playing around to see that subgroups of \( \Gamma \) are either finite, isomorphic to \( \mathbb{Z} \) or isomorphic to \( B \wr \mathbb{Z} \) for some subgroup \( B \) of \( A_5 \). In all cases, there are not
of finite index (unless they are isomorphic to $\Gamma$). So $\Gamma$ should be solvable. But a computation of its commutator subgroup leaves $\oplus ZA_5$. This group is stable under further commutators, so $\Gamma$ is not solvable.

The group is however residually finite.

Another interesting example is the following (it is neither linear, residually finite nor solvable).

**Example 4.3.** Take the group of all finitely supported permutation of the integers, $S_\infty$. It is a direct limit of $S_{2n+1}$ (all permutations supported on $[-n,n]$), hence amenable (but not finitely generated). To make it into a finitely generated group, add the shift: the infinite cyclic permutation which sends $k$ to $k+1$. It fairly easy to see the resulting group is an extension of $S_\infty$ by $Z$: $1 \to S_\infty \to \Gamma \to Z \to 1$. [In fact, it’s a semi-direct product.] So $\Gamma$ is EA.

$\Gamma$ is finitely generated: take one transposition $(12)$, and, upon conjugation by the shift, one may get any other transposition. [It’s easy to believe that $\Gamma$ is not finitely presented.]

It’s also fairly easy to check that $\Gamma$ is neither virtually-solvable nor residually finite. The same argument as in the previous example shows it is not linear.

4.B ... equivalences; combinatorial

Here are a few less-known equivalent conditions to amenability. The third is (surprisingly) due to Følner (in the same 1955) paper.

**Theorem 4.4**

The following are equivalent:

(i) There exists a sequence $\{F_n\}$ of finite sets such that $\lim_{n \to \infty} \frac{|F_n \Delta F_n|}{|F_n|} = 0$. [Følner 1955]

(i-2) There exists $c \in [0, 2]$ and a sequence $\{F_n\}$ such that $\forall \gamma \in \Gamma$, $\limsup_{n \to \infty} \frac{|F_n \Delta F_n|}{|F_n|} \leq c$. [Nagnibeda-Juschenko 2013]

(i-3) There exists $c \in [0, 2]$ such that $\forall \mu \in P_r(\Gamma)$, $\exists F \subset G$ finite with $\sum_{\gamma} \mu(\gamma) \frac{|F \Delta F|}{|F|} \leq c$. [Følner 1955]

(i-4) There exists $c \in [0, 2]$ such that $S \subset G$ finite symmetric generating, $\exists F \subset G$ finite with $\sum_{\gamma \in S} \frac{|F \Delta F|}{|S||F|} \leq c$. [Thom 2013]

(i-3) has been somehow largely forgotten (which explains the strange chronology). It actually suffices to consider $\mu$ with rational values. It’s easy to do (i) $\implies$ (i-2) $\implies$ (i-3) $\implies$ (i-4). However, it seems at first difficult to believe that (i-4) or even (i-2) imply (i). To show (i-3) $\implies$ (i), we will follow Følner’s original argument\(^{22}\). He uses a preliminary result:

\(^{22}\)In a modern perspective, the underlying idea in this proof is to show that $\inf_{F} \|\nu - \mu_{i} \ast \nu\|_{E} = 0$ for any finite number of $\mu_{i} \in P_{r}(\Gamma)$. From there, one can use the arguments in (iii) $\implies$ (vi).
Lemma 4.5 (Dixmier 1950\textsuperscript{23})

\(\Gamma\) is amenable if and only if \(\sup_x H(x) \geq 0\) for any \(H\) of the form

\[H(x) = \sum_{i=1}^{n} h_i(x) - h_i(x\gamma_i)\]

where \(h_i \in \ell^\infty \Gamma\) (for \(i = 1, \ldots, n\)) and \(\gamma_i \in \Gamma\)

\textbf{Proof:} If there exists a function \(H\) which is written as in (D) and \(\sup_x H(x) \leq c\) for some \(c < 0\), then there can be no invariant mean: \(m(H) = 0\) but by positivities \(m(H) \leq c < 0\).

On the other hand, if \(H\) satisfies (D) implies \(\sup_x H(x) \geq 0\), one can define “explicitly” an invariant mean as follows. Consider \(m(f) = \inf_H \sup_x (f(x) + H(x))\). Then \(m(\lambda f) = \lambda m(f)\) (for \(\lambda \geq 0\)) and \(m(f + g) \leq m(f) + m(g)\). Furthermore, \(m\) is invariant (\(m(f)\) is clearly 0 if \(f = g - g \star \delta\)). Thus, by Hahn-Banach’s extension, there exists a linear functional (which will still be invariant) such that \(m(f) \leq m(f)\).

The function \(m\) is called upper mean value. The idea stems from a simpler upper mean defined (on Abelian groups) via \(m(f) = \inf_{\mu \in \mathcal{M}(\Gamma)} \sup_x f \star \mu(x)\). It is worth mentioning that Banach proved the Hahn-Banach extension theorem to show that \(m\) can be turned into a linear map \(m\) (i.e. he proved Abelian groups are amenable). There are recent work (by Cannon, Floyd and Parry) in which functions of the form \(H\) are reinterpreted as cooling functions.

