Positive Solutions of the Equation
\[ \dot{x}(t) = -c(t)x(t - \tau) \] in the Critical Case

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The equation \( \dot{x}(t) - c(t)x(t - \tau) \) is considered in the critical case. For it, the asymptotic behavior of dominant and subdominant solutions is studied. A generalization is made and connections with known results are discussed.

1. INTRODUCTION AND MAIN RESULT

The present article is devoted to the problem of the asymptotic behavior of solutions of delayed equations of the type

\[ \dot{x}(t) = -c(t)x(t - \tau) \] (1.1)

with a positive continuous coefficient \( c \) on the set \( \mathcal{I}(t_0) := [t_0, \infty), \ t_0 \in \mathbb{R}, \ 0 < \tau = \text{const,} \) in the “nonoscillatory” case in the situation when the coefficient \( c \) has a prescribed form. A function \( x \) is called a solution of (1.1) on \( \mathcal{I}_-(t_0) := [t_0 - \tau, \infty) \) if \( x \) is defined and continuous on \( \mathcal{I}_-(t_0) \), differentiable on \( \mathcal{I}(t_0) \), and satisfies (1.1) on \( \mathcal{I}(t_0) \). As is customary, a solution is called oscillatory on \( \mathcal{I}_-(t_0) \) if it has arbitrary large zeros.
Otherwise it is called nonoscillatory on \( \mathcal{J}_{-1}(t_0) \). The delayed equation (1.1) is called oscillatory on \( \mathcal{J}_{-1}(t_0) \) if all solutions are oscillatory on \( \mathcal{J}_{-1}(t_0) \), and nonoscillatory on \( \mathcal{J}_{-1}(t_0) \) if there is a nonoscillatory solution on \( \mathcal{J}_{-1}(t_0) \).

Equation (1.1) (or more common with variable delay \( \tau \)) plays a crucial role in many investigations and therefore is always in the center of interest.

Following the historical order of investigation, Domshlak noticed (see [11, 13]) that among the equations of the form (1.1) with

\[
\lim_{t \to \infty} c(t) = \frac{1}{\tau e},
\]

there exist equations such that all their solutions are oscillatory on \( \mathcal{J}_{-1}(t_0) \) in spite of the fact that the corresponding “limiting” equation

\[
\dot{x}(t) = -\frac{1}{\tau e} x(t - \tau)
\] \hspace{1cm} (1.2)

admits a nonoscillatory solution. Let us remark that (1.2) has positive solutions \( x_1, x_2 \) on \( \mathcal{J}_{-1}(t_0) \) with \( t_0 > 1 \) of the form

\[
x_1(t) = t \exp\left(-\frac{t}{\tau}\right), \quad x_2(t) = \exp\left(-\frac{t}{\tau}\right).
\] \hspace{1cm} (1.3)

Before, by Kozakiewicz [21–23] (see also [25] containing a survey of Kozakiewicz’s results), a basic phenomenon was established that if (1.1) permits two positive solutions \( x_1, x_2 \) on \( \mathcal{J}_{-1}(t_0) \) with

\[
\lim_{t \to \infty} \frac{x_1(t)}{x_2(t)} = 0,
\] \hspace{1cm} (1.4)

then the representation

\[
x(t) = K x_1(t) + O(x_2(t)), \quad t \in \mathcal{J}_{-1}(t_0),
\] \hspace{1cm} (1.5)

holds for any solution \( x \) of the delayed equation (1.1) on \( \mathcal{J}_{-1}(t_0) \). Here \( K \) is a suitable constant dependent on \( x \) and \( O \) is the Landau order symbol. (Kozakiewicz’s results are formulated for more common classes of linear delayed equations containing Stieltjes integrals.)

The representation (1.5) is a typical situation which occurs in the nonoscillatory case. Namely, using a different method, Diblík [10] proved the following results for the equation

\[
\dot{x}(t) = -c(t)x(t - \tau(t))
\] \hspace{1cm} (1.6)
with a positive coefficient $c$ and a continuous bounded positive delay
$\tau(t) \leq r$, where the difference $t - \tau(t)$ has to be an increasing function
(these conditions we will suppose throughout this paper). For variable
delay, we define
\[
\mathcal{I}_-(t_0) := \left[ t_0 - \tau(t_0), \infty \right).
\]
(The definitions given above can be easily adapted for this equation.)

Studying the proof of Theorem 16 in [10], we get the following stronger
formulation of it:

**Theorem 1.1.** Let us suppose the existence of a positive solution $\tilde{x}$ of (1.6)
on $\mathcal{I}_-(t_0)$. Then either every solution $x$ of (1.6) on $\mathcal{I}_-(t_0)$ is represented in a
unique way by the formula
\[
x(t) = \tilde{x}(t)(K + \zeta(t)),
\]
where $K \in \mathbb{R}$ depends on $x$, and $\zeta$ is a continuous, vanishing function
dependent on $x$, or every solution $x$ of (1.6) on $\mathcal{I}_-(t_0)$ is represented in a
unique way by the formula
\[
x(t) = \tilde{x}(t)(KY(t) + \delta(t)),
\]
where $Y$ is a continuous, increasing function which is the same for each $x$,
satisfies $\lim_{t \to \infty} Y(t) = \infty$, $K \in \mathbb{R}$ dependent on $x$, and $\delta$ is a bounded
continuous function dependent on $x$.

Further, we have

**Theorem 1.2 (Theorem 18 in [10]).** Let there exist a positive solution $\tilde{x}$ of (1.6)
on $\mathcal{I}_-(t_0)$. Then there are positive solutions $x_1$ and $x_2$ of (1.6) on $\mathcal{I}_-(t_0)$ satisfying the relation (1.4). Moreover, every solution $x$ of (1.6) on $\mathcal{I}_-(t_0)$ is represented by the formula (1.5), where $K \in \mathbb{R}$ depends on $x$.

**Theorem 1.3.** Let $(x_1, x_2)$ be a fixed pair of solutions of (1.6) on $\mathcal{I}_-(t_0)$ with the properties as indicated in Theorem 1.2. Then there is no positive solution $x_3$ of (1.6) on $\mathcal{I}_-(t_0)$ satisfying
\[
\lim_{t \to \infty} \frac{x_3(t)}{x_2(t)} = 0.
\]

**Proof.** Assume there is a positive solution $x_3$ of (1.6) on $\mathcal{I}_-(t_0)$
satisfying (1.9). Then with the aid of Theorem 1.1 with $\tilde{x} = x_3$, we conclude
the following: The relation (1.7) cannot be used for the representation of
$x_1$ and $x_2$. Therefore, there are constants $K_1, K_2$, and a function $Y$ with
the properties as indicated in Theorem 1.1, and bounded functions $\delta_1, \delta_2$
such that formula (1.8) gives
\[ x_1(t) = x_3(t)(K_1Y(t) + \delta_1(t)), \]
\[ x_2(t) = x_3(t)(K_2Y(t) + \delta_2(t)), \]
for \( t \to \infty \). Thus
\[ 0 = \lim_{t \to \infty} \frac{x_2(t)}{x_1(t)} = \lim_{t \to \infty} \frac{K_2Y(t) + \delta_2(t)}{K_1Y(t) + \delta_1(t)}. \]
From this \( K_2 = 0 \) follows. We obtain
\[ x_2(t) = x_3(t)\delta_2(t) \]
which contradicts (1.9). \( \Box \)

