Adaptive Parallel-in-Time Integration (APTI) for Second-order Ordinary Differential Equations

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Outline

- Membrane Problem: a $2^{nd}$ ODE on long time
- Multiple Shooting and Coarse Grid generation
- Rescaling: Invariance and Similarity
- (Coarse Grid) Adaptive Parallel Time Integration (APTI Algorithm)
- Implementation and Results
- Follow-up...
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Implementation and Results

Follow-up...
Membrane Problem: Second Order ODE


\[
\begin{aligned}
&y'' - b|y'|^{q-1} \ y' + |y|^{p-1} \ y = 0, \ \ t > 0, \\
y(0) = y_{1,0}, \\
y'(0) = y_{2,0}.
\end{aligned}
\]

\[
p \leq q \leq \frac{2p}{p+1},
\]
Membrane Problem: First-order System

\[
(S) \begin{cases}
    y_1' = y_2 \\
    y_2' = b|y_1|^{q-1}y_1 - |y_1|^{p-1}y_1 = 0, \quad t > 0, \\
    y_1(0) = y_{1,0}, \\
    y_2(0) = y_{2,0}.
\end{cases}
\]

\[p \leq q \leq \frac{2p}{p+1},\]

(b) Phase plane
Sliced-Time Computations


**P. Chartier, B. Philippe.** *A parallel shooting technique for solving dissipative ODE’s*, 1993

\[ \begin{align*}
\frac{dy}{dt} &= F(y), & 0 < t \leq T, \\
y(0) &= y_0,
\end{align*} \]

Seek a sequence of “Time Slices”: \( \bigcup_{n \geq 1} [T_{n-1}, T_n] \) such that:

\[ \bigcup_{n \geq 1} [T_{n-1}, T_n] = [0, T] \]

\( (S) \) is equivalent to a Sequence of Initial Value Shooting Problems:

On each \( n^{th} \) slice \( [T_{n-1}, T_n] \) \((n \geq 1)\), seek for \( \{y, T_n\} \):

\[ \begin{align*}
\frac{dy}{dt} &= F(y), & T_{n-1} < t \leq T_n \\
y(T_{n-1}) &= y_{n-1}, \\
\forall t \in (T_{n-1}, T_n), & E[y(t)] \neq 0 \text{ and } E[y(T_n)] = 0.
\end{align*} \]
Membrane Problem: Non-uniform Sliced-Times generation

\[
\begin{aligned}
(S_n) & \quad y_1' = y_2, \quad T_{n-1} < t \leq T_n \\
y_2' &= b|y_1|^{q-1}y_1 - |y_1|^{p-1}y_1, \\
y_1(T_{n-1}) &= y_{1,n-1}, \quad y_2(T_{n-1}) = y_{2,n-1}. \\
\forall t \in (T_{n-1}, T_n), \quad E[y(t)] &= y_2(t) - |y_1(t)|^{p+1} \neq 0 \\
E[y(T_n)] &= y_2(T_n) - |y_1(T_n)|^{p+1} = 0
\end{aligned}
\]
Another non-uniform time-slices Generation

A slice $[T_{n-1}, T_n]$ is determined when $(y_1, y_2)$ completes a full, rotation.

at $t = T_n$, $\theta_n [y(T_n)] = 2\pi$,
if $T_{n-1} < t < T_n$, $0 < \theta_n [y(t)] < 2\pi$.

where $\theta_n [y(t)] = |(\omega \vec{M}_{n-1}, \omega \vec{M}(t))|$. 
Another non-uniform time-slices Generation

A slice \([T_{n-1}, T_n]\) is determined when \((y_1, y_2)\) completes a full, rotation.

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if \(T_{n-1} < t < T_n\), \(0 < \theta_n[y(t)] < 2\pi\).

where \(\theta_n[y(t)] = \left|\left(\omega\vec{M}_{n-1}, \omega\vec{M}(t)\right)\right|\).
Rescaling: Change of variables

