Constraint Satisfaction Problems over the Integers with Successor

Manuel Bodirsky, Barnaby Martin, Antoine Mottet
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Let $\sigma$ be a relational signature. A primitive positive $\sigma$-formula is a formula $\phi(x_1, \ldots, x_n)$ of the form

$$\exists x_{n+1}, \ldots, x_{n+r} \bigwedge_{i=1}^{m} R_i(x_{i1}, \ldots, x_{ik}),$$

where $R_i \in (\sigma \cup \{=\})$.

What is the complexity of deciding satisfiability of pp-formulas over $\mathbb{Z}$, depending on $\sigma$?

Examples:

- $\{<, +\}$: in P (consequence of LP tractability and scalability)
- $\{<, +, 1\}$: NP-complete (integer linear programming)
- $\{+, \times\}$: undecidable (Hilbert's Tenth Problem)
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Motivation
Satisfiability of arithmetic formulas

Let $\sigma$ be a relational signature. A primitive positive $\sigma$-formula is a formula $\phi(x_1, \ldots, x_n)$ of the form

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Definition (CSP(Γ))

Let Γ be a relational structure with a finite signature. The constraint satisfaction problem of Γ is the following decision problem:

**Input**: a primitive positive sentence φ in the language of Γ,

**Question**: is φ true in Γ?

The structure Γ is called the template of CSP(Γ).
The CSP of $\langle \mathbb{Z}; \text{succ} \rangle$: 

$\exists x_1, \ldots, x_6 (x_2 = \text{succ}(x_1) \land x_4 = \text{succ}(x_2) \land \ldots) \supseteq = x_1 x_2 x_3 x_4 x_5 x_6$ satisfying assignment $\supseteq$ homomorphism to $\langle \mathbb{Z}; \text{succ} \rangle$.
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- Complexity:

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\exists x_1, \ldots, x_6 (x_2 = \text{succ}(x_1) \land x_4 = \text{succ}(x_2) \land \ldots) \equiv \text{satisfying assignment} \equiv \text{homomorphism to} \ (\mathbb{Z}; \text{succ})
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Definition

Let $\Gamma$ be a relational structure and $R \subseteq \Gamma^n$ be a relation.

Examples:

- The unary $R = \{ x \in \mathbb{Z} : x \text{ is even} \}$ is definable in $(\mathbb{Z}; +)$.
- $\{ (x, y) \in \mathbb{Z}^2 : x \text{ divides } y \}$ is definable in $(\mathbb{Z}; \times)$. 
Definition

Let $\Gamma$ be a relational structure and $R \subseteq \Gamma^n$ be a relation. We say that $R$ is first-order definable in $\Gamma$ if there exists a first-order formula $\phi(x_1, \ldots, x_n)$ in the language of $\Gamma$ such that

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New relations from old
Definable relations and reducts

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Definition
Let $\Gamma, \Delta$ be structures over the same domain. We say that $\Gamma$ is a reduct of $\Delta$ when all the relations of $\Gamma$ are (fo-)definable in $\Delta$. 
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A distance CSP is a CSP whose template is a reduct of \((\mathbb{Z}; \text{succ})\).
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- CSP\((\mathbb{Z}; \text{succ})\): in P,
- Let \(R\) be the ternary relation that contains
  \[(a + 1, a, a), (a, a + 1, a), (a + 1, a + 1, a)\]
  for all \(a \in \mathbb{Z}\).
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Distance Constraint Satisfaction Problems

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- \(\text{CSP}\left(\mathbb{Z}; |x - y| = 1, |x - y| = 5\right):\) \(NP\)-complete,
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**Problem (Complexity classification project for \((\mathbb{Z}; \text{succ})\))**

Give a complete classification of the complexity of distance CSPs.
Previous Work

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Theorem (Bodirsky, Dalmau, Martin, Pinsker '10)

Let $\Gamma$ be a locally finite reduct of $(\mathbb{Z}; \text{succ})$. Then $\text{CSP}(\Gamma)$ is in $P$ or NP-complete.
A reduct $\Gamma$ of $\langle \mathbb{Z}; \text{succ} \rangle$ is **locally finite** if every $x \in \mathbb{Z}$ is contained in finitely many tuples of relations of $\Gamma$.