Let \((x_1, x_2)\) be a pair of positive solutions of the delayed equation (1.6) on \( \mathcal{I}_-(t_0) \) with the properties as indicated in Theorem 1.2. Let \((\tilde{x}_1, \tilde{x}_2)\) be another pair of positive solutions of (1.6) on \( \mathcal{I}_-(t_0) \) satisfying
\[ \lim_{t \to \infty} \frac{\tilde{x}_2(t)}{\tilde{x}_1(t)} = 0. \] (1.10)
Then by Theorem 1.2, there are \( K_1, K_2 \in \mathbb{R} \) with
\[ \dot{x}_1(t) = K_1x_1(t) + O(x_2(t)), \quad \dot{x}_2(t) = K_2x_1(t) + O(x_2(t)), \quad t \in \mathcal{I}_-(t_0). \]
Assume \( K_1 = 0 \). Then
\[ \lim_{t \to \infty} \frac{\dot{x}_1(t)}{\dot{x}_2(t)} = \lim_{t \to \infty} \frac{O(x_2(t))}{K_2x_1(t) + O(x_2(t))}, \]
and therefore \( K_2 = 0 \). Thus \( \dot{x}_1(t) = O(x_2(t)), \quad \dot{x}_2(t) = O(x_2(t)), \) and the existence of \( x_1, x_2 \) and \( x_3 = \tilde{x}_2 \) contradicts Theorem 1.3. Hence \( K_1 \neq 0 \), and \( K_2 \neq 0 \) follows from (1.10). Choose now \( \tilde{x} = \tilde{x}_2 \) in Theorem 1.1. Then the representation (1.8) has to hold for all solutions \( x \) of (1.6) on \( \mathcal{I}_-(t_0) \). Specifying \( x \) to \( x_2 \), we get
\[ x_2(t) = \tilde{x}_2(t)(KY(t) + \delta(t)), \quad t \in \mathcal{I}_-(t_0), \]
with a constant \( K \in \mathbb{R} \) and a bounded function \( \delta \). We can conclude that \( K = 0 \): In the opposite case there would be three positive solutions \( x_1, x_2, \)
$x_2 = \bar{x}_2$ and we can proceed as above. Therefore,

$$x_2(t) = O(\bar{x}_2(t)), \quad t \in \mathcal{I}_-(t_0).$$

Summarizing, we have

$$\bar{x}_1(t) = K_1x_1(t) + O(x_2(t)) \quad \text{with} \quad K_1 \neq 0,$$

$$x_2(t) = O(\bar{x}_2(t)), \quad t \in \mathcal{I}_-(t_0),$$

such that (1.5) gives

$$x(t) = \tilde{K}\bar{x}_1(t) + O(\bar{x}_2(t)), \quad t \in \mathcal{I}_-(t_0),$$

for any solution $x$ of (1.6) on $\mathcal{I}_-(t_0)$ with a constant $\tilde{K} \in \mathbb{R}$ depending on $x$.

This is the reason for the following definition:

**Definition 1.4.** Let $x_1$ and $x_2$ be fixed positive solutions of the delayed equation (1.6) on $\mathcal{I}_-(T)$, $T \geq t_0$, with the property (1.4). Then $(x_1, x_2)$ is called a pair of dominant and subdominant solutions on $\mathcal{I}_-(T)$.

**Remark 1.5.** A pair $(x_1, x_2)$ of dominant and subdominant solutions for (1.2) is given by (1.3).

With respect to conditions for the nonoscillatory case, we refer, e.g., to the books by Erbe, Kong, and Zhang [15] and Győri and Ladas [16] and to the papers by Diblík [6, 9], Elbert and Stavroulakis [14], Győri and Pituk [17], Koplatadze and Chanturija [20], and Zhou [33].

Let us note that a common criterion, given in the book by Erbe, Kong, and Zhang [15] (see the paper by Zhou [33] too) is formulated (with respect to our conditions) as follows.

**Theorem 1.6.** The delayed equation (1.6) has a positive solution on $\mathcal{I}_-(t_0)$ if and only if there exists a continuous function $\lambda$ on $\mathcal{I}_-(t_0)$ such that $\lambda(t) > 0$ on $\mathcal{I}(t_0)$ and

$$\lambda(t) \geq c(t) \exp\left(\int_{t-\tau(t)}^{t} \lambda(s) \, ds\right), \quad t \in \mathcal{I}(t_0).$$

(1.11)

The following well-known sufficient condition for (1.6) is a consequence of this criterion: If

$$\int_{t-\tau(t)}^{t} c(s) \, ds \leq 1/e, \quad t \in \mathcal{I}(t_0),$$

then (1.6) has a positive solution on $\mathcal{I}_-(t_0)$. 

Therefore, the inequality
\[ c(t) \leq 1/(\tau e), \quad t \in \mathcal{J}(t_0), \]
is sufficient for existence of a positive solution of the delayed equation
(1.1) on \( \mathcal{J}_{-1}(t_0) \) in the case \( \tau(t) = \tau = \text{const} \). The expression \( 1/(\tau e) \) in the
right-hand side of this inequality is not the best possible one: In the paper
by Domshlak [12], it was shown that if
\[ \liminf_{t \to \infty} c(t) = \frac{1}{\tau e} \]
and
\[ \liminf_{t \to \infty} \left( c(t) - \frac{1}{\tau e} \right) \cdot t^2 = D > \frac{\tau}{8e} \]
(where the inequality \( D > \tau/(8e) \) cannot be improved), then all solutions
of the delayed equation (1.1) oscillate. Moreover, in the paper by Domshlak
and Stavroulakis [13], a further improvement was given: All solutions of
(1.1) oscillate if
\[ \liminf_{t \to \infty} c(t) = \frac{1}{\tau e}, \quad \liminf_{t \to \infty} \left( c(t) - \frac{1}{\tau e} \right) \cdot t^2 = \frac{\tau}{8e}, \]
and
\[ \liminf_{t \to \infty} \left[ \left( c(t) - \frac{1}{\tau e} \right) t^2 - \frac{\tau}{8e} \right] \cdot \ln^2 t = C > \frac{\tau}{8e} \]
(where the inequality \( C > \tau/(8e) \) cannot be improved). In this direction, a
final generalization of these results (in terms of inequalities for function \( c \))
was given in the paper by Diblik [9].

Let us formulate this generalization. For this at first we introduce some
necessary abbreviations. Denote:
\[ \ln_0 t := t, \quad \ln_p t := \ln(\ln_{p-1} t) \quad \text{for} \quad p \geq 1, \quad t > \exp_{p-2} 1, \]
where
\[ \exp_{-1} t := 0, \quad \exp_0 t := t, \quad \exp_p t := \exp(\exp_{p-1} t) \quad \text{for} \quad p \geq 1. \]

From the analysis of the above results, it can be expected that the functions
\[ c_p : (\exp_{p-1} 1, \infty) \to \mathbb{R}, \quad p \in \{0\} \cup \mathbb{N}, \]
which we call critical and which are defined by
\[
\begin{align*}
    c_p(t) := & \frac{1}{\tau e} + \frac{\tau}{8e t^2} + \frac{\tau}{8e(t \ln t)^2} + \frac{\tau}{8e(t \ln t \ln^2 t)^2} \\
    & + \cdots + \frac{\tau}{8e(t \ln t \ln^2 t \cdots \ln^p t)^2},
\end{align*}
\]
will play a fundamental role in its generalization. Indeed, the following holds.