On each slice \([T_{n-1}, T_n]\),

\[
t = T_{n-1} + \beta_n s
\]

\[
y(t) = y_{n-1} + D_n Z_n(s)
\]

\[
y_{n-1} = y(T_{n-1}), \quad \beta_n > 0, \text{ time-rescaling factor}
\]

\[
D_n = \begin{pmatrix}
y_{1,n-1} & 0 \\
0 & y_{2,n-1}
\end{pmatrix}, \quad y_{n,i} \neq 0, \forall n, \forall i = 1, 2.
\]

\[
\forall n, \forall i = 1, 2, \quad Z_{n,i}(s) = \frac{y_i(t) - y_{i,n-1}}{y_{n-1,i}}.
\]
Membrane Problem

\[
(S_n) \begin{cases}
    y'_1 = y_2, & T_{n-1} < t \leq T_n \\
    y'_2 = b|y_1|^{q-1}y_1 - |y_1|^{p-1}y_1, & y_1(T_{n-1}) = y_{1,n-1}, \; y_2(T_{n-1}) = y_{2,n-1}.
\end{cases}
\]

\[
\forall t \in (T_{n-1}, T_n), \quad E[y(t)] = y_2(t) - |y_1(t)|^{\frac{p+1}{2}} \neq 0
\]

\[
E[y(T_n)] = y_2(T_n) - |y_1(T_n)|^{\frac{p+1}{2}} = 0
\]
Rescaled Initial Value Shooting Problems

On each $n^{th}$ slice $[0, s_n]$, seek for $\{Z_n(s), s \in [0, s_n]\}$:

\[
(S'_n) \quad \begin{cases}
\frac{dZ_n}{ds} = G_n(Z_n), & 0 < s \leq s_n, \\
Z_n(0) = 0, \\
H(Z_n(s_n)) = 0, & H(Z_n(s)) \neq 0, \forall s \in (0, s_n)
\end{cases}
\]

where: $G_n(Z_n) = \beta_n D_n^{-1} F(y_{n-1} + D_n Z_n)$

\[
\forall \beta_n \left\{ \\
y_n = y(T_n) = (I + \text{diag}(Z_n(s_n))) y_{n-1}, \ y_{n-1} = y(T_{n-1}) \\
T_n = T_{n-1} + \beta_n s_n
\right. 
\]
Rescaled Initial Value Shooting Membrane Problem

\[
\begin{align*}
\left( S'_n \right) \quad \begin{cases} 
\frac{dZ_1}{ds} &= \beta_n \frac{y_{2,n-1}}{y_{1,n-1}} (1 + Z_2) \quad 0 < s \leq s_n \\
\frac{dZ_2}{ds} &= \beta_n \left( b \frac{|y_{1,n-1}|^{q-1} y_{1,n-1}}{y_{2,n-1}} |1 + Z_1|^{q-1} (1 + Z_1) + \frac{|y_{1,n-1}|^{p-1} y_{1,n-1}}{y_{2,n-1}} |1 + Z_1|^{p-1} (1 + Z_1) \right), \\
Z_1(0) &= Z_2(0) = 0. 
\end{cases}
\end{align*}
\]

\[\forall s \in (0, s_n), \ E [y(t)] = y_2(t) - |y_1(t)|^{\frac{p+1}{2}} \neq 0 \]

\[E [y(T_n)] = y_2(T_n) - |y_1(T_n)|^{\frac{p+1}{2}} = 0 \]
Rescaled Systems for Membrane Problems

\[ \beta_n = \frac{y_{1,n-1}}{y_{2,n-1}} \quad \text{and as} \quad y_{2,n-1} = \left| y_{1,n-1} \right|^{\frac{p+1}{2}}, \text{then:} \]

\[ \beta_n \frac{\left| y_{1,n-1} \right|^{p-1} y_{1,n-1}}{y_{2,n-1}} = 1 \text{ and} \]

\[ \beta_n \frac{\left| y_{1,n-1} \right|^{q-1} y_{1,n-1}}{y_{2,n-1}} = \gamma_n = \left| y_{n-1,1} \right|^{\frac{p+1}{2}} \left( q - \frac{2p}{p+1} \right) \]
Rescaled Systems for Membrane Problems

Resulting rescaled system:

\[
\begin{align*}
\frac{dZ_{n,1}}{ds} &= G_{n,1}(Z_n) = 1 + Z_{n,2}, \quad 0 < s \leq s_n, \\
\frac{dZ_{n,2}}{ds} &= G_{n,2}(Z_n) = b\gamma_n|1 + Z_{n,2}|^{q-1}(1 + Z_{n,2}) - |1 + Z_{n,1}|^{p-1}(1 + Z_{n,1}), \\
Z_{n,1}(0) &= Z_{n,2}(0) = 0, \\
H(Z_n(s)) \neq 0, \quad s \in (0, s_n) \text{ and } H(Z_n(s_n))) = 0.
\end{align*}
\]

\[
H(Z) = \frac{1 + Z_2}{|1 + Z_1|^{\frac{p+1}{2}}} - 1
\]
Sequential Solver using Rescaling

\[ S'_n \begin{cases} \frac{dZ_n}{ds} = G_n(Z_n), & 0 < s \leq s_n \\ Z_n(0) = 0, \\ H(Z_n(s)) \neq 0, \forall s < s_n \text{ and } H[Z_n(s_n)] = 0. \end{cases} \]