\textbf{Proof of (i-3) \(\implies\) (iii):} Introduce

\((D_e)\) \(\exists H\) such that \(H = \sum_{i=1}^{n} h_i - h_i \star \delta\), \(2n\max_i \|h_i\|_{\ell^\infty} \leq 1\) and \(\sup_x H(x) \leq -\epsilon\).

and

\((l_k)\) \(\forall \mu, \exists F\) such that \(\sum_{\gamma \in \Gamma} \mu(\gamma) \frac{|F\gamma \cap F|}{|F|} \geq k\).

The proof goes by contradiction: \((D_e)\) holds (for some \(\epsilon > 0\)) even though \((l_k)\) is true. So, take \(H_0\) as in \((D_e)\). Let \(F\) associated to the measure \(\mu = \frac{1}{n} \sum_{i=1}^{n} \delta\). Define \(H_1\) as follows:

\[H_1 = \frac{1}{|F|} \sum_{i=1}^{n} h_i \star 1_F - h_i \star 1_F \star \delta\]

This sum is apparently a sum on \(n|F|\) terms of functions with \(\|h\|_{\ell^\infty} \leq 1/|F|\), and \(\sup_x H_1(x) \leq -\epsilon\). However, by the choice of \(F\), many terms in the sum cancel out: there are at most \((1 - k)n|F|\) in the sum. This means, \(\|H_1\|_{\ell^\infty} \leq (1 - k)\). This implies \(-(1 - k) \leq -\epsilon\), i.e. \(\epsilon \leq 1 - k\).

But one can iterate this process: define \(H_2\) from \(H_1\) as \(H_1\) was defined from \(H_0\). This implies \(\epsilon \leq (1 - k)^2\). Constructing a sequence \(H_m\) in a similar fashion implies \(\epsilon \leq (1 - k)^m\). This contradicts \((D_e)\) (for any \(\epsilon > 0\)). By Lemma 4.5, there is an invariant mean. \(\blacksquare\)

\textsuperscript{23}Følner seems to have been unaware of the result of Dixmier and proved this result in a previous paper in 1954. Both were very probably inspired by von Neumann 1929 (Hilfsatz 2, p.90 in §1 ¶1). Greenleaf calls it the “Dixmier condition".
For the record, here is a proof of (i-2) $\implies$ (v). It requires some familiarity with ultrapowers.

**Proof of (i-2) $\implies$ (v):** Let $\xi_i$ be the $\ell^2$-normalised characteristic function of the sets $F_n$, i.e. $\xi_n = \mathbb{1}_{F_n}/\sqrt{|F_n|}$. So by assumption:

$$
\langle \lambda_\gamma \xi_i - \xi_i | \lambda_\gamma \xi_i - \xi_i \rangle = \|\lambda_\gamma \xi_i - \xi_i\|^2 = \frac{|\gamma_{F_i} \triangle F_i|}{|F_i|} \leq c
$$

Using this equality and $\langle \xi_i | \xi_i \rangle = \|\xi_i\|^2 = 1$, one gets

$$
\langle \lambda_\gamma \xi_i - \xi_i | \lambda_\gamma \xi_i - \xi_i \rangle = 2 - 2\langle \lambda_\gamma \xi_i | \xi_i \rangle.
$$

For $i$ big enough, this implies that

$$
0 < 2 - c \leq \langle \xi_i | \lambda_\gamma \xi_i \rangle
$$

Consider $(\xi_i)_{i \in \mathbb{N}}$ as an element in some ultrapower $\ell^2(\Gamma)^\omega$ of $\ell^2(\Gamma)$. Then the above inequality implies that any translate of this element is separated by a hyperplane away from $0 \in \ell^2(\Gamma)^\omega$. Thus, if $X$ is the closure of the convex hull of the $\Gamma$-orbit of $(\xi_i)_{i \in \mathbb{N}}$, then $X$ is also separated away from 0 by this hyperplane. Since the space is uniformly convex, there is a closest point projection on closed convex sets. Thus the nearest point to 0 in $X$ is a fixed point under the action of $\Gamma$. This means that the left regular representation in $\ell^2(\Gamma)$ admits almost fixed points and implies the equivalent condition (v). \hfill \Box

### 4.C ... equivalences; paradoxical

**Definition 4.6.** A group $\Gamma$ is said to have a paradoxical decomposition if there exists

- $s, t \in \mathbb{N}$,
- $\{A_i\}_{i=1}^s$ and $\{B_i\}_{i=1}^t$ collections of mutually disjoint subsets of $\Gamma$
- $\{\gamma_i\}_{i=1}^s$ and $\{\eta_j\}_{j=1}^t$ elements of $\Gamma$

so that

$$
\Gamma = \left( \bigsqcup_{i=1}^{t} A_i \right) \cup \left( \bigsqcup_{j=1}^{t} B_j \right) = \bigsqcup_{i=1}^{s} A_i \gamma_i = \bigsqcup_{j=1}^{t} B_j \eta_j
$$

where $\sqcup$ stresses that the unions are disjoint. The Tarski number of a group, say $\tau(\Gamma)$ is the minimum of $s + t$ over all paradoxical decompositions of $\Gamma$ (and $\infty$ if $\Gamma$ has no paradoxical decompositions).