**Theorem 1.7** (see [9]).

(a) Let us assume that \( c(t) \leq c_p(t) \) for \( t \to \infty \) and an integer \( p \geq 0 \). Then there is a \( T \geq t_0 \) and a positive solution \( x \) of the delayed equation (1.1) on \( \mathcal{I}_{-1}(T) \). Moreover,
\[
x(t) < e^{-t/\tau} \sqrt{t \ln t \ln^2 t \cdots \ln^p t}, \quad t \geq T - \tau.
\]

(b) Let us assume that
\[
c(t) \geq c_{p-1}(t) + \frac{\partial \tau}{8e(t \ln t \ln^2 t \cdots \ln^p t)^2}
\]
for \( t \to \infty \), an integer \( p \geq 1 \), and a constant \( \partial > 1 \). Then all solutions of the delayed equation (1.1) oscillate.

The main aim of the present paper is to study the asymptotic behavior of a pair \((x_1, x_2)\) of dominant and subdominant solutions for the delayed equation (1.1) when the coefficient \( c \) is equal to a critical function, i.e., in the case of equation
\[
\dot{x}(t) = -c_p(t) x(t - \tau), \quad p \in \{0\} \cup \mathbb{N}, \quad t \geq t_0 > \exp_{p-1} 1,
\]
where \( p \) is fixed and the critical function \( c_p \) was defined by (1.12).

Some remarks and comparisons are given and some open questions are formulated. An extension to more common classes of equations of the type (1.1) is given, too. As auxiliary results we claim two theorems on the asymptotic behavior of solutions of nonlinear delayed systems of differential equations (see Theorems 2.3 and 2.4 below) and corresponding “linear” corollaries which are interesting in their own right. The proof of Theorem 2.3 is based on the topological principle of Ważewski (see [6]). For the proof of Theorem 2.4, we have to use another method: The topological principle of Ważewski cannot be applied since the considered inequalities
are reversed. Theorems on differential inequalities cannot be applied since the right-hand side of the differential equation will not be monotonously increasing.

We state the main result:

**Theorem 1.8.** Let \( p \in (0) \cup \mathbb{N} \) be fixed. Then for any fixed constants \( \delta_1 > 2 \) and \( \delta_2 < 0 \), there are a \( T \geq t_0 \) and a pair \((x_1, x_2)\) of dominant and subdominant solutions of (1.14) on \( \mathcal{I}_1(T) \) satisfying the two-sided estimates

\[
e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln^{\delta_1}_{p+1} t} < x_1(t) < e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln^{\delta_1}_{p+1} t}
\]

(1.15)

and

\[
e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t \ln^{\delta_2}_{p+1} t} < x_2(t) < e^{-t/\tau} \sqrt{t \ln t \ln_2 t \cdots \ln_p t}
\]

(1.16)

on \( \mathcal{I}_1(T) \).

2. PRELIMINARY—NONLINEAR AUXILIARY RESULTS

In this section, some results concerning the asymptotic behavior of delayed systems of differential equations will be given.

Let \( \mathbb{R}^n \) be equipped with the maximum norm. With \( \mathbb{R}_{\geq 0}^n \) \( (\mathbb{R}_0^n) \) we denote the set of all component-wise nonnegative (positive) vectors \( v \) in \( \mathbb{R}^n \), i.e., \( v = (v^1, \ldots, v^n) \) with \( v^i \geq 0 \) \( (v^i > 0) \) for \( i = 1, \ldots, n \). For \( u, v \in \mathbb{R}^n \), we say \( u \preceq v \) if \( v - u \in \mathbb{R}_{\geq 0}^n \), \( u \ll v \) if \( v - u \in \mathbb{R}_{> 0}^n \), \( u < v \) if \( u \preceq v \) and \( u \neq v \).

Let \( C([a, b], \mathbb{R}^n) \), where \( a, b \in \mathbb{R}, a < b \), be the Banach space of the continuous functions from the interval \( [a, b] \) into \( \mathbb{R}^n \) equipped with the supremum norm. In the case \( a = -r < 0, b = 0 \), we shall denote this space as \( C_r \), that is, \( C_r = C([0, 0], \mathbb{R}^n) \).

Let us consider a system of retarded functional differential equations

\[
y'(t) = f(t, y_t),
\]

(2.1)

where \( f: \Omega \rightarrow \mathbb{R}^n \) is a continuous map which satisfies a local Lipschitz condition with respect to the second argument and \( \Omega \) is an open subset in \( \mathbb{R} \times C_r \).
If \( \sigma \in \mathbb{R}, A \geq 0, \) and \( y \in C([\sigma - r, \sigma + A], \mathbb{R}^n) \), then for each \( t \in [\sigma, \sigma + A] \), we define \( y_i \in C \) by means of relation \( y_i(\vartheta) = y(t + \vartheta), \) \( \vartheta \in [-r, 0] \). If necessary, we shall assume that the derivatives in the system (2.1) are right-sided.

In accordance with [19], a function \( y \) is said to be a solution of system (2.1) on \([\sigma - r, \sigma + A] \) if there are \( \sigma \in \mathbb{R} \) and \( A > 0 \) such that \( y \in C([\sigma - r, \sigma + A], \mathbb{R}^n) \), \( (t, y) \in \Omega \), and \( y(t) \) satisfies the system (2.1) for \( t \in [\sigma, \sigma + A] \). For given \( \sigma, \varphi \in C_r \), we say \( y(\sigma, \varphi) \) is a solution of the system (2.1) through \((\sigma, \varphi) \in \Omega \) if there is an \( A > 0 \) such that \( y(\sigma, \varphi) \) is a solution of the system (2.1) on \([\sigma - r, \sigma + A] \) and \( y(\sigma, \varphi) = \varphi \). In view of the above conditions, each element \((\sigma, \varphi) \in \Omega \) determines a unique solution \( y(\sigma, \varphi) \) of the system (2.1) through \((\sigma, \varphi) \in \Omega \) on its maximal interval of existence, which depends continuously on the initial data [19].

For fixed \( t^* \), let us introduce functions
\[
\varphi, \delta \in C^i((t^*, \infty), \mathbb{R}^n) \quad \text{with} \quad \varphi(t) \ll \delta(t) \quad \text{for} \quad t \in (t^*, \infty).
\]

Let us, moreover, define the sets
\[
\Omega := (t^*, \infty) \times C_r,
\]
\[
\omega := \{(t, y) : t > (t_0 - r + t^*)/2, \varphi(t) \ll y \ll \delta(t)\},
\]
with fixed \( t_0 > t^* + r \).

In the sequel we employ the following result which is proved in [6]:

**THEOREM 2.1.** Suppose that:

(i) For all \( t \geq t_0 \) and \( \varphi \in C_r \) for which \( (t + \vartheta, \varphi(\vartheta)) \in \omega \) for any \( \vartheta \in [-r, 0] \), it follows that
\[
\hat{\varphi}'(t) < f^i(t, \varphi) \quad \text{when} \quad \varphi'(0) = \delta'(t) \tag{2.2}
\]
and
\[
\hat{\varphi}'(t) > f^i(t, \varphi) \quad \text{when} \quad \varphi'(0) = \varphi'(t) \tag{2.3}
\]
for any \( i = 1, 2, \ldots, p, 0 \leq p \leq n \).