(S\(_n\))

**(Seq-Alg)**

Input: \{\(y_0, F(.), H(.), T\}\)

\[ T_n = 0; n = 0 \textbf{ while } T_n \leq T \]

\[ n = n + 1 \]

Determine \(\{\beta_n, G_n\}\)

Apply Fine Solver on \(S'_n\), obtain:

\[ \{(s_n, Z_n(s_n))\} \]

\[ Z_n(s), 0 \leq s \leq s^c_n \]

\[ y_n = y_{n-1} + \text{diag}(y_{n-1})Z_n(s_n) \]

\[ T_n = T_{n-1} + \beta_nZ_n(s_n) \]

end

\[ N = n \]

If \(\max_n ||Z_n(s_n) - Z^c_n(s^c_n)|| \leq \epsilon^\text{loc}_{tol}\) then:

\[ \max_n \frac{||y_n(T_n) - y^c_n(T^c_n)||}{||y^c_n(T^c_n)||} \leq \epsilon^\text{glob}_{tol} = C(N)\epsilon^\text{loc}_{tol} \]

To parallelize (Seq-Alg), choose \(\beta_n\) to predict: \(\{(s_n, Z_n(s_n))\}, \{[T_{n-1}, T_n]\}\) and

\[ \{(y(T_{n-1}), y(T_n))\} \]
Invariance of the Rescaled Systems

\[ (S'_n) \]
\[
\begin{align*}
\frac{dZ_n}{ds} &= G_n(Z_n), \quad 0 < s \leq s_n, \\
Z_n(0) &= 0, \\
H(Z_n(s_n)) &= 0, \quad H(Z_n(s)) \neq 0, \quad \forall s \in (0, s_n)
\end{align*}
\]

**Theorem 1** If \( q = \frac{2p}{p+1} \), \( \forall p \leq 1 \), then \((S'_{n})\) are invariant:

\[ G_n(Z) = G(Z), \; \forall n: \]

\[ G(Z) = \begin{cases} 
G_1(Z) = 1 + Z_2, \; 0 < s \leq s_n, \\
G_2(Z) = b|1 + Z_2|^{q-1}(1 + Z_2) - |1 + Z_1|^{p-1}(1 + Z_1)
\end{cases} \]

\[ \implies \forall n, \; s_n = s_1 \text{ and } Z_n(s) = Z_1(s), \; s \in [0, s_1]. \]
Asymptotic Similarity of the Rescaled Systems

\[ (S'_n) \begin{align*}
\frac{dZ_n}{ds} &= G_n(Z_n), \quad 0 < s \leq s_n, \\
Z_n(0) &= 0, \\
H(Z_n(s_n)) &= 0, \quad H(Z_n(s)) \neq 0, \quad \forall s \in (0, s_n)
\end{align*} \]

**Theorem 2** If \( 0 < p \leq q < \frac{2p}{p+1} \leq 1 \), then as \( \lim_{n \to \infty} \gamma_n = 0 \), \((S'_n)\) are asymptotically similar to:

\[ (S_L) \begin{align*}
\frac{dZ_{L,1}}{ds} &= 1 + Z_{L,2}, \quad 0 < s \leq s_L \\
\frac{dZ_{L,2}}{ds} &= -|1 + Z_{L,1}|^{p-1}(1 + Z_{L,1}), \\
Z_{L,1}(0) &= Z_{L,2}(0) = 0, \\
H[Z_L(s_L)] &= 0 \quad \text{and} \quad \forall s \in (0, s_L), \quad H[Z_L(s)] \neq 0.
\end{align*} \]

\[ \implies \lim_{n \to \infty} s_n = s_L \quad \text{and} \quad \lim_{n \to \infty} Z_n(s_n) = Z_L(s_L). \]
Weak Similarity