Note the convention whether $\gamma_i$ and $\eta_j$ are on the left is not important (one could put it also on the right by sending all elements to their inverse). Since the Cayley graphs are here defined by multiplication on the right, the convention on the left would be more geometric. Indeed, multiplying on the left is a graph automorphism. Hence one cuts the Cayley graph in finitely many pieces (the $A_i$s and $B_j$s). Upon moving (by automorphisms) the $A_i$s one recovers the full Cayley graph. Same goes for the $B_j$s.

It is fairly easy to check that $s, t \geq 2$ (see exercises) so $\tau(\Gamma) \geq 4$. A well-known result is that $\tau(\Gamma) = 4$ if and only if $\Gamma$ admits $\mathcal{F}_2$ (the free group on 2 generators) as a subgroup (see exercises).

The Banach-Tarski paradox (which actually goes back to Hausdorff) stems from the fact that $\mathcal{F}_2$ has a paradoxical decomposition and that it is a subgroup of $SO_3(\mathbb{R})$.

---

24 This is an unpublished work of Jonsson in the 40’s and a published work of Dekker in the 50’s.
Theorem 4.7

\(\Gamma\) is amenable if and only if

(vii) \(\Gamma\) does not have a paradoxical decomposition, i.e. \(\tau(\Gamma) = \infty\) \hspace{1cm} (Tarski, 1938)

To be more precise (iii) \(\Rightarrow\) (vii) is due to von Neumann (1929). The other direction is due to Tarski\(^{25}\).

**Proof of (iii) \(\Rightarrow\) (vii):** Assume \(\Gamma\) has a paradoxical decomposition and let \(m\) be the invariant mean, then

\[ 1 \text{ norm.} \cdot m(1_\Gamma) = m\left(\sum_i 1_{A_i} + \sum_j 1_{B_j}\right) \overset{\text{lin.}}{=} \sum_i m(1_{A_i}) + \sum_j m(1_{B_j}) \]

where “norm.” stands for normalisation and “lin.” for linearity (it is crucial that the sums are finite!). On the other hand,

\[ 1 \text{ norm.} \cdot m(1_\Gamma) = m\left(\sum_i 1_{A_i} \gamma_i\right) \overset{\text{lin.}}{=} \sum_i m(1_{A_i}) \overset{\text{inv.}}{=} \sum_i m(1_{\gamma_i}) \]

where “inv.” stands for the invariance. The same goes with the \(B_i\)s. Putting these 3 equations together gives \(1 = 2\) which is a contradiction. \(\blacksquare\)

For the other direction, the following theorem will be required:

**Theorem 4.8 (Hall’s marriage Lemma)**

... 

**Proof of (vii) \(\Rightarrow\) (i):** The proof will actually be \(\neg(i) \Rightarrow \neg(vii)\), i.e. the absence of Følner sequence implies paradoxical decomposition. Indeed, \(\neg(i)\) means (using Lemma 1.3)

\[ \exists C > 0 \text{ so that } \forall F \subset \Gamma \text{ finite}, |F| \leq C |F + 1 \setminus F| \]

where \(F^+ = \{\gamma \in \Gamma \mid d(\gamma, F) \leq r\}\). So \(|F^+| \geq (1 + C)|F|\). But by a relative simple iteration argument,

\[ |F^{+r}| \geq (1 + C)^r |F^+| \geq \ldots \geq (1 + C)^r |F| \]

Just pick \(r\) so that \((1 + C)^r \geq 2\). Consider the bipartite graph with vertices on one side \(F\) and the other \(F^{+r}\). Put an edge between vertices if the two elements are related by an element in \(B_r = \{\gamma \in \Gamma \mid d(\gamma, e) \leq r\}\).

Using Hall’s marriage Lemma, one finds two way of sending \(F\) to \(F^{+r}\). These partial maps use at most finitely many elements (because \(B_r\) is finite). By looking at all those maps for an increasing family of sets \(F_k\),\(^{26}\) one gets a family of such maps.

Now this sequence could be quite chaotic. Note for each \(B_i\) there is however a subsequence which is eventually constant when restricted to \(B_i\). Restrict further this subsequence for larger ball and use a diagonal argument to conclude. \(\blacksquare\)

It might be good to underline that one does not need the axiom of choice\(^{27}\) to see that (vii) \(\Rightarrow\) (i).

\(^{25}\)Since there was no other criterion for amenability at the time of Tarski, his proof was certainly non-trivial. I could not get my hands on the original paper...

\(^{26}\)To use Hall’s marriage Lemma, we really need that all subsets \(F\) of \(F_k\) have the property that there are enough elements in \(F^{+r}\).

\(^{27}\)Of course, the axiom of countable choice is used, but that one was never controvesial.
4.D ... equivalences; fixed point
In construction....

5 Varia

5.A Reiter and Liouville

As mentioned before, one could think that some $\mu^{(n)}$ could be good choices for the Reiter sequence. Recall that a function $f$ is $\mu$-harmonic (for some $\mu \in \mathcal{P}_c(\Gamma)$) if $f * \mu = f$. $\Gamma$ is said to be $\mu$-Liouville if there are no $\mu$-harmonic functions in $\ell^\infty \Gamma$ except constant functions.