(ii) For all \( t \geq t_0 \) and \( \varphi \in C_r \) for which \( (t + \vartheta, \varphi(\vartheta)) \in \omega \) for any \( \vartheta \in [-r, 0] \), it follows that
\[
\hat{\varphi}'(t) < f^i(t, \varphi) \quad \text{when} \quad \varphi'(0) = \varphi'(t)
\]
and
\[
\hat{\varphi}'(t) > f^i(t, \varphi) \quad \text{when} \quad \varphi'(0) = \delta'(t)
\]
for any \( i = p + 1, p + 2, \ldots, n \).
Then there exists a uncountable set \( \mathcal{Y} \) of solutions of system (2.1) on \([t_0 - r, \infty)\) such that, for each \( y \in \mathcal{Y} \),

\[
\varrho(t) \ll y(t) \ll \delta(t), \quad t \in [t_0 - r, \infty). \tag{2.4}
\]

For given \( k \in \mathbb{R}^n_0 \), let us consider the integro-functional inequalities

\[
\lambda_i(t) \geq -\text{diag}(I(k, \lambda_1)(t))^{-1}f(t, I(k, \lambda_1)), \tag{2.5}
\]

\[
\lambda_2(t) \leq -\text{diag}(I(k, \lambda_2)(t))^{-1}f(t, I(k, \lambda_2)), \tag{2.6}
\]

on \([t_0, \infty)\) for \( \lambda_j \in C([t_0 - r, \infty), \mathbb{R}^n) \), \( j = 1, 2 \), where

\[
I : \mathbb{R}^n_{\geq 0} \times C([t_0 - r, \infty), \mathbb{R}^n) \to C([t_0 - r, \infty), \mathbb{R}^n)
\]

is defined by

\[
I(k, \lambda)(t) := k^i \exp \left( -\int_{t_0 - r}^t \lambda_i(s) \, ds \right)
\]

for \( i = 1, \ldots, n \), \( t \in [t_0 - r, \infty) \), \( k \in \mathbb{R}^n_{\geq 0} \), \( \lambda \in C([t_0 - r, \infty), \mathbb{R}^n) \).

**Definition 2.2.** We say that the function \( f \) is decreasing with respect to the second argument on \( \Omega \) if, for any \( (t, \varphi) \in \Omega \) and \( (t, \psi) \in \Omega \) such that \( \varphi(\vartheta) \leq \psi(\vartheta), \vartheta \in [-r, 0) \), the inequality \( f(t, \varphi) \geq f(t, \psi) \) holds. We say that the function \( f \) is strongly decreasing with respect to the second argument on \( \Omega \) if, for any \( (t, \varphi) \in \Omega \) and \( (t, \psi) \in \Omega \) such that \( \varphi(\vartheta) \ll \psi(\vartheta), \vartheta \in [-r, 0) \), the inequality \( f(t, \varphi) \gg f(t, \psi) \) holds.

Let us prove the following two theorems concerning the asymptotic behavior of the solutions of system (2.1).

**Theorem 2.3.** Let us suppose that

(i) The function \( f \) is strongly decreasing with respect to the second argument on \( \Omega \).

(ii) There are \( k \in \mathbb{R}^n_{\geq 0} \) and continuous functions \( \lambda_j : [t_0 - r, \infty) \to \mathbb{R}^n \), \( j = 1, 2 \), satisfying \( \lambda_i(t) \ll \lambda_2(t) \) and the inequalities (2.5) and (2.6) for \( t \in [t_0, \infty) \).

Then there is a solution \( y \) of the system (2.1) on \([t_0 - r, \infty)\) satisfying

\[
I(k, \lambda_2)(t) \ll y(t) \ll I(k, \lambda_1)(t) \quad \text{for } t \in [t_0 - r, \infty). \tag{2.7}
\]

**Proof.** For the proof, we use Theorem 2.1. Let us put \( p = n \), \( \delta = I(k, \lambda_1) \), and \( \varrho = I(k, \lambda_2) \). Indeed, in this case the inequality (2.2) holds since, for \( i = 1, 2, \ldots, n \), \( \varphi^i(\vartheta) = \delta^i(\vartheta) \) in view of (2.5) and condition (i) of
Theorem 2.3, we get
\[
\dot{\xi}(t) - f'(t, \varphi) = -\lambda_i(t) I'(k, \lambda_i)(t) - f'(t, \varphi) \\
\leq f'(t, I(k, \lambda_i)) - f'(t, \varphi) < 0
\]
for \( t \geq t_0 \). The inequality (2.3) holds, too, since, for \( i = 1, 2, \ldots, n, \varphi'(0) = \varphi(t) \), in view of condition (i) of Theorem 2.3 and inequality (2.6), we get
\[
\dot{\xi}(t) - f'(t, \varphi) = -\lambda_i(t) I'(k, \lambda_i)(t) - f'(t, \varphi) \\
\geq f'(t, I(k, \lambda_i)) - f'(t, \varphi) > 0
\]
for \( t \geq t_0 \). The inequalities (2.7) follow now immediately from (2.4). The theorem is proved.

Now we consider the integro-functional inequalities
\[
\lambda_i(t) \leq -\text{diag}(I(k, \lambda_i)(t))^{-1}f(t, I(k, \lambda_i)), \quad (2.8)
\]
\[
\lambda_i(t) \geq -\text{diag}(I(k, \lambda_i)(t))^{-1}f(t, I(k, \lambda_i)), \quad (2.9)
\]
on \([t_0, \infty)\) for \( \lambda_j \in C([t_0 - r, \infty) , \mathbb{R}^n) \), \( j = 1, 2 \), and fixed \( k \in \mathbb{R}^n_{>0} \).

**THEOREM 2.4.** Let us suppose that
(i) For any \( M \geq 0, \theta \geq t_0 \), there are \( K_1, K_2 \) with
\[
|f(t, y) - f(t', y')| \leq K_1|t - t'| + K_2\|y - y'\|
\]
for any \( t, t' \in [t_0, \theta] \), and any \( y, y' \in C \), with \( \|y\|, \|y'\| \leq M \), where \( \|\cdot\| \) denotes the norm in \( C \).

(ii) There are \( k \in \mathbb{R}^n_{>0} \) and functions \( \lambda_j \in C([t_0 - r, \infty) , \mathbb{R}^n) \), \( j = 1, 2 \), satisfying \( \lambda_j(t) \leq \lambda_j(t) \) on \([t_0 - r, \infty) \), and the inequalities (2.8) and (2.9) on \([t_0, \infty) \),\), satisfying \( \varphi(t_0) = 0 \) and
\[
\lambda_i(t) \leq -\text{diag}(I(k, \lambda_i)(t))^{-1}f(t_0, I(k, \lambda_i)) + \varphi(t), \\
\lambda_i(t) \geq -\text{diag}(I(k, \lambda_i)(t))^{-1}f(t_0, I(k, \lambda_i)) + \varphi(t),
\]
on \([t_0 - r, t_0] \).
(iv) For any functions \( \Lambda_j \in C([t_0 - r, \infty), \mathbb{R}^n) \), \( j = 1, 2 \), with \( \Lambda_1(t) \leq \Lambda_2(t) \) for \( t \geq t_0 - r \), we have

\[
\mathrm{diag}(I(k, \Lambda_1)(t))^{-1} f(t, I(k, \Lambda_1)),
\]

\[
\geq \mathrm{diag}(I(k, \Lambda_2)(t))^{-1} f(t, I(k, \Lambda_2)), \quad t \in [t_0, \infty).
\]

Then there is a solution \( y \) of the system (2.1) on \([t_0 - r, \infty)\) satisfying

\[
I(k, \Lambda_2)(t) \leq y(t) \leq I(k, \Lambda_1)(t) \quad \text{for } t \in [t_0 - r, \infty).
\]

Proof. First we show that the equation

\[
\lambda(t) = -\mathrm{diag}(I(k, \lambda)(t))^{-1} f(t, I(k, \lambda)), \quad t \in [t_0, \infty),
\]

has a solution \( \lambda \in C([t_0 - r, \infty), \mathbb{R}^n) \) satisfying \( \lambda_1(t) \leq \lambda(t) \leq \lambda_2(t) \) for \( t \geq t_0 - r \). Then the proof will be ended if we put \( y = I(k, \lambda) \).