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Our result:

- Complete classification of the complexity of distance CSPs.
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*Let $\Gamma$ be a locally finite reduct of $(\mathbb{Z}; \text{succ})$. Then CSP(\Gamma) is in P or NP-complete.*

Our result:

- Complete classification of the complexity of distance CSPs.
- Systematic approach using universal algebraic methods.
Fact

Let $\Gamma$ be a relational structure, and let $R$ be a relation that has a primitive positive definition in $\Gamma$. Then $\text{CSP}(\Gamma)$ and $\text{CSP}(\Gamma, R)$ are polynomial-time equivalent.
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Let \( f : D^n \rightarrow D \) and let \( R \subseteq D^k \) be a relation. We say that \( f \) preserves \( R \) iff
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a_2^1 & a_2^2 & \cdots & a_2^n \\
\vdots & \vdots & \ddots & \vdots \\
a_k^1 & a_k^2 & \cdots & a_k^n \\
\end{pmatrix} \in R
$$

$\in R$ for all $a_1^1, a_1^2, \cdots, a_1^n, a_2^1, a_2^2, \cdots, a_2^n, \cdots, a_k^1, a_k^2, \cdots, a_k^n$.
The Algebraic Approach to Constraint Satisfaction

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A polymorphism of $\Gamma$ is a function that preserves all the relations of $\Gamma$.

**Lemma**

*If $\Gamma$ is a finite structure, a relation $R$ is pp-definable in $\Gamma$ iff $R$ is preserved by all the polymorphisms of $\Gamma$.***
A **polymorphism** of $\Gamma$ is a function that preserves all the relations of $\Gamma$.

**Lemma**

*If $\Gamma$ is a finite structure, a relation $R$ is pp-definable in $\Gamma$ iff $R$ is preserved by all the polymorphisms of $\Gamma$.***

- The previous lemma generalizes to infinite structures which have many **automorphisms**.
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- The previous lemma generalizes to infinite structures which have many **automorphisms**.
- In general, a reduct of $(\mathbb{Z}; \text{succ})$ does not satisfy this condition.
- Solution: we can recover a part of the connection if $\Gamma$ has enough **elements**.
Definition
Let $\Gamma$ be a reduct of $(\mathbb{Z}; \text{succ})$. There exists a unique countable model of Th($\Gamma$) which contains every other countable model. We call this structure the $\omega$-saturated model of $\Gamma$, denoted by $\omega.\Gamma$. $\omega$-saturation
ω-saturation

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\[
\begin{align*}
\rightarrow & \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \\
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Let $\Gamma$ be a reduct of $(\mathbb{Z}; \text{succ})$. Let $R$ be a relation fo-definable in $(\omega.\mathbb{Z}; \text{succ})$ that consists of $n$ orbits under $\text{Aut}(\omega.\Gamma)$. Then $R$ is pp-definable in $\omega.\Gamma$ iff $R$ is preserved by all the polymorphisms of arity $n$ of $\omega.\Gamma$. 
The Result

Theorem (Bodirsky, Martin, AM ’15)

Let $\Gamma$ be a reduct of $(\mathbb{Z}; \text{succ})$ with a finite signature. There exists a structure $\Delta$ with $\text{CSP}(\Gamma) = \text{CSP}(\Delta)$ and such that one of the following cases applies.

1. $\Delta$ is a finite structure. In this case, $\text{CSP}(\Gamma)$ is conjectured to be in P or NP-complete.
2. $\Delta$ is a reduct of $(\mathbb{Z}; =)$. In this case, $\text{CSP}(\Gamma)$ is either in P or NP-complete (Bodirsky, K´ara ’08).
3. $\Delta$ is a reduct of $(\mathbb{Z}; \text{succ})$ whose endomorphisms are all isometries. In this case, $\text{CSP}(\Gamma)$ is in P or NP-complete. Moreover, the tractability of $\text{CSP}(\Gamma)$ is characterized by the existence of certain polymorphisms of finite arity.
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Further projects

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- More ambitious project: classify the complexity of reducts of \((\mathbb{Z}; <, +)\), i.e., reducts of Presburger Arithmetic.