Discrete Temporal CSPs

Manuel Bodirsky, Barnaby Martin, Antoine Mottet

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Summary

- Complexity of all decision problems of the form: “does there exist integers satisfying some constraints expressed in some subset of first-order logic with <.”
- Rich class of problems, exhibits a complexity dichotomy.
- Interesting point of our proof: use of saturation (from model theory).
Temporal Reasoning

The Right Template

Preservation Theorem

(Complexity)
Definition (CSP(Γ))

Γ = (D; φ₁, . . . , φₙ), called the template of the problem.
Input: a sentence Φ := ∃x₁, . . . , xₙ. ∧ ψᵢ(yᵢ), ψᵢ ∈ {φ₁, . . . , φₙ}.
Question: Is Φ true in Γ?
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▶ 3-SAT: D = {0, 1}, clauses with 3 literals
▶ k-colouring: D = {1, . . . , k}, φ = "≠"
▶ Digraph acyclicity: D = ℤ, φ = "<"
Definition (CSP(Γ))

\( \Gamma = (D; \phi_1, \ldots, \phi_s) \), called the template of the problem.

**Input:** a sentence \( \Phi := \exists x_1, \ldots, x_n \land \psi_i(y_i), \psi_i \in \{\phi_1, \ldots, \phi_s\} \).

**Question:** Is \( \Phi \) true in \( \Gamma \)?

**Example**

- 3-SAT: \( D = \{0, 1\} \), clauses with 3 literals
- \( k \)-colouring: \( D = \{1, \ldots, k\} \), \( \phi = "\neq" \)
- Digraph acyclicity: \( D = \mathbb{Z} \), \( \phi = "<" \)

**Our work:** the complexity of discrete temporal satisfiability.
Framework: first-order logic with $y \leq x + k$ for all $k \in \mathbb{Z}$. 
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**Definition**

$(D; \phi_1, \ldots, \phi_s)$ temporal template if $D$ is $\mathbb{Q}$ or $\mathbb{Z}$ and $\phi_1, \ldots, \phi_s$ are first-order over $\prec$. 
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**Example (Definitions over $\mathbb{Z}$)**

- $x \leq y$: $x < y \lor x = y$.  

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**Example (Definitions over $\mathbb{Z}$)**

- $x \leq y$: $x < y \lor x = y$.
- $y = x + 1$: $x < y \land \forall z(x < z \Rightarrow y \leq z)$. 
Framework: first-order logic with \( y \leq x + k \) for all \( k \in \mathbb{Z} \).

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\((D; \phi_1, \ldots, \phi_s)\) temporal template if \( D \) is \( \mathbb{Q} \) or \( \mathbb{Z} \) and \( \phi_1, \ldots, \phi_s \) are first-order over \( < \).

**Example (Definitions over \( \mathbb{Z} \))**

- \( x \leq y : x < y \lor x = y \).
- \( y = x + 1 : x < y \land \forall z(x < z \implies y \leq z) \).
- \( y = x + k : \exists z_0, \ldots, z_k(\bigwedge z_{i+1} = z_i + 1 \land z_0 = x \land z_k = y) \).
**Framework:** first-order logic with $y \leq x + k$ for all $k \in \mathbb{Z}$.

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- $y \leq x + k$: $\exists z (z = x + k \land y \leq z)$.
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- $y = x + 1$: $x < y \land \forall z(x < z \Rightarrow y \leq z)$.
- $y = x + k$: $\exists z_0, \ldots, z_k(\bigwedge z_{i+1} = z_i + 1 \land z_0 = x \land z_k = y)$
- $y \leq x + k$: $\exists z(z = x + k \land y \leq z)$
- $x \leq \max(y + k, z + k')$: $x \leq y + k \lor x \leq z + k'$
Feasibility in $\mathbb{Z}^n$ of a system of constraints of the form:

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- Equivalent to deciding winner in deterministic mean-payoff games.
- In P, if $k$ given in unary.
Temporal Reasoning

Difference logic with modular constraints

Fix $d \in \mathbb{N}$, $d \geq 1$.
Feasibility in $\mathbb{Z}^n$ of a system of constraints of the form:

$$a \leq x - y \leq b, x = y \mod d$$
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Feasibility in \( \mathbb{Z}^n \) of a system of constraints of the form:

\[
a \leq x - y \leq b, \; x = y \mod d
\]

▶ If \( d = 1 \), difference logic.