Can be tested numerically

\[ \left\{ \begin{array}{l}
\frac{dZ_n}{ds} = G_n(Z_n), \\
Z_n(0) = 0, \\
H(Z_n(s_n)) = 0, \\
H(Z_n(s)) \neq 0, \forall s \in (0, s_n)
\end{array} \right. \]

The rescaled systems \((S'_n)\) are said to be weakly similar up to a tolerance \(\epsilon\) on \(n_r\) consecutive slices, starting at slice \(n_0\) and reached at \(n_s = n_0 + n_r\), if:

\[ \forall n \in \{n_0 + 1, \ldots, n_0 + n_r\}, \quad \|Z_n(s_n) - Z_{n-1}(s_{n-1})\|_{\infty} < \epsilon. \]
Similarity Properties: Theorem

**Theorem**

*Under some boundedness conditions on \{s_n\} and \{Z_n(s)\}, and Lipschitz condition on \(G_L\), if \((S_n')\) are asymptotically similar to \((S_L)\), then \(\{Z_n(.)\}\) converges uniformly to \(Z_L(.)\):

\[
\lim_{n \to \infty} \max_{s \in [0, \hat{s}]} \|Z_n(s) - Z_L(s)\|_\infty = 0.
\]

**Corollary**

*If also \(H\) satisfies some bijective properties mapping onto an interval \([-\epsilon, \epsilon] \subset \mathbb{R}\) containing 0, then:

\[
\begin{align*}
\lim_{n \to \infty} s_n &= s_L, \\
\lim_{n \to \infty} Z_n(s_n) &= Z_L(s_L).
\end{align*}
\]
Rescaling: Invariance and Similarity

Similarity Properties: Theorem

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Under some boundedness conditions on \{s_n\} and \{Z_n(s)\}, and Lipschitz condition on \(G_L\), if \((S'_n)\) are asymptotically similar to \((S_L)\), then \(\{Z_n(.)\}\) converges uniformly to \(Z_L(.)\):

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\lim_{n \to \infty} \left[ \max_{s \in [0, \hat{s}]} \|Z_n(s) - Z_L(s)\|_{\infty} \right] = 0.
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\begin{aligned}
\lim_{n \to \infty} s_n &= s_L, \\
\lim_{n \to \infty} Z_n(s_n) &= Z_L(s_L).
\end{aligned}
\]
Convergence Analysis

Assumption

\[ Y_{n-1}^p \approx Y_{n-1} \implies \begin{cases} Z_n^c(.) \approx Z_n(.) \\ S_n^c \approx S_n \end{cases} \]

Theorem

\[ Y_{n-1}^p \approx Y_{n-1} \implies Y_n^c \approx Y_n \]

and if \( \beta_n \) is a continuous function of \( Y_{n-1} \), verifying a Lipschitz condition:

\[ T_{n-1}^p \approx T_{n-1} \implies T_n^c \approx T_n \]

If \( Y_{n-1}^p \) is accurate enough, then \( Y_n^c \) and \( T_n^c \) are close enough to the exact values.

Corollary

**Test of convergence:**

\[ \begin{cases} Y_{n-1}^p \approx Y_{n-1} \\ Y_n^p \approx Y_n^c \end{cases} \implies Y_n^p \approx Y_n \]

If also \( Y_n^p \) is close enough to \( Y_n^c \), the predictions keep being accurate for solving next slice.
Convergence Analysis

Assumption

\[ Y_{n-1}^p \approx Y_{n-1} \implies \begin{cases} Z_n^c(\cdot) \approx Z_n(\cdot) \\ S_n^c \approx S_n \end{cases} \]

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\[ Y_{n-1}^p \approx Y_{n-1} \quad \Rightarrow \quad \begin{cases} Z_n^c(\cdot) \approx Z_n(\cdot) \\ S_n^c \approx S_n \end{cases} \]

Theorem

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If also \( Y_n^p \) is close enough to \( Y_n^c \), the predictions keep being accurate for solving next slice.
Adaptive Coarse Grid Parallel Time Integration

Choice of EOS condition:

\[ \beta_n, G_n(.), H(.) \]

1. **Sequential run on** \( n_s \) **slices**, \( \epsilon_{tol}^{n_s} \)

\[ n_0 = 0: \]

\[ D^{(0)} = \{ Z_n(s_n) | n = n_0 + 1, \ldots, n_0 + n_s \} \]

\[ T_c = T_{ns} \]

2. **while** \( T_c \leq T \)

- **Predict:** \( \{ \{ s_n, Z_n(s_n) \} | n > n_s \} = Fit(D^{(0)}) \)