Before moving on to the proof, a small lemma will be required:

**Lemma 5.1**

In $\ell^1\mathbb{N}$, weak and strong convergence coincide.

**Proof:** It is classical that strong convergence implies weak convergence. Assume the sequence $x_n$ converges weakly but not strongly. WLOG, one may assume it converges to 0 (if it converges to $x$ look at the sequence $x_n - x_0$ instead). WLOG, one may also assume that $\|x_n\|_\ell^1 \geq 1$ (because it does not converge strongly, this is $\geq \delta$ for some subsequence; just restrict to this subsequence and multiply by a scalar to get $\geq 1$).

Since all functions with values $\pm$ on a finite set belong to $\ell^\infty\mathbb{N}$, weak convergences implies that the $x_n$ tend to 0 on finite sets. On the other hand, for each $x_n$ there is a finite set $A$ so that $\|x_n\|_{\ell^1(A^c)} < \epsilon$ (for some fixed and small $\epsilon \to 0$. Hence one may find sequences $n_i$ and $N_i$ so that $\forall i \|x_{n_i}\|_{\ell^1(\mathbb{N} \setminus N_{i-1}, N_i)} < \epsilon$

Pick $y \in \ell^\infty\mathbb{N}$, given by

$$y(n) = \text{sgn } x_{n_i}(n)$$

with $\text{sgn } r = \frac{r}{|r|}$ if $r \neq 0$ and 0 if $r = 0$. Then, notice that, $\forall i$,

$$\langle y, x_{n_i} \rangle \geq \|x_{n_i}\|_{\ell^1(\mathbb{N} \setminus N_{i-1}, N_i)} - 2\epsilon \geq 1 - 2\epsilon.$$  

Hence, the sequence cannot converge weakly to 0.

Recall that $\mathcal{P}_c(\Gamma)$ is the space of finitely supported probability measures. $\mathcal{P}_r(\Gamma)$ is the norm-closure of $\mathcal{P}_c(\Gamma)$ in $\ell^1 \Gamma$, i.e. probability measures in $\ell^1 \Gamma$.

**Proposition 5.2**

Let $\Gamma$ be finitely generated and let $\mu \in \mathcal{P}_r(\Gamma)$ be symmetric and irreducible. $\Gamma$ is $\mu$-Liouville if and only if the sequence $\xi_N = \frac{1}{N} \sum_{i=0}^{N-1} \mu^{(i)}$ is a Reiter sequence.

**Proof:** $(\Rightarrow)$ Assume $f$ is $\mu$-harmonic and $\xi_n$ is Reiter. Since $\xi_n$ are convex combinations of $\mu^{(n)}$, $f * \xi_n = f$. So for any $x, s \in \Gamma$

$$|f(xs) - f(x)| = |f * \delta_s(x) - f(x)| = |f * \xi_n * \delta_s(x) - f * \xi_n(x)|$$

$$= |f * (\xi_n * \delta_s - \xi_n)(x)| \leq \|f\|_{\ell^\infty} \|\xi_n * \delta_s - \xi_n\|_{\ell^1}$$
Since $n$ is arbitrary, $f(xs) = f(x)$. But $x$ and $s$ are also arbitrary so $f$ is constant.

(⇒) Assume $\xi_N = \frac{1}{N} \sum_{i=1}^{N-1} \mu(i)$ (or any of its subsequence) is not Reiter. Pick some $s \in \Gamma$. Then $\exists \epsilon > 0$ so that, $\forall N$, $|\delta_s \ast \xi_N - \xi_N|_{\ell^1} > \epsilon$. Since this is norm-separated from 0, there is an element in the dual of $\ell^\ast \Gamma$ (i.e. $\ell^\infty \Gamma$) which also does so. So, there exists $g \in \ell^\infty \Gamma$ such that, $\forall n$,

\[
(\ast\ast) \quad \left| \int_{\Gamma} g(\delta_s \ast \xi_N) - \int_{\Gamma} g(\xi_N) \right| > \epsilon.
\]

Put $f(x) = \lim_n \int_{\Gamma} g(y) \delta(x \ast \xi_n)(y)$ where the limit is given by some diagonal subsequence or a weak* accumulation point. Then $f$ is bounded (obvious) and harmonic:

\[
|f - f \ast \mu|(x) = |\lim_N \int_{\Gamma} g(y) \delta(x \ast \xi_N - \delta_s \ast \xi_N \ast \mu)(y)|
\]

\[
= |\lim_N \int_{\Gamma} g(y) \delta(x - \delta_s \ast \mu(N))(y)|
\]

\[
\leq |g|_{\ell^1} \lim_N \|\delta(x - \delta_s \ast \mu(N))\|_{\ell^1}
\]

\[
\leq |g|_{\ell^1} \lim_N \frac{2}{N} \epsilon.
\]

Note that it is crucial that the limit tends to 0 whatever the subsequence chosen. Lastly, $| f(s^{-1}) - f(e) | > \epsilon$ thanks to equation $(\ast\ast)$, so we have that $f$ is not a constant.