▶ For all \( d \geq 1 \), in P.
Theorem (Bodirsky, Kára, JACM 2010)

Let \( \Gamma \) be a continuous temporal template. Then \( \text{CSP}(\Gamma) \) is in \( P \) or \( NP \)-complete.
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Example

- $(\mathbb{Q}, <)$ itself: in $P$, digraph acyclicity
- $(\mathbb{Q}, x = y \Rightarrow u = v, \leq, \neq)$: in $P$, Ord-Horn (Nebel, Bürckert)
- $(\mathbb{Q}, x < y < z \lor z < y < x)$: NP-complete, Betweenness
Theorem (Bodirsky, Martin, M)

Let $\Gamma$ be a discrete temporal constraint. Then $\text{CSP}(\Gamma)$ is in $P$, $NP$-complete (or is $\text{CSP}(\Delta)$ for a finite structure $\Delta$).
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Theorem (Bodirsky, Martin, M)

Let $\Gamma$ be a \textit{discrete} temporal constraint. Then CSP($\Gamma$) is in P, NP-complete (or is CSP($\Delta$) for a finite structure $\Delta$).

Example

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It can happen that \( \text{CSP}(\Gamma) = \text{CSP}(\Delta) \) for distinct structures.
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- It can happen that $\text{CSP}(\Gamma) = \text{CSP}(\Delta)$ for distinct structures. Example: $\text{CSP}(\mathbb{Z}, <) = \text{CSP}(\mathbb{Q}, <)$.

- For every continuous temporal template $\Gamma$, there exists a discrete temporal template $\Delta$ such that $\text{CSP}(\Gamma) = \text{CSP}(\Delta)$.

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**Theorem**

$\Gamma$ discrete template with finite signature.
It can happen that \( \text{CSP}(\Gamma) = \text{CSP}(\Delta) \) for distinct structures. Example: \( \text{CSP}(\mathbb{Z}, <) = \text{CSP}(\mathbb{Q}, <) \).

\[ \forall \text{ continuous temporal template } \Gamma, \exists \Delta \text{ discrete temporal template s.t. } \text{CSP}(\Gamma) = \text{CSP}(\Delta). \]

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\( \Gamma \) discrete template with finite signature. \( \exists \Delta \) with \( \text{CSP}(\Delta) = \text{CSP}(\Gamma) \) and at least one of the following cases applies:
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First step: characterise the discrete templates that don’t have the same CSP as a finite structure or a continuous template.

**Theorem**

$\Gamma$ discrete template with finite signature. $\exists \Delta$ with $\text{CSP}(\Delta) = \text{CSP}(\Gamma)$ and at least one of the following cases applies:

1. $\Delta$ has a finite domain.
2. $\Delta$ is a continuous template.
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Definition (Endomorphism)

Γ = (D, φ₁, . . . , φₛ) a structure, f : D → D. f is an endomorphism if it maps edges to edges.
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Remark: \( \Gamma \) and \( f(\Gamma) \) have the same CSP.

Proposition

\( \Gamma \) infinite structure. Then \( \Gamma \) has the same CSP as a finite structure if and only if there exists an endomorphism of \( \Gamma \) whose range is finite.
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Example

\( \Gamma = (\mathbb{Z}, |x - y| = 1) \). Then \( f: x \mapsto x \mod 2 \) is an endomorphism.
Definition

\[ f : \mathbb{Z} \to \mathbb{Z}, \ t \geq 1. \text{ } f \text{ is tightly-}t\text{-bounded if} \]

\[ \forall x \in \mathbb{Z}, |f(x + t) - f(x)| \leq t. \]
Definition

$f : \mathbb{Z} \to \mathbb{Z}, \ t \geq 1$. $f$ is tightly-$t$-bounded if

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- Fix \( \Gamma = (\mathbb{Z}; \ldots) \) discrete temporal template.
- Suppose that for each \( t \), there exists an endomorphism \( f_t \) of \( \Gamma \) which is not tightly-\( t \)-bounded.
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- Then \( \forall x \neq y \in \mathbb{Z} \) and \( \forall k \in \mathbb{Z}, \exists e \in \text{End}(\Gamma) \) such that \( |e(x) - e(y)| > k \).
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- So what?
\textbf{Definition}