- **Parallel computations:**
  - on \( Z_{n}^{new}(s) | n > n_s \) using **Fine solver** \( \epsilon_{tol}^{loc} \)
  - \( y_n^{new} = y_{n-1}^{pred} + D_{n}^{pred} Z_{n}^{new}(s_n) \ n > n_s \)
  - \( gap_n = \frac{||y_n^{new} - y_{n}^{pred}||}{||y_n^{new}||}, n > n_s \)
  - Update \( n_0 \) and \( T_c \): \( \max_{n_s < n \leq n_0} \{ gap_n \} \leq \epsilon_{tol}^{glob} \ T_c = T_{s}^{new} \).
  - Update \( D^{(n_0)} \).
Our Main References in Time Parallelism

- **J.Nievergelt.** *Parallel methods for integration ordinary differential equations*, 1964
- **P.Chartier, B.Philippe.** *A parallel shooting technique for solving dissipative ODE’s*, 1993
- **J.Erhel, S.Rault.** *Algorithme parallèle pour le calcul d’orbites*, 2000
- **J.L.Lions, Y.Maday, G.Turinici.** *Résolution d’EDP par un schéma en temps “pararéel”*, 2001
Overview of APTI Algorithm

**Initialize**

- Rescaled Problems with Ratio Property and Predictive Model for ratios
  - REACH_RATIOPROPERTY_UP_TO_e^\sqrt{\lambda}

  - \( k = 1 \) and \( n_z^{(k-1)} = n_z \)

**Iterate**

- Predictions: Duplicated on all processors
  - GET_MODEL_PARAMETERS
  - PREDICT

- Corrections: Parallel execution
  - SOLVE_SLICE
  - TEST_SLICE_CONVERGENCE
  - TRUE
  - FALSE

- Communicate: \( N^{(0)} \), \( \{ n_z^{(k)} \} \)
  - \( T_c^{(g)} \geq T \)
  - \( T_c^{(g)} < T \)

**Conclude**
Asymptotic similarity of the rescaled problems

Rescaled Solution in the Phase Plane

Case of Asymptotic Similarity
Asymptotic similarity of the rescaled problems

Case of Asymptotic Similarity
Asymptotic similarity of the rescaled problems

Rescaled Solution in the Phase Plane

Case of Asymptotic Similarity
The following tolerances have been used:

Fine solver: $\epsilon_{tol}^\tau = 10^{-14}$

Tolerance for stopping sequential runs: $\epsilon_{tol}^{n_s} = 10^{-5}$ (for getting $n_s$)

Global tolerance for computing gaps: $\epsilon_{tol}^g = 5 \times 10^{-6}$.

<table>
<thead>
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<th>Case</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</table>
Application to a Reaction-Diffusion Problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u^m &= au^p, \ x \in \Omega \subset \mathbb{R}^d, \ t > 0 \\
u(x, t) &= 0, \ x \in \partial \Omega, \ t \geq 0, \\
u(x, 0) &= u_0(x) > 0, \ x \in \Omega.
\end{align*}
\]

where \(a > 0, \ m > 0, \ p > 0\).

When \(0 < m \leq p \leq 1\) and for some initial conditions, the solution goes monotonously to infinity in infinite time and the blow-up occurs on all the components of the solution.
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Choice of EOS condition & \( \{ \beta_n \} \)

EOS condition:

\[
\begin{align*}
& \text{at } t = T_n, \quad \left\| D_n^{-1} (Y(T_n) - Y_{n-1}) \right\|_\infty = S \\
& \forall t \in (T_{n-1}, T_n), \quad \left\| D_n^{-1} (Y(t) - Y_{n-1}) \right\|_\infty < S
\end{align*}
\]

Critical choice of \( \beta_n \):

\[
\beta_n = \frac{1}{\left\| (Y_{n-1})^{pq-q} \right\|_\infty}
\]

Resulting rescaled systems:

\[
(S'_{n}) \quad \begin{cases} 
\frac{dZ_n}{ds} = G_n(Z_n) & 0 < s \leq s_n, \\
Z_n(0) = 0, \\
\forall s < s_n, \ |Z_n(s)|_\infty < S, \text{ and } |Z_n(s_n)|_\infty = S.
\end{cases}
\]
Implementation and Numerical Results

Choice of EOS condition & \( \{ \beta_n \} \)

EOS condition:

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\begin{cases}
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Choice of EOS condition & \( \{ \beta_n \} \)