The first proof that $h(\mu) = 0$ (the entropy) gives a Reiter sequence is due to Avez 1974. He used it to show that $h(\mu) = 0$ is equivalent to $\mu$-Liouville. This has been revisited by Kaimanovich & Vershik: $h(\mu) = 0$ implies that $\xi_N = \frac{1}{2} \mu(N) + \frac{1}{2} \mu(N+1)$ is Reiter.

Note that, even on $Z$ and for a symmetric finitely supported $\mu$, $\mu(n)$ is in general not a Reiter sequence. Indeed, if the Cayley graph generated by $S = \text{Support}(\mu)$ is bipartite, it is easy to show that $\mu(n)$ is supported on elements $n$ such that $|\gamma|_S \equiv n \mod 2$. In particular, for any $s \in S$, the support of $\delta_s \ast \mu(n)$ and $\mu(n)$ are disjoint.

Proposition 5.2 also implies that non-amenable groups are not Liouville (for symmetric irreducible probability measures). Here is another argument (taken from U. Bader’s lectures) which does not require symmetry:

**Proposition 5.3**

If $\Gamma$ is not amenable, then it is not $\mu$-Liouville for any irreducible $\mu \in \mathcal{P}(\Gamma)$.

**Proof:** Recall $M$ is the convex set of means inside the LCTVS $(\ell^\infty \Gamma)^{\ast}$, and, as $\Gamma$ is not amenable, none of them is invariant. $\mu$ acts on function in $\ell^\infty \Gamma$ by convolution and hence defines a dual action on its dual. Because $\mu$ is a probability measure this action (is given by taking a limit of convex combinations, and hence) preserves $M$. Now, by the Markov-Kakutani theorem this action will have a fixed point.

Pick $m$ to be such a fixed point. Since it is not invariant, $\exists \gamma \in \Gamma$ so that $\gamma \cdot m \neq m$ (again the action on means is dual to the right-regular action on $\ell^\infty \Gamma$). $\exists g \in \ell^\infty \Gamma$ so that

---

28 We use the absence of invariance on the left instead of the right. But since $\mu$ is symmetric, this is equivalent.

29 This is where Lemma 5.1 comes in: the sequence is also weakly separated from $0$.

30 Since $g$ is bounded by $c = |g|_{\ell^\infty}$, this is a sequence of functions in $[-c, c]$.

31 In this case, see the comments after Proposition 5.3.

32 The semi-group $\mathbb{N}$ acts on $M$ by applying $n$ times $\mu$. The proof goes exactly as in (i) $\implies$ (ii).
\((\gamma \cdot m)(g) \neq m(g)\). Define \(h(\gamma) = (\gamma \cdot m)(g)\). Then \(h\) is non-constant (because \(m\) is not invariant) but is harmonic (because \(m\) is fixed by \(\mu\)).

The previous proof also indicates how to modify the proof of \((\Rightarrow)\) in Proposition 5.2 to more general groups. One needs to replace the particular convex combination \(\xi_N\) by \(X\), the convex set generated by the sequence \(\mu^{(n)}\). Separation in the weak topology follows from Hahn-Banach (instead of Lemma 5.1). \(Y\) is then defined as a weak\(^*\) accumulation point of \(f_n(x) := \int_{\Gamma} g(y) \delta_x * \mu^{(n)}(y)\). The Markov-Kakutani theorem is applied to find a \(\mu\)-invariant element in \(Y\).

Note that neither the Proposition 5.2 nor Proposition 5.3 require that \(\mu\) is finitely supported. A construction of Kaimanovich & Vershik 1983 shows that, in an amenable group, one can always construct a (in general, not finitely supported) probability measure \(\mu\) so that the group is \(\mu\)-Liouville (this was a conjecture of Furstenberg). Hence

**Theorem 5.4 (Kaimanovich & Vershik 1983)**

A group \(\Gamma\) is amenable if and only if there is a symmetric and irreducible probability measure \(\mu\) so that \(\Gamma\) is \(\mu\)-Liouville.

**Proof:** to do...

However, there are amenable groups which are not \(\mu\)-Liouville (for a symmetric finitely supported measure \(\mu\)). The simplest example is the lamplighter group on \(\mathbb{Z}^3 : \Gamma = \mathbb{Z}_2 \wr \mathbb{Z}^3\). This was shown by A.Ershler

**Example 5.5.** Let \(\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}^3\) be the lamplighter group on \(\mathbb{Z}^3\) (there is a lamp at each vertex in \(\mathbb{Z}^3\)). To define an explicit bounded harmonic function, first notice that since the space where the “lamplighter” moves is \(\mathbb{Z}^3\). Up to forgetting the lamps, one has a (lazy) random walk on \(\mathbb{Z}^3\). It is transient, so with probability 1 the lamplighter will visit finitely many a finite subset \(F\) of \(\mathbb{Z}^3\). Fix some lamp states on \(F\), i.e. fix an element \(a\) of \(\oplus_{x \in F} \mathbb{Z}_2\). Define, for \(x \in \Gamma\),

\[
h(x) = \mathbb{P}(\text{for } n \text{ large enough, a SRW starting at } x \text{ has lamp state equal to } a \text{ on } F).
\]

Transience is crucial for this probability to be defined. It’s obviously bounded. By construction (and elementary probability), it is \(\mu\)-harmonic. Remains to check it is non-constant. To see this take \(x, y \in \Gamma\) so that the lamplighter position \(x \in \mathbb{Z}^3\) is the same in \(x\) and \(y\) but the lamp states on \(F\) are equal to \(a\) in \(x\) and not equal to \(a\) in \(y\). Let \(z\) go to infinity, then \(h(x) \to 1\) while \(h(y) \to 0\). Hence \(h\) is a non-constant bounded harmonic function.