\((\mathbb{Q}, \mathbb{Z}, <)\) is the structure on \(\mathbb{Q} \times \mathbb{Z}\) with the lexicographic ordering.
\(\Gamma = (\mathbb{Z}; \phi_1, \ldots, \phi_s)\) a discrete temporal template,
\(\mathbb{Q}.\Gamma = (\mathbb{Q}, \mathbb{Z}, \phi_1, \ldots, \phi_s)\) corresponding template on \(\mathbb{Q} \times \mathbb{Z}\).
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Fact: $\text{CSP}(\Gamma) = \text{CSP}(\mathbb{Q}.\Gamma)$. 
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- Fix \(\Gamma\) a discrete template.
- Suppose that for each \(t\), there exists an endomorphism \(f_t\) of \(\mathbb{Q}.\Gamma\) which is not tightly-\(t\)-bounded.
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- Then \(\forall x \neq y \in \mathbb{Q} \times \mathbb{Z}, \forall k, \exists e \in \text{End}(\mathbb{Q}.\Gamma)\) such that \(|e(x) - e(y)| > k|\).
Definition

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- Let \(k \rightarrow +\infty\) (using König’s tree lemma)
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- Fix \Gamma a discrete template.
- Suppose that for each t, there exists an endomorphism \( f_t \) of \( \mathbb{Q}.\Gamma \) which is not tightly-\( t \)-bounded.
- Then \( \forall x \neq y \in \mathbb{Q} \times \mathbb{Z}, \forall k, \exists e \in \text{End}(\mathbb{Q}.\Gamma) \) such that \( |e(x) - e(y)| > k \).
- Let \( k \to +\infty \) (using König’s tree lemma)
- \( \exists e \in \text{End}(\mathbb{Q}.\Gamma) \) such that 

\[ \forall x, y \in \mathbb{Q} \times \mathbb{Z}, e(x) \neq e(y) \Rightarrow e(x) - e(y) = \infty. \]
Let $\Gamma$ be a discrete template with finite signature and without finite-range endomorphisms. Exactly one of the following applies:

- There exists a continuous template $\Delta$ with $\text{CSP}(\Gamma) = \text{CSP}(\Delta)$.
- There exists $t > 0$ such that every endomorphism of $\mathbb{Q}.\Gamma$ is tightly-$t$-bounded.
Theorem

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$f$ is tightly-$t$-bounded if $\forall x \in \mathbb{Z}, |f(x + t) - f(x)| \leq t$. 

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The Right Template

Petrus

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Lemma

If $f \in \text{End}(\mathbb{Q}.\Gamma)$ is tightly-$t$-bounded and $\mathbb{Q}.\Gamma$ does not have finite-range endomorphisms, then we have

$$|f(x + t) - f(x)| = t.$$
Definition

Let $\Gamma$ be a discrete template, and let $t \geq 1$. $(\mathbb{Q}.\Gamma)/t$ is the structure induced by $\mathbb{Q}.\Gamma$ on $\{t \cdot z : z \in \mathbb{Z}\}$. 
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**Fact:** $(\mathbb{Q}.\Gamma)/t$ is isomorphic to $\mathbb{Q}.$\Delta for some discrete template $\Delta$. 
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\[\text{Diagram}\]
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![Diagram](image-url)
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The endomorphisms of $(\mathbb{Q}.\Gamma)/t$ are tightly-1-bounded.
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- The endomorphisms of $(\mathbb{Q}.\Gamma)/t$ are tightly-1-bounded.
- $\text{CSP}(\Gamma) = \text{CSP}(\Delta)$. 
We proved:

**Theorem**

Γ a discrete template with finite signature. ∃Δ with CSP(Δ) = CSP(Γ) and at least one of the following cases applies:

1. Δ has a finite domain.
2. Δ is a continuous template.
3. The endomorphisms of \( \mathbb{Q}.\Delta \) are isometries.
Temporal Reasoning

The Right Template

Preservation Theorem

(Complexity)
A structure is **saturated** if every “imaginary” element that can be described by a finitely consistent set of first-order formulas exists.
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Theorem

Δ discrete template, $R \subseteq (\mathbb{Q} \cdot \mathbb{Z})^k$ relation with a first-order definition in $\prec$. If $R$ is generated by $n$ tuples, then

$$\{\exists, \land\}$$-definability in $\mathbb{Q} \cdot \Delta$

$\uparrow$

invariance under $n$-ary polymorphisms of $\mathbb{Q} \cdot \Delta$
A structure is saturated if every “imaginary” element that can be described by a finitely consistent set of first-order formulas exists.

