A physics perspective on Algebraic Graph Theory (AGT)

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Our knowledge and intuition about electrical circuits can provide an interesting insight on AGT. One of the prime concepts adopted from electric circuits is the equivalent resistance, $R_{\rm eq}$ (*resistance distance* in [1]). A particularly simple expresssion for $R_{\rm eq}$, recently obtained in [2], yields a convenient tool to

- Investigate and make analytical statements about connectivity of graphs.
- Count the number of spanning trees and forests of certain type.
- Compute the resistance distance for generic graphs of finite size, as well as for infinite or large graphs (with explicit dependance on the graph size) that exhibit some symmetry or pattern [5].
- Allow for complex valued edge weights by considering the complex impedance of AC-circuits. The expression for the equivalent impedance readily allows to investigate the resonance phenomena in AC-circuits.
- Given the analogy between electric circuits and random walks on graphs [3], one can readily obtain the corresponing quantities of interest for the latter, such as, for instance, the *escape probability*.

Consider a graph G with n vertices and designate the edge conductance (inverse resistance) between vertices i and j as $\sigma_{ij} = 1/R_{ij} = \sigma_{ji}$. Without loss of generality, assume that every vertex is connected to every other vertex. If, in reality, some vertices are not connected by an edge, we simply put the corresponding edge conductance to zero. The weighted Laplacian (Kirchhoff) matrix for G is given by

$$L_{ij} = -\sigma_{ij} \quad \text{for} \quad i \neq j, \quad \text{and} \quad L_{ii} = \sum_{j=1}^{n} \sigma_{ij}. \tag{1}$$

The equivalent resistance between vertices i and j can be written as [2]

$$R_{\rm eq}(i,j) = \frac{\Delta''_{ij}}{\Delta'}, \tag{2}$$

where Δ' is (any) co-factor of the Laplacian matrix and Δ''_{ij} the determinant of L with rows and columns i and j removed. These determinants have several key properties. Both Δ' and Δ''_{ij} are polynomials (of degree $n-1$ and $n-2$ respectively) in the edge conductances σ_{ij} and contain only positive monomials (n^{n-2} and $2n^{n-3}$ respectively) which are linear in each particular σ_{ij}.

Furthermore, the set of edges appearing in each such monomial of Δ' represents a spanning tree of graph G. Putting each non-zero σ_{ij} to 1, yields the Kirchhoff theorem ($\Delta' =$ number of spanning trees). The set of edges in each monomial of Δ''_{ij} represents a *forest* of two trees in G: one connected to vertex i and the other one to vertex j. (One of the trees to be just vertex i or just vertex j.) By putting each non-zero σ_{ij} to 1, Δ''_{ij} would count the number of ways to have all vertices of G connected (through a path) to either i or j. We can define analogous determinants Δ'''_{ijk} and so on, by removing from L the rows and columns i, j, and k and so on. Δ'''_{ijk} correspond to forests with trees connecting all vertices in G to either i, j, or k.

In the physics (electric) context, the two special vertices i and j in Δ''_{ij} are understood as the terminals of the voltage source (battery). If $\Delta''_{ij} = 0$, it follows from the Kirchhoff's vertex equations [4] that some vertex potentials cannot be determined, which implies that there are components of G that are not connected to the battery. Moreover, the multiplicity of zero eigenvalue in Δ''_{ij} gives the number of such disconnected components. Since any circuit with finite (or zero) values of edge conductance must have a finite value of equivalent conductance, it follows from Eq. (2) that if $\Delta''_{ij} = 0$ then so is Δ'. Also on the grounds of equivalent conductance, if $\Delta''_{ij} \neq 0$, G is connected (disconnected) if and only if $\Delta' \neq 0$ ($\Delta' = 0$).

Finally, it can be shown that

$$\Delta''_{ij} = \frac{\partial \Delta'}{\partial \sigma_{ij}}, \quad \Longrightarrow \quad R_{\text{eq}}(i,j) = \frac{\partial \ln \Delta'}{\partial \sigma_{ij}}. \tag{3}$$

Thus for many analytical purposes it is sufficient to know Δ' as a function of the edge conductances σ_{ij}. For relatively small graphs, such explicit expressions can be obtained using widely avilable mathematical packages.

References

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