For \( \theta \in (t_0, \infty) \), we introduce the Banach space

\[
\mathcal{L}_\theta := C([t_0 - r, \theta], \mathbb{R}^n)
\]

defined by the continuous functions on \([t_0 - r, \theta]\) into \( \mathbb{R}^n \) equipped with the maximum norm. Further, we introduce the closed, normal cone

\[
\mathcal{F}_\theta := C([t_0 - r, \theta], \mathbb{R}^n_{\geq 0})
\]

defined by the continuous functions on \([t_0 - r, \theta]\) into \( \mathbb{R}^n_{\geq 0} \). By the cone \( \mathcal{F}_\theta \), a partial ordering \( \leq \) in \( \mathcal{L}_\theta \) is given: For \( \lambda, \mu \in \mathcal{L}_\theta \), we say \( \lambda \leq \mu \) if and only if \( \mu - \lambda \in \mathcal{F}_\theta \). We introduce the operator \( T_\theta : \mathcal{L}_\theta \to \mathcal{L}_\theta \) defined by

\[
(T_\theta \lambda)(t) = \begin{cases} 
-\mathrm{diag}(I(k, \lambda)(t))^{-1} f(t, I(k, \lambda)), & t \in [t_0, \theta], \\
-\mathrm{diag}(I(k, \lambda)(t_0))^{-1} f(t_0, I(k, \lambda)_0) + \varphi(t) & \text{for } t \in [t_0 - r, t_0).
\end{cases}
\]

Let \( \lambda, \mu \in \mathcal{L}_\theta \) with \( \lambda \leq \mu \). Condition (iv) implies that \( T_\theta \) is monotone increasing, i.e., \( T_\theta \lambda \leq T_\theta \mu \) if \( \lambda \leq \mu \). Let \( \mu_{\theta, j} := \lambda_j \mid_{[t_0 - r, \theta]} \). The conditions (ii), (iii) imply

\[
\mu_{\theta, 1} \leq T_\theta \mu_{\theta, 1} \leq T_\theta^2 \mu_{\theta, 1} \leq \cdots \leq T_\theta^2 \mu_{\theta, 2} \leq T_\theta \mu_{\theta, 2} \leq \mu_{\theta, 2}.
\]

In order to show the convergence of \( (T_\theta^n \mu_{\theta, 1})_{m=0}^n, (T_\theta^n \mu_{\theta, 2})_{m=0}^n \) to fixed points of \( T_\theta \), we need the compactness of \( T_\theta \). For it, let \( \mathcal{L} \) be a bounded subset of \( \mathcal{F}_\theta \). We have to show that \( T_\theta \mathcal{L} \) is a relatively compact subset of \( \mathcal{L}_\theta \). Due to the Theorem of Arzelà and Ascoli, it is enough to show the
boundedness and equicontinuity of $T_0 \mathcal{L}$. The boundedness of $T_0 \mathcal{L}$ follows from condition (i) with $t = t'$. Now we want to find a number $K$ with
\[
\| (T_0 \lambda)(t) - (T_0 \lambda)(t')\| \leq K |t - t'|
\]
for any $\lambda \in \mathcal{L}$ and any $t, t' \in [t_0 - r, \theta]$. Thanks to condition (iii), we can restrict us to $t, t' \geq t_0$. Thanks to the boundedness of $\mathcal{L}$, there is a $K_0$ with $\|\lambda\| \leq K_0$ for any $\lambda \in \mathcal{L}$. It is easy to show that there are numbers $K_3, K_4, K_5, K_6$ with
\[
\begin{align*}
|I'(k, \lambda)(t)| &\leq K_3, \\
|I'(k, \lambda)(t) - I'(k, \lambda)(t')| &\leq K_4 |t - t'|, \\
|I'(k, \lambda) - I'(k, \lambda)'| &\leq K_5 |t - t'|, \\
|f'(t, I(k, \lambda))| &\leq K_6,
\end{align*}
\]
for any $\lambda \in \mathcal{L}$ and any $t, t' \in [t_0, \theta]$. Thanks to condition (i), we have
\[
\begin{align*}
|\text{diag}(I(k, \lambda)(t)) &- \text{diag}(I(k, \lambda)(t'))| \\
&\leq \max_i |I'(k, \lambda)(t) - I'(k, \lambda)(t')| |f'(t, I(k, \lambda))| \\
&\quad + \max_i |I'(k, \lambda)(t')| |f'(t, I(k, \lambda)) - f'(t', I(k, \lambda))| \\
&\leq (K_4 K_6 + K_5 (K_1 + K_2 K_3)) |t - t'|.
\end{align*}
\]
Thus, $T_0 \mathcal{L}$ is equicontinuous and the compactness of $T_0$ is shown.

Now we are in position to apply Theorem 7.A in [32] about the monotone iterative method in order to infer the existence of fixed points $\mu_{\varrho, 1}$ and $\mu_{\varrho, 2}$ of $T_0$ with $T_0^m \mu_{\varrho, 1} \succ \mu_{\varrho, 1}, T_0^m \mu_{\varrho, 2} \succ \mu_{\varrho, 2}$ as $m \to \infty$ and
\[
\mu_{\varrho, 1} \leq \mu_{\varrho, 1} \leq \mu_{\varrho, 2} \leq \mu_{\varrho, 2}.
\]
Since $T_0 \lambda \mid_{[t_0 - r, \theta]} = (T_0 \lambda \mid_{[t_0 - r, \theta]})$ for $\Theta \geq \varrho$ and $\lambda \in C([t_0 - r, \infty), \mathbb{R}^n)$, we have $\mu_{\varrho,j} = \mu_{\varrho,j} \mid_{[t_0 - r, \theta]}$ for $\Theta \geq \varrho$, $j = 1, 2$. Thus, the functions $\lambda_1, \lambda_2 \in C([t_0 - r, \infty), \mathbb{R}^n)$ defined by
\[
\lambda_j(t) = \begin{cases} 
\mu_{t+1,j}(t) & \text{for } t < t_0 + 1, \\
\mu_{t,j}(t) & \text{for } t \geq t_0 + 1,
\end{cases}
\]
satisfy
\[
\lambda_1(t) \leq \lambda_1(t) \leq \lambda_2(t) \leq \lambda_2(t), \quad t \in [t_0 - r, \infty),
\]
and

$$\bar{\lambda}_j(t) = -\text{diag}(I(k, \bar{\lambda}_j)(t))^{-1} f(t, I(k, \bar{\lambda}_j)), \quad j = 1, 2, \quad t \in [t_0, \infty).$$

Choosing $\lambda = \bar{\lambda}_i$, the proof is completed.

3. THE LINEAR CASE

Theorems 2.3 and 2.4 will be applied to the investigation of the delayed equation (1.14) with critical coefficient. Therefore, we reformulate them as corollaries for linear equations of the type (1.6). [For this in the above theorems, put $\nu = 1$, $k = 1$, and $f(t, x) = -c(t)x(t - \tau(t))$.]