One can also extend these harmonic functions by using any \(f \in \ell^\infty L\) (where \(L = \oplus_{g \in \mathbb{Z}^d} \mathbb{Z}_2\)) and then \(h\) being the expected value of \(f\) after the walker has gone to \(\infty\). These functions are dubbed “limit configurations”. A. Ershler\(^{33}\) showed these are actually the only bounded non-constant harmonic functions on \(\mathbb{Z}_2 \wr \mathbb{Z}^d\) if \(d > 4\). From this description, can you find the \(g\) in proposition 5.2 (and \(s\), depending on \(a\)) which separates the convex hull of the \(\mu^{(i)}\) from their translates (it might be hard to prove it actually does!).

\(^{33}\)This has recently been proved for any \(d > 2\) by Y. Peres.
5.B The Mazur-Ulam theorem

This proof follows [a presentation of N. Monod of] a proof due to B. Nica [12].

Definition 5.6. Let $(X,d)$ be a metric space. A mid-point is a map $m : X \times X \to X$ satisfying

(A) $d(x,m_{x,y}) = d(y,m_{x,y}) = \frac{1}{2}d(x,y)$.

(B) $m_{x,y} = m_{y,x}$.

A symmetry is an isometry $\sigma \in \text{Isom}(X)$ such that $\exists p \in X$ with $\forall z \in X, m_{z,\sigma z} = p$. A convex symmetric space is a space with mid-point $(X,d,m)$ such that

(C) $d(m_{x,y},z) \leq \frac{1}{2}(d(x,z) + d(y,z))$.

(D) $\forall x, y \in X$, there is a symmetry $\sigma$ such that $\sigma x = y, \sigma y = x$.

Note that the symmetry in (D) corresponding to $x$ and $y$ has necessarily $p = m_{x,y}$. In some Banach space, there are many mid-point maps different from $\frac{x+y}{2}$ (although this is the simplest one).

Theorem 5.7

If $X$ is a LCTVS which is also a convex symmetric space, then every isometry preserves $m$.

Proof: Let $g \in \text{Isom}(X)$. Fix $x, y \in X$ and let $\Delta_g := \Delta_g(x,y) := d(m_{gx,gy}, gm_{x,y})$. Then

(*) $\Delta_g \leq \frac{1}{2}(d(gm_{x,y}, gx) + d(gm_{x,y}, gy)) = \frac{1}{2}(d(m_{x,y}, x) + d(m_{x,y}, y)) = \frac{1}{2}d(x,y)$.

Using (D), take $\sigma$ a symmetry corresponding to $gx$ and $gy$ (so that $p = m_{gx,gy}$) and let $h = g^{-1}\sigma g \in \text{Isom}(X)$, then $hx = y$ and $hy = x$. As a consequence,

$$
\begin{align*}
\Delta_h &= d(m_{hx,hy}, g^{-1}\sigma gm_{x,y}) = d(m_{y,z}, g^{-1}\sigma gm_{x,y}) \\
&\leq d(m_{x,y}, g^{-1}\sigma gm_{x,y}) = d(z, \sigma z) \quad \text{where } z = gm_{x,y} \\
&= 2d(z,p) = 2d(gm_{x,y}, m_{gx,gy}) \quad \text{since } p = m_{gx,gy} \\
&= 2\Delta_g.
\end{align*}
$$

Now, the upper bound (*) is independent of the isometry chosen. Thus, one gets a contradiction (by repeating this “doubling”) unless $\Delta_g = 0$.

A map is affine if it preserves convex combinations. Actually, if the vector space is over $\mathbb{R}$, $f$ is affine if and only if $f(\frac{x+y}{2}) = \frac{1}{2}(f(x) + f(y))$. Since $m_{x,y} = \frac{1}{2}(x+y)$ is always a mid-point map in a Banach space, one gets:

Corollary 5.8 (Mazur-Ulam, 1932)

Every isometry of a Banach space is affine.
5.C More fixed points properties

6 Exercises

Everywhere: $\Gamma$ is a finitely generated (discrete) group. For hints look at the end.

**Exercise 1** Show that a group $\Gamma$ is finite if and only if every affine action on a vector space admits a fixed point. [It might make life easier to use the Mazur-Ulam theorem: an isometric action on a Banach space is affine.]

**Exercise 2** Show that a sequence $\{F_n\}$ of finite sets is Følner (i.e. $\forall U \subseteq \Gamma$ finite, $\frac{|F_n \Delta F_n|}{|F_n|} \to 0$) if and only $\frac{|\partial F_n|}{|F_n|} \to 0$.