**Theorem**

$\Delta$ discrete template, $R \subseteq (\mathbb{Q} \times \mathbb{Z})^k$ relation with a first-order definition in $\prec$. If $R$ is generated by $n$ tuples, then

\[
\{\exists, \land\}-\text{definability in } \mathbb{Q}.\Delta \quad \uparrow \quad \text{invariance under } n\text{-ary polymorphisms of } \mathbb{Q}.\Delta
\]

A polymorphism of is a homomorphism $(\mathbb{Q}.\Delta)^n \to \mathbb{Q}.\Delta$ for $n \in \mathbb{N}$. 
A structure is **saturated** if every “imaginary” element that can be described by a finitely consistent set of first-order formulas exists.

**Theorem**

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A **polymorphism** of is a homomorphism \( (\mathbb{Q},\Delta)^n \to \mathbb{Q}.\Delta \) for \( n \in \mathbb{N} \).

**Consequence:** when endos of \( \mathbb{Q}.\Delta \) are isometries, \( y = x + 1 \) is \( \{\exists, \land\}\text{-definable} \) or \( |y - x| = k \) is, for all \( k \in \mathbb{Z} \).
Temporal Reasoning

The Right Template

Preservation Theorem
Theorem (Bodirsky, Martin, M)

Γ a discrete temporal template with finite signature. ∃Δ with CSP(Δ) = CSP(Γ) and at least one of the following cases applies:
Theorem (Bodirsky, Martin, M)

$\Gamma$ a discrete temporal template with finite signature. $\exists \Delta$ with $\text{CSP}(\Delta) = \text{CSP}(\Gamma)$ and at least one of the following cases applies:

1. $\Delta$ has a finite domain.
Theorem (Bodirsky, Martin, M)

Γ a discrete temporal template with finite signature. ∃Δ with CSP(Δ) = CSP(Γ) and at least one of the following cases applies:

1. Δ has a finite domain.
2. Δ is a continuous template.
Theorem (Bodirsky, Martin, M)

\( \Gamma \) a discrete temporal template with finite signature. \( \exists \Delta \) with \( \text{CSP}(\Delta) = \text{CSP}(\Gamma) \) and at least one of the following cases applies:

1. \( \Delta \) has a finite domain.
2. \( \Delta \) is a continuous template.

Or \( \Delta \) is a discrete template containing \( y = x + 1 \) or \( |y - x| = k \) for all \( k \in \mathbb{Z} \), and:

3. The relations of \( \Delta \) are Horn-definable, and \( \text{CSP}(\Delta) \) is in P.
4. The relations of \( \Delta \) are max-closed, and \( \text{CSP}(\Delta) \) is in P.
5. \( \text{CSP}(\Delta) \) is NP-complete.
Theorem (Bodirsky, Martin, M)

Γ a discrete temporal template with finite signature. ∃Δ with CSP(Δ) = CSP(Γ) and at least one of the following cases applies:

1. Δ has a finite domain.
2. Δ is a continuous template.
Or Δ is a discrete template containing y = x + 1 or |y − x| = k for all k ∈ ℤ, and:

3. The relations of Δ are Horn-definable, and CSP(Δ) is in P.
4. The relations of Δ are max-closed, and CSP(Δ) is in P.
5. CSP(Δ) is NP-complete.
Theorem (Bodirsky, Martin, M)

Γ a discrete temporal template with finite signature. ∃Δ with CSP(Δ) = CSP(Γ) and at least one of the following cases applies:

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Complexity dichotomy modulo the Feder-Vardi conjecture (Bulatov-Zhuk-... theorem).