Prof. Dr. Martin Keller-Ressel

TU Dresden
Institute of mathematical stochastics

Exercises in Financial Mathematics

Exercise sheet 1

Due date: November 1, 2017

Exercise 1.

a) Complete the proof of Lemma 0.1 from the lecture: use an elementary no-abitrage argument to show that $C_t \leq S_t$, where C_t denotes the price of a European call option and S_t the price of the underlying asset.

b) Derive bounds for the price P_t of a European put option along the lines of Lemma 0.1

c) Let $X_t(K)$ be the price of a forward contract on an asset S with strike price K. The value F_t solving

$$X_t(F_t) = 0$$

is called forward price of S at time t. Using an elementary replication argument, show that

$$F_t = \frac{S_t}{B(t,T)},$$

where $B(t,T)$ denotes the price of a zero-coupon bond.

Exercise 2. Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subseteq \mathcal{F}$. The *conditional variance* of X given $\mathcal{G}$ is defined as

$$\mathbb{V}ar\left[X \middle| \mathcal{G}\right] := \mathbb{E}\left[\left(X - \mathbb{E}\left[X \middle| \mathcal{G}\right]\right)^2 \middle| \mathcal{G}\right].$$

Show that

$$\mathbb{V}ar\left[X \middle| \mathcal{G}\right] = \mathbb{E}\left[X^2 \middle| \mathcal{G}\right] - \left(\mathbb{E}\left[X \middle| \mathcal{G}\right]\right)^2$$

and that

$$\mathbb{V}ar\left[X\right] = \mathbb{E}\left[\mathbb{V}ar\left[X \middle| \mathcal{G}\right]\right] + \mathbb{V}ar\left[\mathbb{E}\left[X \middle| \mathcal{G}\right]\right].$$

Exercise 3.

a) Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G} \subseteq \mathcal{F}$ and let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent probability measure. Show the *abstract Bayes formula:*

$$\mathbb{E}^{\mathbb{Q}}\left[X \middle| \mathcal{G}\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[X \cdot \frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{G}\right]}{\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{G}\right]}$$

b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t \in I}$ and let $\mathbb{Q} \ll \mathbb{P}$. Show that the 'density process' $M_t := \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t\right]$ is an $\mathcal{F}_t$-martingale.

c) Show Theorem $1.7\frac{1}{2}$(a) from the lecture: Let X be an adapted stochastic process. If

$$\mathbb{E}\left[(H \circ X)_T\right] = 0$$

holds for any locally bounded predictable process H, then $(X_n)_{n \in \{0,\dots,T\}}$ is a martingale.

Exercise 4. Consider the discounted price process $X_t := \frac{S_t^1}{S_t^0}$ in the CRR (Cox-Ross-Rubinstein)-model. Which assumptions on the returns R_n are needed, in order to make (X_t) a martingale?