**Exercise 3** The aim is to compute the isoperimetric profile of $\mathbb{Z}^d$ with a non-optimal constant $34$: $|F|^{(d-1)/d} \leq \frac{1}{3}|\partial F|$. To do so, let $\pi_i : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ be the projection orthogonal to the $i$th coordinate. Let $p : \mathbb{Z}^d \to \mathbb{Z}$ be the projection on the first coordinate. Take $F \subset \mathbb{Z}^d$ finite and let $F_i := \pi_i(F)$. Let $a_x = |p^{-1}(x)|$ and (for $j \geq 2$) $a_{x,j} = |\pi_j(p^{-1}(x))|$. 

1. Assuming the Loomis-Whitney inequality, i.e.

\[
(LW) \quad |F|^{d-1} \leq \prod_{i=1}^{d} |F_i|,
\]

deduce that $|F|^{(d-1)/d} \leq \frac{1}{3}|\partial F|$.

2. When $d = 2$, show directly that $|F| \leq |F_1| \cdot |F_2|$.

3. To show $(LW)$ for $d > 2$, induction will be done. First, show that $|F| \overset{\dagger}{=} \sum_{x \in \mathbb{Z}} a_x$, $a_x \overset{\dagger}{=} |F_1|$ and $\sum_{x \in \mathbb{Z}} a_{x,j} \overset{\dagger}{=} |F_j|$.

4. Use induction and $\dagger$ to get $a_x^{d-1} \overset{\S}{\leq} |F_1| \prod_{j=2}^{d} a_{x,j}$.

5. Use $\ast$, $\S$, Hölder's inequality and $\dagger$ to conclude that $(LW)$ holds.

6. Show that the profile is sharp (up to the constant).

**Exercise 4** The infinite permutation group is

$$S_\infty = \{\sigma : \mathbb{N} \to \mathbb{N} \mid \sigma \text{ is bijective and there is a finite set } F \text{ such that } \sigma_{|\mathbb{N}\setminus F} = \text{Id}\}.$$ 

Show that it is amenable and is not finitely generated. Show that $A_\infty$, the even permutations in $S_\infty$, is simple.

**Exercise 5** Prove the “closure properties” as follows:

1. “Subgroup” (a) using the “Følner sequences” (i) definition.

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34 For many application, having the optimal constant is not so important. This argument is taken from Loomis-Whitney 1940’s.
2. “Extensions” (b) using the “invariant means” (iii) definition.

3. “Quotients” (c) using the “probability of return” (iv) definition [Kesten’s criterion]

4. “Direct limits” (d) using the “fixed point property” (ii) definition.

5. Show that a semi-direct product of amenable groups is amenable (a subcase of (b)) using the “Følner sequences” definition (i).

Exercise 6 Assume $\Gamma$ is infinite. Let $c_0\Gamma$ be the norm closure of finitely supported functions in $\ell^\infty\Gamma$. Show that if $m$ is an invariant mean, then $m(f) = 0$ for all $f \in c_0\Gamma$. Conclude that the kernel of any35 invariant mean contains (the norm closure of)

$$c_0\Gamma + \{ f * \delta_\gamma - f \mid f \in \ell^\infty\Gamma \text{ and } \gamma \in \Gamma \}.$$

Can you show this is indeed the intersection of the kernel of all means?

Exercise 7 The goal is to show there always exists Følner sequences which are right AND left-invariant.

1. Show that if there is a right-Reiter sequence, then there is a left-Reiter sequence. Deduce any amenable group has a sequence which is both left- and right-Reiter.

2. Conclude that there is always a left- and right-Følner sequence.

Exercise 8 Show directly that the “Følner sequence” definition (i) implies the “invariant mean” definition (iii). [You will need to know what an ultralimit is...]

Exercise 9 Show that \( \lim \inf_{n \to \infty} \frac{|R_{n+1}|}{|Z_n|} = 0 \) implies the group is amenable.

Exercise 10 Here are steps to show: \( \tau(\Gamma) = 4 \iff \Gamma \) has a subgroup isomorphic to \( F_2 \).

1. Prove that, WLOG, \( \gamma_1 = \eta_1 = e_r \).

2. Deduce that \( s \geq 2, t \geq 2 \) and \( \tau(\Gamma) \geq 4 \).

3. Use the Ping-Pong-Lemma to show that \( \tau(\Gamma) = 4 \implies \Gamma \) has a subgroup isomorphic to \( F_2 \).

4. Show that \( \tau(F_2) = 4 \).

5. If \( H \) is a subgroup of \( \Gamma \), show that \( \tau(H) \geq \tau(\Gamma) \). [This is also a way of doing (vii) \( \implies (a) \).]

Exercise 11 Assume \( H \) is a subgroup of \( \Gamma \) and \( Q = \Gamma/N \) for some normal subgroup \( N \). Show that \( \tau(H) \geq \tau(G) \) and \( \tau(Q) \geq \tau(G) \). [This is also a way of doing (vii) \( \implies (c) \).]

