Canonical Functions and Constraint Satisfaction

Antoine Mottet
Workshop {Symmetry, Logic, Computation}
Finding general conditions for tractability of infinite-domain CSPs, akin to the finite case
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If possible find decidable conditions.
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Proving complete complexity classifications:
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Proving complete complexity classifications:

**Theorem**

Assume that the finite-domain tractability conjecture holds. If the relations of $\mathbb{A}$ are definable in a unary language, then CSP($\mathbb{A}$) is in $P$ or NP-complete.
Outline

Computation: Constraint Satisfaction

Symmetry: Canonical Functions

Logic Computation
Computation: Constraint Satisfaction

Symmetry: Canonical Functions

Logic Computation
Relational structure: $\mathbb{A} = (A, R_1^A, \ldots, R_k^A)$ with $R_i^A \subseteq A^{r_i}$
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A homomorphism $f : \mathbb{A} \to \mathbb{B}$ is a function such that

$$\forall R_i, \forall (a_1, \ldots, a_{r_i}) \in R_i^A, \ (f(a_1), \ldots, f(a_{r_i})) \in R_i^B$$
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Let $\mathbb{A}$ be a relational structure, in a fixed finite signature $\tau$.

Definition (CSP($\mathbb{A}$))

**Input:** a finite $\tau$-structure $\mathbb{B}$

**Question:** $\exists$ homomorphism $h: \mathbb{B} \rightarrow \mathbb{A}$?
Example (CSP($K_3$))

**Input:** a finite graph $B$

**Question:** Is $B$ 3-colourable?
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Complexity: NP-complete
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Example (CSP($\mathbb{Z}$, $<$))

**Input:** a finite directed graph $\mathbb{B}$

**Question:**
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Example (CSP($\mathbb{Z}$, $<$))

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### Example ($\text{CSP}(K_3)$)

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### Example ($\text{CSP}(\mathbb{Z}, <)$)

**Input:** a finite directed graph $B$

**Question:** Is $B$ acyclic?

**Complexity:** linear time
Example (CSP(\(\mathbb{Z}, +, \times\)))

**Input:** a hypergraph with vertices \(V\) and hyperedges \(E_+(x, y, z)\) and \(E_\times(x, y, z)\)

**Question:**

∃ assignment \(s: V \rightarrow \mathbb{Z}\) such that

\[
\begin{align*}
\{ s(x) + s(y) &= s(z) \} &\in E_+ \\
\{ s(x) \times s(y) &= s(z) \} &\in E_\times
\end{align*}
\]

**Complexity:** undecidable.

**Theorem (Matiyasevich-Davis-Robinson-Putnam)**

Every recursively enumerable set \(S \subseteq \mathbb{Z}\) is the projection on one variable of the set of solutions of some instance of CSP(\(\mathbb{Z}, +, \times\)).
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Conjecture (Feder-Vardi, ’93)

Let $\mathbb{A}$ be a structure with a finite domain. Then $\text{CSP}(\mathbb{A})$ is in P or NP-complete.
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Transition to infinite domains:

- Find a reasonable class $\mathcal{A}$ of infinite structures,
- Classify the complexity of $\text{CSP}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{A}$, assuming the tractability conjecture.
Definition

\( \mathcal{B} \) is \textit{finitely bounded} if there exists a finite family \( \mathcal{F} \) of finite structures such that for all finite \( \mathcal{C} \),

\[
\mathcal{C} \text{ substructure of } \mathcal{B} \iff \forall F \in \mathcal{F}, F \text{ not a substructure of } \mathcal{C}
\]

So the question “\( \mathcal{C} \) substructure of \( \mathcal{B} \)” is decidable.