**Corollary 3.1.** Let us suppose that there are continuous functions

$$\lambda_j: \mathcal{I}_{-}(t_0) \to \mathbb{R}, \quad j = 1, 2,$$

satisfying $\lambda_1(t) < \lambda_2(t)$ for $t \in \mathcal{I}_{-}(t_0)$ and the inequalities

$$\lambda_1(t) \geq c(t) \exp \left( \int_{t-\tau(t)}^{t} \lambda_1(s) \, ds \right), \quad \text{for } t \in \mathcal{I}(t_0).$$

$$\lambda_2(t) \leq c(t) \exp \left( \int_{t-\tau(t)}^{t} \lambda_2(s) \, ds \right), \quad \text{for } t \in \mathcal{I}_{-}(t_0).$$

Then there is a solution $x$ of (1.6) on $\mathcal{I}_{-}(t_0)$ such that

$$\exp \left( -\int_{t_0-\tau(t_0)}^{t} \lambda_2(s) \, ds \right) < x(t)$$

$$< \exp \left( -\int_{t_0-\tau(t_0)}^{t} \lambda_1(s) \, ds \right), \quad t \in \mathcal{I}_{-}(t_0).$$

**Corollary 3.2.** Let us suppose that there are continuous functions

$$\lambda_j: \mathcal{I}_{-}(t_0) \to \mathbb{R}, \quad j = 1, 2,$$

and a Lipschitz continuous function $\varphi: [t_0 - \tau(t_0), t_0] \to \mathbb{R}$ satisfying $\varphi(t_0) = 0$, $\lambda_1(t) \leq \lambda_2(t)$ for $t \in \mathcal{I}_{-}(t_0)$, the inequalities

$$\lambda_1(t) \leq c(t) \exp \left( \int_{t-\tau(t)}^{t} \lambda_1(s) \, ds \right), \quad \text{for } t \in \mathcal{I}(t_0).$$

$$\lambda_2(t) \leq c(t) \exp \left( \int_{t-\tau(t)}^{t} \lambda_2(s) \, ds \right), \quad \text{for } t \in \mathcal{I}_{-}(t_0).$$
on $\mathcal{I}(t_0)$, and the inequalities
\begin{align}
\lambda_1(t) &\leq c(t_0) \exp\left(\int_{t_0}^{t} \lambda_1(s) \, ds\right) + \varphi(t), \quad (3.6) \\
\lambda_2(t) &\geq c(t_0) \exp\left(\int_{t_0}^{t} \lambda_2(s) \, ds\right) + \varphi(t), \quad (3.7)
\end{align}
on $[t_0 - \tau(t_0), t_0]$, where $c$ is locally Lipschitz continuous. Then there is a solution $x$ of (1.6) on $\mathcal{I}_-(t_0)$ such that
\begin{align}
\exp\left(-\int_{t_0}^{t} \lambda_2(s) \, ds\right) \leq x(t) \\
\leq \exp\left(-\int_{t_0}^{t} \lambda_1(s) \, ds\right), \quad t \in \mathcal{I}_-(t_0).
\end{align}

4. PROOF OF THE MAIN RESULT

The following lemmas (the proofs of which can be made easily by use of the binomial formula and the method of induction and therefore are omitted) concerning asymptotic expansions of indicated functions will be used also in the proof of the main result. All computations will have sufficient accuracy for further application (everywhere we suppose $0 < \tau = \text{const}$). We shall, as usual, write $g(t) = o(\sigma(t))$ if the function $g(t)$ is of higher order than $\sigma(t)$ as $t \to \infty$, i.e., $g(t) = e(t)\sigma(t)$, where $e(t)$ tends to zero as $t \to \infty$.

**Lemma 4.1.** For $t \to \infty$ and $\sigma \in \mathbb{R}$, the following asymptotic representation holds:

\[
(t - \tau)^\sigma = t^\sigma \left[1 - \frac{\sigma \tau}{t} + \frac{\sigma(\sigma - 1)\tau^2}{2t^2} - \frac{\sigma(\sigma - 1)(\sigma - 2)\tau^3}{6t^3} + o\left(\frac{1}{t^3}\right)\right].
\]

**Lemma 4.2.** For $k \geq 1$, $\sigma \in \mathbb{R}$, and $t \to \infty$, the following asymptotic representation holds:

\[
\ln_k^\sigma(t - \tau) = \ln_k t^\sigma \left[1 - \frac{\sigma \tau}{t \ln t \ln_2 t \cdots \ln_{k-1} t \ln_k t}\right].
\]
Now we are in position to prove Theorem 1.8.

**Proof of Theorem 1.8.** Define an auxiliary function \( \lambda_p^\delta : (\exp_{p-1} \cdot \infty) \to \mathbb{R} \), where \( p \in \{0, 1, 2, \ldots\} \) and \( \delta \in \mathbb{R} \), by means of the relation

\[
\lambda_p^\delta(t) = \frac{1}{\tau} - \frac{1}{2t} - \frac{1}{2t \ln t} - \cdots - \frac{1}{2t \ln t \ln \cdots \ln_{p-1} t} - \frac{\delta}{2t \ln t \ln \cdots \ln_{p} t}.
\]

We study the relation between the expressions

\[
\lambda_{p+1}(t) \quad \text{and} \quad R_p^\delta(t) := c_p(t) \exp \left( \int_{t-\tau}^t \lambda_{p+1}(s) \, ds \right)
\]

for \( t \to \infty \). For \( R_p^\delta \), we obtain

\[
R_p^\delta(t) = e \cdot c_p(t) \cdot \left( \frac{t - \tau}{t} \right)^{1/2} \cdot \left( \frac{\ln(t - \tau)}{\ln t} \right)^{1/2} \cdot \cdots \cdot \left( \frac{\ln_p(t - \tau)}{\ln_p t} \right)^{1/2} \cdot \left( \frac{\ln_{p+1}(t - \tau)}{\ln_{p+1} t} \right)^{\delta/2}.
\]
Now we develop an asymptotic representation of \( R_p^\delta(t) \) for \( t \to \infty \). With the aid of Lemmas 4.1 and 4.2, we get

\[
R_p^\delta(t) = \left[ \frac{1}{\tau} + \frac{\tau}{8t^2} + \frac{\tau}{8(t \ln t)^2} + \frac{\tau}{8(t \ln t \ln_2 t)^2} + \cdots \right]
\]

\[
\times \left[ \frac{\tau}{8(t \ln t \ln_2 t \cdots \ln_p t)^2} \right]
\]

\[
\times \left[ 1 - \frac{\tau}{2t} - \frac{\tau^2}{8t^2} - \frac{\tau^3}{16t^3} + o\left( \frac{1}{t^3} \right) \right]
\]

\[
\times \left[ 1 - \frac{\tau}{2t \ln t} - \frac{\tau^2}{4t^2 \ln t} - \frac{\tau^2}{8(t \ln t)^2} + o\left( \frac{1}{t^3} \right) \right]
\]

\[
\times \left[ 1 - \frac{\tau}{2t \ln t \ln_2 t} - \frac{\tau^2}{4t^2 \ln t \ln_2 t} - \frac{\tau^2}{4(t \ln t)^2 \ln_2 t}
\]

\[
- \frac{\tau^2}{8(t \ln t \ln_2 t)^2} + o\left( \frac{1}{t^3} \right) \right]
\]

\[
\times \left[ 1 - \frac{\tau}{2t \ln t \ln_2 t \cdots \ln_{p-1} t \ln_p t}
\]

\[
- \frac{\tau^2}{4t^2 \ln t \ln_2 t \cdots \ln_{p-1} t \ln_p t}
\]

\[
- \frac{\tau^2}{4(t \ln t)^2 \ln_2 t \cdots \ln_{p-1} t \ln_p t} - \cdots
\]