Exercise 12 Show that any group with \( \tau(\Gamma) < \infty \) has a finitely generated subgroup \( H \) with \( \tau(H) = \tau(\Gamma) \).

\[35\text{Even on } \mathbb{Z}, \text{ there are uncountably many invariant means...} \]
\( Q = \{ f \in \mathbb{C}(\Gamma) \mid \sum_{\gamma \in \Gamma} f = 0 \} \), take an affine map \( \phi \) into \( P \) (i.e. add \( \delta_e \) to each element). Act on \( v \in Q \) as follows: \( v \cdot \gamma = \alpha^{-1}(a(v) \cdot \gamma) \). Either compute this map to see it’s affine, or use Mazur-Ulam (both \( \alpha \) and the action are isometric).

**Hint[s] for 2:** Let \( \partial^r F \) be the \( r \)-boundary of \( F \): look at all edges with some extremity [or all vertices] at distance \( \leq r \) from \( F \). Give a rough bound: \( |\partial^r F| \leq (|S| + 1)^r |\partial F| \).

**Hint[s] for 3:** (1) for any \( i \), \( 2|F_i| \leq |\partial F| \). (5) the form of Hölder’s inequality required here is \( \sum_{\gamma \in \Gamma} |\sum_{\rho \in \partial_{ij},\gamma} a_{\rho}| \leq \prod_{j=2}^{d} \left( \sum_{\gamma \in \Gamma} |a_{\gamma}|^{d-1} \right)^{1/(d-1)} \).

**Hint[s] for 5:** (1) For each coset, one gets a sequence (the intersection of the Følner sequence with the coset). What happens if none of these satisfy the Følner condition? (2) Average over cosets to be able to project the function to the quotient, then average there. (3) Show that the probability of return is bigger in the quotient group than in the quotiented group. (4) Look at the limit of the fixed point sets. (5) If \( \Gamma = G \times H \), \( F_n \) is Følner for \( G \) and \( E_n \) is Følner for \( H \), try either \( (\{e_G\}, F_n) (E_n, \{e_H\}) \) or \( (E_n, \{e_H\}) (\{e_G\}, F_n) \) (to get either a left- or a right-Følner sequence).

**Hint[s] for 6:** \( m \) is continuous and the kernel is closed. To find the kernel of a mean it is easier to think in terms of Følner sequences. Let \( f_n = |F_n|^{-1} f * 1_{F_n} \) (this is a convex combination of translates of \( f \)). \( f_n \) has the same invariant mean as \( f \) and \( m(f) = 0 \) if and only if \( \lim_{n \to \infty} ||f_n||_\infty = 0 \).

**Hint[s] for 7:** If \( \xi_n \) is right-Reiter, \( \xi_n^{-1} \) is left-Reiter and \( \xi_n^{-1} \) is both.

**Hint[s] for 8:** Define \( f_n(x) = \frac{1}{|F_n|} \sum_{\gamma \in F_n} f(x \gamma) = f * 1_{F_n}(x) \). Show this function becomes essentially constant and define \( \tilde{f} = \lim f_n \) (where this is an ultralimit). Put \( m(f) = \tilde{f}(x) \) for any \( x \).

**Hint[s] for 9:** \( \frac{|R_{n+1} \setminus R_n|}{|R_n|} = |R_{n+1} / R_n| - 1 \).

**Hint[s] for 10:** 3: Ping-pong lemma: Let \( H \) be a group, generated by two elements \( a \) and \( b \) of infinite order. Suppose there is a \( H \)-action on a set \( X \) such that there are non-empty subsets \( A B \subset X \) with \( B \) not included in \( A \) and such that for all \( n \in \mathbb{Z} \setminus \{0\} \) one has \( a^n \cdot B \subset A \) and \( b^n \cdot A \subset B \). Then \( H \) is freely generated by \( \{a, b\} \).

4: Say the generators of the free group are \( a \) and \( b \). A first try would be \( A_1 = \{a^n \cdot b \} \subset A \) and \( B_1 = \{b^n \cdot a \} \subset B \). But the identity is not included. Add the identity to one set (and change the place of infinitely many other elements) in order to fix this.

5: Each [right- or left- ?] cosets of the subgroup has a paradoxical decomposition with the same translating elements. Put them together.

**Hint[s] for 11:** Very similar to 10.5. If \( \tau(Q) = \infty \) there is nothing to prove. Say \( Q \) has a paradoxical decompositions with \( q_i \) and \( r_j \) as translating elements and \( A_i, B_i \) as sets. For the paradoiscal decomposition of \( \Gamma \), pull the sets back, i.e. use \( \pi^{-1}(A_i) \) where \( \pi : \Gamma \to Q \) and pick the translating elements in \( \pi^{-1}(q_i) \) and \( \pi^{-1}(r_j) \) (it does not matter which one you pick).

**Hint[s] for 12:** Look at the subgroup generated by the elements used to make the paradoxical decomposition. One inequality comes from exercise 10.

### 7 References

There are many book which treat these equivalences, the two standard references are Pier “Amenable locally compact groups” and Paterson “Amenability”. Greenleaf “Invariant means on topological groups” is actually quite nice to read (but not so easy to find). The appendix in the book by Bekka, de la Harpe & Valette “Kazhdan property (T)” (appendix G) is also a great source. Some lecture notes by Anne Thomas can be another good source. All these
do the “general case” of locally compact groups (which might make a first reading harder). The above proofs were obtained by successive simplifications of the proofs in these various sources.

Further references