Example ▶ (\( \mathbb{Q}, < \)): \( \mathcal{F} = \) all 3-element structures that are not linear orders

Example ▶ Universal triangle-free graph: \( \mathcal{F} = \{ \} \).
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**Example**

- $(\mathbb{Q}, <)$: $\mathcal{F} = \text{ all 3-element structures that are not linear orders}$
- Universal triangle-free graph: $\mathcal{F} = \{ \bullet \triangle \bullet \}$.
Definition

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- \((\mathbb{Q}, <)\): \( \mathcal{F} = \) all 3-element structures that are not linear orders
- Universal triangle-free graph: \( \mathcal{F} = \{\begin{array}{c}
\bullet \\
\rightarrow \\
\bullet
\end{array}\}. \)

Definition

\( \mathcal{B} \) is homogeneous if every partial isomorphism with finite domain can be extended to an automorphism.
Definition

A is a reduct of B if the relations of A have a fo-definition in B.
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Conjecture (Bodirsky-Pinsker)

Let A be a reduct of a finitely bounded homogeneous structure. Then CSP(A) is in P or NP-complete.
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Why “reduct of finitely bounded homogeneous structure”:

- $\omega$-categorical structures
- CSP is guaranteed to be in NP
- false if we drop “finitely bounded”
**Definition**

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**Question:** How to prove it?
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Why “reduct of finitely bounded homogeneous structure”:

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**Question:** How to prove it, assuming the finite-domain conjecture?
BP conjecture is confirmed for:

- Reducts of \((\mathbb{N},=)\) (Bodirsky, Kára, ’06)
- Reducts of the Rado graph (Bodirsky, Pinsker, JACM’15)
- Reducts of a homogeneous graph (Bodirsky, Martin, Pinsker, Pongrácz, ICALP’16)
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In the first 3 cases, the classification is of the form:

**Theorem (xxx)**

\(\mathbb{A}\) has a *canonical polymorphism* and CSP(\(\mathbb{A}\)) is in P, or CSP(\(\mathbb{A}\)) is NP-complete.
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In the first 3 cases, the classification is of the form:

**Theorem (xxx)**

\(\mathbb{A}\) has a *canonical* polymorphism and CSP(\(\mathbb{A}\)) is in \(P\), or CSP(\(\mathbb{A}\)) is NP-complete.

Not true for \((\mathbb{Q}, <)\).
Computation: Constraint Satisfaction

Symmetry: Canonical Functions

Logic Computation
G ≤ Sym(X), orbit of a ∈ X^m is \{(α · a_1, ..., α · a_m) : α ∈ G\}
\[ G \leq \text{Sym}(X), \text{ orbit of } a \in X^m \text{ is } \{(\alpha \cdot a_1, \ldots, \alpha \cdot a_m) : \alpha \in G\} \]

\[ G \text{ is oligomorphic} \text{ if for all } m \geq 1, \text{ there are finitely many orbits of } m\text{-tuples of } X \text{ under } G. \]
Oligomorphic groups, clones

- \( G \leq \text{Sym}(X) \), orbit of \( a \in X^m \) is \( \{(\alpha \cdot a_1, \ldots, \alpha \cdot a_m) : \alpha \in G\} \)

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- A function clone \( \mathcal{C} \) is a subset of \( \bigcup_{n \geq 1} X^{X^n} \) closed under composition and containing projections.
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Natural topology: \( (f_i) \rightarrow f \) iff for arbitrarily large finite sets \( X' \subset X \), there is \( i_0 \) such that \( f_j|_{X'} = f|_{X'} \) for \( j \geq i_0 \).
- $G \leq \text{Sym}(X)$, orbit of $a \in X^m$ is $\{(\alpha \cdot a_1, \ldots, \alpha \cdot a_m) : \alpha \in G\}$
- $G$ is oligomorphic if for all $m \geq 1$, there are finitely many orbits of $m$-tuples of $X$ under $G$.
- A function clone $\mathcal{C}$ is a subset of $\bigcup_{n \geq 1} X^X$ closed under composition and containing projections.
- Natural topology: $(f_i) \rightarrow f$ iff for arbitrarily large finite sets $X' \subset X$, there is $i_0$ such that $f_j|_{X'} = f|_{X'}$ for $j \geq i_0$.
- $\phi : \mathcal{C} \rightarrow \mathcal{P}$ is continuous iff for every $g \in \mathcal{C}$, there is a finite set $X' \subset X$ such that $g|_{X'} = h|_{X'} \Rightarrow \phi(g) = \phi(h)$. 
Fix:

- $G \leq \text{Sym}(X)$,
- $f : X^n \to X$
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**Definition**

$f$ is $G$-canonical if

$$\forall \alpha_1, \ldots, \alpha_n \in G, f \circ (\alpha_1, \ldots, \alpha_n) \in \overline{G \cdot f}$$
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Equivalently: \( f \) induces an action on \( G \)-orbits of \( m \)-tuples, for all \( m \geq 1 \). If \( G \) is oligomorphic then \( f \) acts naturally on finite sets.