\[
- \frac{\tau^2}{4(t \ln t \ln_2 t \cdots \ln_{p-1} t \ln_p t)^2 \ln_2 t}
\]

\[
- \frac{\tau^2}{8(t \ln t \ln_2 t \cdots \ln_{p-1} t \ln_p t)^2} + o\left( \frac{1}{t^3} \right) \right]
\]

\[
\times \left[ 1 - \frac{\delta \tau}{2t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1} t} \right]
\]
After a necessary computation, we have

\[
R_p^\delta(t) = \frac{1}{\tau} - \frac{1}{2t} - \frac{1}{2t \ln t} - \cdots - \frac{1}{2t \ln t \ln_2 t \cdots \ln_p t} - \frac{1}{2t \ln t \ln_2 t \cdots \ln_{p+1} t} \]

\[
+ \frac{\delta}{8(t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1} t)^2} \left[ \tau^2 - \frac{1}{16t^3} + o\left(\frac{1}{t^3}\right) \right].
\]

or

\[
R_p^\delta(t) = \lambda_{p+1}^\delta(t) + \frac{\delta(\delta - 2)\tau}{8(t \ln t \ln_2 t \cdots \ln_p t \ln_{p+1} t)^2} - \frac{\tau^2}{16t^3} + o\left(\frac{1}{t^3}\right),
\]

(4.1)

Now, on the basis of relation (4.1), we conclude that

\[
\lambda_{p+1}^\delta(t) > R_p^\delta(t) \quad \text{if } 0 \leq \delta \leq 2 \quad (4.2)
\]

and

\[
\lambda_{p+1}^\delta(t) < R_p^\delta(t) \quad \text{if } \delta \in (-\infty, 0) \cup (2, +\infty) \quad (4.3)
\]

for \( t \to \infty \). Let us end the proof with the aid of Corollaries 3.1 and 3.2.

Because of (4.2) and (3.1), there is a \( T_1 \geq t_0 \) such that the integral inequalities (3.1), (3.2) are valid for \( c(t) := c_p(t), \quad \tau(t) := \tau = \text{const, } t \in \mathcal{T}(T_1) \) and for \( \lambda_1 := \lambda_{p+1}^0 |_{\mathcal{T}(T_1)}, \quad \lambda_2 := \lambda_{p+1}^\delta |_{\mathcal{T}(T_1)} \), where the constant
\( \delta_2 \) is taken from the interval \((-\infty,0)\). Clearly, \( \lambda_1(t) < \lambda_2(t) \) for \( t \in \mathcal{I}_1(T_1) \). All conditions of Corollary 3.1 hold with \( T_1 \) instead of \( t_0 \).

Further, there is a \( T_2 \geq t_0 \) such that the integral inequalities (3.4), (3.5) are valid for \( t \in \mathcal{A}(T_2) \), \( \lambda_1 := \lambda_{p+1}^{\delta_1} \big| \mathcal{A}(T_2) \), and \( \lambda_2 := \lambda_{p+1}^{\delta_2} \big| \mathcal{A}(T_2) \), where \( \delta_1 \) is taken from the interval \((2,\infty)\). We see that \( \lambda_1(t) < \lambda_2(t) \) on \( \mathcal{I}_1(T_2) \). The initial inequalities (3.6), (3.7) with \( t_0 = T_2 \) hold on \([T_2 - \tau, T_2] \) with \( \varphi(t) = \lambda_1(t) - \lambda_2(t_0) \). The first inequality (3.6) follows from (3.4) with \( t = T_2 \). In order to show (3.7), we proceed as follows. In view of the definition of \( R_p, \lambda_1, \lambda_2 \), and (4.1), we have

\[
c_p(T_2) \exp \left( \int_{T_2}^{T_2} \lambda_2(s) \, ds \right) = R_p^2(T_2) = \lambda_2(T_2) - \frac{\tau^2}{16T_2^2} + o \left( \frac{1}{T_2^3} \right)
\]

and

\[
\lambda_1(t) = \lambda_2(t) - \frac{-2 + \delta_1}{2t \ln t \cdots \ln_{p+1} t}, \quad t \in \mathcal{I}_1(T_2),
\]

such that

\[
c_p(T_2) \exp \left( \int_{T_2}^{T_2} \lambda_2(s) \, ds \right) + \lambda_1(t) - \lambda_1(T_2)
\]

\[
= R_p^2(T_2) + \lambda_1(t) - \lambda_1(T_2)
\]

\[
= \lambda_2(t) - \frac{-2 + \delta_1}{2t \ln t \cdots \ln_{p+1} t} + \frac{-2 + \delta_1}{2T_2 \ln T_2 \cdots \ln_{p+1} T_2}
\]

\[
- \frac{\tau^2}{16T_2^2} + o \left( \frac{1}{T_2^3} \right)
\]

\[
\leq \lambda_2(t) - \frac{\tau^2}{16T_2^2} + o \left( \frac{1}{T_2^3} \right)
\]

on \([T_2 - \tau, T_2]\). Choosing \( T_2 \) large enough, the inequality (3.7) follows on \([T_2 - \tau, T_2]\). Since \( c_p \) and \( \lambda_1 \) are Lipschitz continuous, all assumptions of Corollary 3.2 are satisfied with \( T_2 \) instead of \( t_0 \).

Without loss of generality we may assume \( T_1 = T_2 = T \geq t_0 \).

Choosing \( T \) large enough, we have \( \ln_{p+1}^{\delta_1}(t - \tau) > 1 \) and \( \ln_{p+1}^{\delta_2}(t - \tau) < 1 \) for \( t \geq T \). The two-sided estimates (1.16) and (1.15) follow now from the inequalities (3.3), (3.8), and the linearity of the considered equation. Thus the proof is finished. \( \square \)
5. GENERALIZATION

Let us consider a more common equation
\[ \dot{x}(t) = -\left(c_p(t) + f_p(t)\right)x(t - \tau). \]  
(5.1)

We will suppose that the function \( f_p \) is locally Lipschitz continuous on \( \mathcal{A}(t_0), \ t_0 > \exp_{p-1} 1 \), and such that (5.1) is nonoscillatory. Then the following generalization of Theorem 1.8 holds:

**Theorem 5.1.** Let \( p \in [0] \cup \mathbb{N} \) be fixed, let \( f_p = F_p^1 + F_p^2, F_p^1(t) \leq 0, \)
\[ F_p^1(t) = o \left( \frac{1}{(t \ln t \ln_2 t \cdots \ln_{p+1} t)^2} \right) \quad \text{for} \ t \to \infty, \]
and, moreover, let there be a constant \( \varepsilon \in (0, 1) \) such that
\[ |F_p^2(t)| < \frac{\tau \varepsilon}{16e t^3}. \]
Then for any fixed constants \( \delta_1 > 2 \) and \( \delta_2 < 0 \), there are \( T \geq t_0 \) and a pair \((x_1, x_2)\) of dominant and subdominant solutions of the delayed equation (1.14) on \( \mathcal{F}_- (T) \) satisfying (1.15) and (1.16) on \( \mathcal{F}_- (T) \).