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\begin{align*}
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\forall s < s_n, \| Z_n(s) \|_{\infty} < S, \text{ and } \| Z_n(s_n) \|_{\infty} = S.
\end{cases}
\]
Rescaling the RD Problem

where:

\[ G_n(Z_n) = G_{n_{diff}}(Z_n) + G_{n_{reac}}(Z_n) \]

with:

\[ G_{n_{diff}}(Z_n(s)) = -\frac{1}{q} \frac{1}{\|(Y_{n-1})\cdot pq-q\|_\infty} D^{-1}(Y_{n-1}) \cdot q D^{-1}(1+Z_n(s)) \cdot pq-q+1 AD_{Y_{n-1}} [1 + Z_n(s)] \]

\[ G_{n_{reac}}(Z_n(s)) = \frac{a}{q} \frac{1}{\|(Y_{n-1})\cdot pq-q\|_\infty} D(Y_{n-1}) \cdot pq-q [1 + Z_n(s)] \cdot pq-q+1 \]

**Theorem**

The rescaled systems \((S'_n)\) are **asymptotically similar** to the limit system:

\[
\begin{cases}
\frac{dZ_L}{ds} = G_L(Z_L), & 0 < s \leq s_L, \\
Z_L(0) = 0, & \\
\forall s < s_L, \|Z_L(s)\|_\infty < S, \text{ and } \|Z_L(s_L)\|_\infty = S.
\end{cases}
\]

defined by the function:

\[ G_L(Z_L) = \frac{a}{q} [1 + Z_L(s)] \cdot pq-q+1. \]
Rescaling the RD Problem

where:

$$G_n(Z_n) = G_{n_{diff}}(Z_n) + G_{n_{reac}}(Z_n)$$

with:

$$G_{n_{diff}}(Z_n(s)) = -\frac{1}{q} \frac{1}{\parallel(Y_{n-1}).pq-q\parallel_{\infty}} D^{-1}(Y_{n-1}).q D^{-1}[1+Z_n(s)].q^{-1} AD_{Y_{n-1}} [1 + Z_n(s)]$$

$$G_{n_{reac}}(Z_n(s)) = \frac{a}{q} \frac{1}{\parallel(Y_{n-1}).pq-q\parallel_{\infty}} D(Y_{n-1}).pq-q [1 + Z_n(s)].pq-q+1$$

Theorem

The rescaled systems ($S'_n$) are **asymptotically similar** to the limit system:

$$\begin{cases} 
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Z_L(0) = 0, & \\
\forall s < s_L, \|Z_L(s)\|_{\infty} < S, \text{ and } \|Z_L(s_L)\|_{\infty} = S.
\end{cases}$$

defined by the function:

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Asymptotic similarity of the rescaled problems

\[ m = 0.7, \, p = 0.9, \, a = 3, \, S = 2, \text{ with } \Omega = [-1, 1] \subset \mathbb{R} \text{ and } u_0(x) = 1 - x^2 \]
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Asymptotic similarity of the rescaled problems

$m = 0.7, \rho = 0.9, a = 3, S = 2$, with $\Omega = [-1, 1] \subset \mathbb{R}$ and $u_0(x) = 1 - x^2$
Application of APTI Algorithm to RD Problem

\[ \Omega = [-1, 1] \subset \mathbb{R}, \; h = 1/8, \; a = 3, \; S = 3, \; u_0(x) = (1 - x^2), \; \tau = \frac{h^2}{2}, \; \epsilon_{\text{tol}}^{\text{eos}} = 10^{-14}, \; \epsilon_{\text{tol}}^{n_s} = \epsilon_{\text{tol}}^{g} = 10^{-8}, \; n_{s_{\text{min}}} = 8. \]

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<th>3</th>
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</table>

![Graph showing average speed-up and maximum speed-up versus number of processors.](chart.png)
Conclusion

1 Constraints:
   - Prior study of the Solution Behavior
   - Existence of an EOS condition inducing an efficient Fit Model for rescaled problems

2 Relevance:
   - Predictions do not require any sequential numerical integration on the coarse grid
   - Case of Asymptotic similarity:
     - Very fast convergence due to accurate predictions
   - Not all the remaining time-slices are solved at each iteration
   - Communications can be minimized in number and size

3 Perspectives
   - Determination of the exact scope of application of APTI
   - Comparison with other time-parallel schemes
   - Experimentation on more application problems
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