Theorem (Bodirsky-Pinsker-Tsankov)

Suppose \( G \) is nice. For all \( f : X^n \to X \), there exists \( g \in \overline{G \cdot f} \) which is \( G \)-canonical.
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**Theorem (Bodirsky-Pinsker-Tsankov)**

*Suppose $G$ is nice. For all $f : X^n \rightarrow X$, there exists $g \in GfG$ which is $G$-canonical.*

**Remark:** $G$-canonical functions form a clone.
Symmetry

Example: \((\mathbb{Q}, <)\)

- unary functions: canonical \(\iff\) monotone
Symmetry

Example: \((\mathbb{Q}, \prec)\)

- **unary functions:** canonical \(\Leftrightarrow\) monotone

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Symmetry

Example: \((\mathbb{Q}, <)\)

 Unary functions: canonical ⇔ monotone

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Example of binary function: the lexicographic order

Non-example: the maximum function
Example: $(\mathbb{Q}, <)$

- **Unary functions**: canonical $\leftrightarrow$ monotone

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- **Non-example:** the maximum function
Definition (Mash-up)

$G$-canonical functions $g, h, O, O'$ $G$-orbits of $m$-tuples. $\omega$ is a mash-up of $g, h$ if it is $G$-canonical and

$$\omega(O, O') = g(O, O')$$
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\[
\begin{array}{|c|c|c|}
\hline
\omega & \ldots & O & O' \\
\hline
\vdots & & \vdots & \vdots \\
O & & O & \vdots \\
O' & & O' & \vdots \\
\vdots & & \vdots & \vdots \\
\hline
\end{array}
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\hline
\vdots & & \ldots & \vdots \\
O & & g(O, O') \\
O' & & h(O', O) \\
\vdots & & \vdots & \\
\hline
\end{array}
\]
Definition

$\mathcal{C}, \mathcal{D}$ clones. $\phi: \mathcal{C} \rightarrow \mathcal{D}$ is a clone homomorphism if $\phi(pr^n_i) = pr^n_i$ and $\phi(f \circ (g_1, \ldots, g_n)) = \phi(f) \circ (\phi(g_1), \ldots, \phi(g_n))$. 

Clone homomorphisms preserve equations:

$\forall x, y, f(x, f(y, z)) = f(f(x, y), z) \Rightarrow \forall x, y, \phi(f)(x, \phi(f)(y, z)) = \phi(f)(\phi(f)(x, y), z)$. 

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\( \mathcal{C}, \mathcal{D} \) clones. \( \phi: \mathcal{C} \rightarrow \mathcal{D} \) is a clone homomorphism if \( \phi(pr_i^n) = pr_i^n \) and \( \phi(f \circ (g_1, \ldots, g_n)) = \phi(f) \circ (\phi(g_1), \ldots, \phi(g_n)) \).

Clone homomorphisms preserve equations:

\[
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Definition (Barto, Opršal, Pinsker)

\( \mathcal{C}, \mathcal{D} \) clones. \( \phi: \mathcal{C} \rightarrow \mathcal{D} \) is an \( h1 \) homomorphism if

\[
\phi(f \circ (pr_{j_1}^n, \ldots, pr_{j_k}^n)) = \phi(f) \circ (pr_{j_1}^n, \ldots, pr_{j_k}^n).
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\( \mathcal{C}, \mathcal{D} \) clones. \( \phi : \mathcal{C} \to \mathcal{D} \) is a clone homomorphism if \( \phi(pr^n_i) = pr^n_i \) and \( \phi(f \circ (g_1, \ldots, g_n)) = \phi(f) \circ (\phi(g_1), \ldots, \phi(g_n)) \).