**Proof.** The proof can be done in the same manner as the proof of Theorem 1.8 if, instead of the function \( c_p \), the function \( c_p + f_p \) is considered, and therefore is omitted. We will focus our attention only on the central moment of the proof. Define
\[ R_p^\delta(t) := (c_p(t) + f_p(t))\exp \left( \int_{t-\tau}^t \lambda_p^\delta(s) \, ds \right). \]
Instead of relation (4.1), we get the relation
\[ R_p^\delta(t) = \lambda_p^\delta(t) + \frac{\delta(\delta - 2)\tau}{8(t \ln t \ln_2 t \cdots \ln_{p+1} t)^2} \]
\[ + eF_p^1(t) + eF_p^2(t) - \frac{\tau^2}{16t^3} + o \left( \frac{1}{t^3} \right). \]
Now in view of the properties of the functions \( F_p^1, F_p^2 \), it is clear that the inequalities (4.2), (4.3) as well as consequent affirmations remain valid.
6. OPEN QUESTIONS

There are some questions concerning an improvement of the inequalities (1.15) and (1.16):

1. Is there a solution \( x_1 \) of the delayed equation (1.14) with the asymptotic behavior
   \[
   x_1(t) = e^{-\frac{t}{\tau}} \sqrt{t \ln t \ln_2 t \cdots \ln_{p-1} t} \cdot (1 + \omega_1(t))
   \]
   for \( t \to \infty \), where the function \( \omega_1 \) is eventually positive and vanishing?

2. Is there a solution \( x_2 \) of (1.14) with the asymptotic behavior
   \[
   x_2(t) = e^{-\frac{t}{\tau}} \sqrt{t \ln t \ln_2 t \cdots \ln_{p-1} t} \cdot (1 - \omega_2(t))
   \]
   for \( t \to \infty \), where the function \( \omega_2 \) is eventually positive and vanishing?

3. Is solving of the previous two questions in a certain sense equivalent? These problems are connected with the investigation of the asymptotic properties of corresponding differential equations for \( \omega_1 \) and \( \omega_2 \).

4. Let \( (x_1, x_2) \) be a pair of dominant and subdominant solutions of (1.14). If \( x_2 \) satisfies the two-sided estimate (1.16), then it satisfies the inequality (1.13). Does it mean that inequality (1.13) always gives only information concerning \( x_2 \) and not about an \( x_1 \)?

7. SOME COMPARISONS AND REMARKS

Let us define the functions \( c_{-\frac{1}{p}}(t) = 1/(\tau e) \) and \( \ln_{-1} t = 1 \). Then (in view of linearity) Theorem 1.8 holds for \( p = -1 \), too. This follows from (1.3).

Every couple of inequalities—(3.1), (3.2) in Corollary 3.1 or (3.4), (3.5) in Corollary 3.2—contains an inequality of the type (1.11) (namely, inequalities (3.1), (3.5)) which gives a guarantee of existence of a positive solution of the delayed equation (1.6). With the aid of these couples, curvilinear tubes containing the graph of corresponding solutions were constructed.

Let \( (x_1, x_2) \) be a fixed pair of dominant and subdominant solutions of the delayed equation (1.6) on \( \mathcal{I}_{-1}(t_0) \). Define the set
   \[
   \mathcal{Z}(t_0) = \{kx_1: k \in \mathbb{R}\},
   \]
and let \( \mathcal{Z}(t_0) \) be the set of all solutions \( x \) of (1.6) on \( \mathcal{I}_{-1}(t_0) \) with \( x(t) = O(x_i(t)) \). As follows from the facts mentioned in the introduction, the set \( \mathcal{Z}(t_0) \) is independent of the concrete choice of the pair \( (x_1, x_2) \),
and any solution \( x \) of (1.6) on \( \mathcal{I}_{-1}(t_0) \) can be represented in a unique way by

\[
x = y_1 + y_2,
\]

with \( y_1 \in \mathcal{I}_1(t_0), \ y_2 \in \mathcal{I}_2(t_0) \). Therefore, any solution \( x = y_1 + y_2 \) with \( y_1 \in \mathcal{I}_1 \setminus \{0\} \), \( y_2 \in \mathcal{I}_2(t_0) \) can serve as an analogy to a “special solution” which is used in many investigations (see, e.g., Pituk [29], Ryabov [30]). Moreover, each solution \( x \in \mathcal{I}_2(t_0) \) of (1.6) on \( \mathcal{I}_{-1}(t_0) \) can serve as an analogy to “small solutions” as they are used, for example, in Hale and Lunel [19]. Note that the asymptotic theory for functional systems of penetration type is contained in the paper by Kozakiewicz [24].

A similar asymptotic behavior of solutions of the delayed equation (1.1) and of second-order ordinary differential equation without delay

\[
y''(t) + a(t)y(t) = 0
\]

with a positive function \( a \) was analyzed in [10] (see also [13]). It is known that all solutions of (7.1) oscillate if \( a(t) \geq a^A_p(t) \) for some integer \( p \geq 0 \), sufficiently large \( t \), and an \( A > 0 \), where the corresponding critical function \( a^A_p(t) \) has the form

\[
a^A_p(t) = \frac{1}{4t^2} + \frac{1}{4(t \ln t)^2} + \frac{1}{4(t \ln t \ln t \cdots \ln_p t)^2} + \frac{1 + A}{4(t \ln t \ln t \cdots \ln_p t)^2},
\]

and are nonoscillatory if \( a(t) \leq a^0_p(t) \) for some integer \( p \geq 0 \). On the basis of the connection between the critical functions \( c_p \) and \( a^0_p \),

\[
c_p(t) = \frac{1}{\tau e} + \frac{\tau}{2e} a^0_p(t), \quad t > \exp_{p-1} 1,
\]

the following property (as a consequence of Theorem 1.7) was formulated by Á. Elbert (oral communication) which we present with his permission:

**THEOREM 7.1.** Suppose

\[
c(t) = \frac{1}{\tau e} + \frac{\tau}{2e} a(t), \quad t \geq t_0 > \exp_{p-1} 1.
\]

Then (1.1), (7.1) are nonoscillatory on \( \mathcal{I}_{-1}(T) \) for a \( T \geq t_0 \) if \( a(t) \leq a^0_p(t) \) for an integer \( p \geq 0 \), \( t \in \mathcal{I}(t_0) \), and they are oscillatory on \( \mathcal{I}_{-1}(T) \) if \( a(t) \geq a^A_p(t) \), for an integer \( p \geq 0 \), \( t \in \mathcal{I}(t_0) \) and an \( A > 0 \).
The equation (7.1) with $a := a_p^0$ admits two positive solutions
\[ x_1(t) = \sqrt{t \ln t \ln^2 t \cdots \ln^p \ln^{p+1} t}, \quad x_2(t) = \sqrt{t \ln t \ln^2 t \cdots \ln^p t}. \]

Even in this direction, in view of the left-hand side of (1.15) and the right-hand side of (1.16), there is a formal connection between (1.14) and (7.1) since the corresponding functions are multiplied by $e^{-t/\tau}$. For establishing further connections it will be, perhaps, useful to substitute $x = e^{-t/\tau} w$ in (1.14) and to investigate the equation for $w$:
\[ \dot{w}(t) = \frac{1}{\tau} [w(t) - w(t - \tau)] - \frac{\tau}{2} a_p^0(t) w(t - \tau). \]

Some classes of equations of similar kind were considered, e.g., by Arino and Győri [1], Atkinson and Haddock [2], Čermák [4, 5], Diblík [7, 8, 10], Győri and Pituk [18], and Slater [31]. Some close questions are discussed in the papers of Čermák [3] and of Pituk [28, 29]. A connection (7.2) between the functions $c$ and $a$ can serve as an inspiration for further investigation of common or similar properties of solutions of (1.1) and (7.1). Note that the papers [3, 4] use the fundamental results of Neuman (see, e.g., [26, 27]) concerning the transformation theory.

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