The Clebsch graph on the crossroads of Algebraic Geometry and Algebraic Graph Theory

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The Clebsch graph Cl is a strongly regular graph (SRG) with the parameters (16,5,10,0,2)S and primitive rank 3 automorphism group of order 1920, isomorphic to the spilt extension $E_{16} : S_5$. It is one of the six known primitive triangle free SRGs with 5,10,16,56,77 and 100 vertices. All these graphs appear as (induced) subgraph of the graph $NL_2(10)$ with 100 vertices, discovered by Dale Mesner in 1956 and also known as the Higman-Sims graph, see [6] for details. The question about the existence of other primitive triangle-free SRGs remains open for a long while and seems to be one of the most challenging problems in AGT.

The name Clebsch graph was coined by J.J. Seidel in [8], sometimes this name is attributed to the complementary to Cl graph of valency 10. Many nice models of Cl appear on the home page of Andries Brouwer [1].

Being originally educated in classical geometry of XIXth century, Seidel was referring to the paper [2]. While the name itself was commonly used for about half a century, it seems that its roots were not discussed properly in literature.

According to procedure, described by A. Rudvalis [7], starting from Cl, one gets a symmetric design on 16 vertices, usually called biplane. All biplanes on 16 points are well-known, see e.g. [5]. The one, which appears from Cl is sometimes called the nicest biplane B_6 (on 16 points). According to the procedure by Rudvalis, which involves polarities of designs, the graph Cl is reconstrutable from B_6.

A remarkable issue is that some of the objects equivalent to biplanes on 16 points were also discoverdd in AG, in the framework of Kummer surfaces, see [4]. The new incaranation of the classical results by Hudson in modern clothes of AG appears in [3]. In this talk we are trying to tie all these loose ends, using in particular facilites of computer packages.

In fact, Clebsch studied in his classical paper [2] a class of quartic surfaces in $\mathbf{P}^3$ obtained as the image of four generic cubic homogeneous polynomials in three variables, that vanish at a given set of of five points in generic position in $\mathbf{P}^2$. In modern terms we can construct the surface by looking at the blowup of $\mathbf{P}^2$ in five points in general position. The blowup surface is smooth and can be embedded in $\mathbf{P}^4$. This is a Del Pezzo surface of degree 4 in $\mathbf{P}^4$ and projecting this surface from a generic point not on the surface, we obtain the Clebsch quartic in $\mathbf{P}^3$. The surface has 16 lines on it obtained as the images of the exceptional divisor, the line connecting two points in the blowup set and the unique conic passing through all of the five points. These are the vertices of the classical copy of *Cl* and the edges connect intersecting pairs of lines. We use Macaulay2 to construct *Cl* using the original Clebsch surface and the routine for constructing the Fano scheme of lines on the surface. An interesting connection, highlighted by Sturmfels and his collaborators, is that the Clebsch graph appears also via tropicalization of a degree four Dnl Pezzo surface, namely the tropicalization is a cone over the Clebsch graph.

One can also consider a Kummer surface, namely a singular quartic in $\mathbf{P}^3$ that has only nodes as singularities and a maximal number of them (sixteen in this case). A Kummer surface gives naturally a rise to a biplane via considering the sixteen nodes and the sixteen curves passing each through exactly six of those nodes. As mentioned above, one can obtain *Cl* from the configuration by the use of polarities.

A main question is can one obtain *Cl* from a Kummer surface using only the algebro-geometric toolkit? It was shown by Skorobogatov in [9], that every Del Pezzo surface of degree 4 admits a degree 2 branched covering map from the desingularization of a Kummer surface, that sends lines on the desingularization to the 16 lines on our Del Pezzo surface. It is now natural to ask whether this gives a geometric picture of the above stated classical combinatorial fact? Thus finally we report about our computer aided efforts to clarify this issue.

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