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h1 homomorphisms preserve equations of height 1.
Theorem

\( \mathcal{C} \) an oligomorphic closed core clone with *mash-ups*. TFAE:

1. there is a continuous \( h_1 \) homomorphism \( \mathcal{C} \to \mathcal{P} \) that preserves left-composition with unary operations;

Any of these properties is decidable!
Theorem

\( \mathcal{C} \) an oligomorphic closed core clone with mash-ups. TFAE:

1. there is a continuous \( h_1 \) homomorphism \( \mathcal{C} \to \mathcal{P} \) that preserves left-composition with unary operations;
2. there is a \( h_1 \) homomorphism \( \mathcal{C} \to \mathcal{P} \) that preserves left-composition with unary operations;
Theorem

$\mathcal{C}$ an oligomorphic closed core clone with mash-ups. TFAE:

1. there is a continuous $h_1$ homomorphism $\mathcal{C} \to \mathcal{P}$ that preserves left-composition with unary operations;

2. there is a $h_1$ homomorphism $\mathcal{C} \to \mathcal{P}$ that preserves left-composition with unary operations;

3. there is a continuous clone homomorphism $\mathcal{C}^{\text{can}} \to \mathcal{P}$.

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Theorem

\( \mathcal{C} \) an oligomorphic closed core clone with \textit{mash-ups}. TFAE:

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2. there is a \( h_1 \) homomorphism \( \mathcal{C} \to \mathcal{P} \) that preserves left-composition with unary operations;
3. there is a continuous clone homomorphism \( \mathcal{C}^{\text{can}} \to \mathcal{P} \);
4. there is no pseudo-cyclic operation in \( \mathcal{C} \);

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4. there is no pseudo-cyclic operation in \( \mathcal{C} \);
5. there is no pseudo-cyclic operation in \( \mathcal{C}^{\text{can}} \).
Theorem

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\( f \) pseudo-cyclic iff there are \( e_1, e_2 \) such that

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e_1 f(x_1, \ldots, x_n) = e_2 f(x_2, \ldots, x_n, x_1)
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Theorem

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Any of these properties is decidable!
Computation: Constraint Satisfaction

Symmetry: Canonical Functions

Logic Computation
Theorem (Bodirsky-M, LICS’16)

Assume the tractability conjecture, and let $\mathbb{A}$ be a reduct of a finitely bounded homogeneous structure $\mathbb{B}$. If $\mathbb{A}$ has a pseudo-cyclic polymorphism that is $\text{Aut}(\mathbb{B})$-canonical, then $\text{CSP}(\mathbb{A})$ is in P.

In fact, any tractability condition from finite-domain CSPs can be lifted to the BP class.

Theorem
Assume the tractability conjecture. If $\text{Pol}(\mathbb{A})$ has mash-ups, then:

- $\text{Pol}(\mathbb{A}) \rightarrow P$, and $\text{CSP}(\mathbb{A})$ is NP-complete,
- or $\text{Pol}(\mathbb{A})$ contains a canonical pseudo-cyclic operation, and $\text{CSP}(\mathbb{A})$ is in $P$.

Consequence: membership in $P$ is decidable! if $P=NP$ or the tractability conjecture is true.
Theorem (Bodirsky-M, LICS’16)

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if P=NP or the tractability conjecture is true
Where are canonical functions enough?

- MMSNP
- Reducts of fin. bounded. homogeneous
- Definable structures over \((\mathbb{N}, =)\)
- Reducts \((\mathbb{Q}, <)\)
- Reducts unary structures
- Reducts \((\mathbb{N}, 0, 1, \ldots)\)
- Finite-domain CSPs
- Reducts of \((\mathbb{N}, =)\)
- Reducts homog. graphs + constants
- Reducts of homogeneous graphs
- Fix relational signature $\tau$
- MMSNP $\tau$-sentences are of the form

$$\exists M_1 \cdots \exists M_k \forall x \bigwedge \neg (\bigwedge \cdots)$$

Example (No monochromatic triangle)

$$\exists \text{Red} \exists \text{Blue} \forall x, y, z (\neg (\cdots) \land \neg (\cdots))$$

Theorem (Feder-Vardi-Kun)

MMSNP has a $\text{P}/\text{NP}$-complete dichotomy iff finite-domain CSPs have a dichotomy.

Theorem

If all obstructions are monochromatic, then tractability is witnessed by canonical functions.
Fix relational signature $\tau$

MMSNP $\tau$-sentences are of the form

$$\exists M_1 \ldots \exists M_k \forall x \bigwedge \neg (\bigwedge \ldots)$$

Example (No monochromatic triangle)

$$\exists Red \exists Blue \forall x, y, z \left( \neg (\square) \land \neg (\triangle) \right)$$
Fix relational signature $\tau$

MMSNP $\tau$-sentences are of the form

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