Cayley graphs based on octonions, and their implementation in MAGMA

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Cayley graphs occupy an important part in algebraic graph theory. Beyond the classical construction that requires groups, it is less known that *quasi-groups* are sufficient [4], *e.g.* to obtain regular graphs (under very mild assumptions). We have constructed new infinite families of regular Cayley graphs based on *Moufang loops* [1]. These loops (non-associative counterpart of a group) arise naturally as the multiplicative subloops $\mathbb{O}^\star(\mathbb{F}_q)$ of octonion algebras over a finite field $\mathbb{F}_q$. There are striking analogies between quotients of these loops by their center $\mathscr{Z}$, denoted $\mathbb{O}^\star(\mathbb{F}_q)/\mathscr{Z}$, and the groups $PGL_2(\mathbb{F}_q)$. This stems for the fact that the 2-by-2 matrices over $\mathbb{F}_p$ (p an odd prime) are isomorphic to some quaternion algebras $\mathbb{H}(\mathbb{F}_p)$, and that octonions are *doubling* algebras of quaternions.

While Cayley graphs on $PSL_2(\mathbb{F}_p)$ have been extensively studied with respect to many aspects, their non-associative counterparts much less (besides [2], we are not aware of concrete examples of construction of Cayley graphs on loops). The construction we have provided [1] is inspired by the famous Ramanujan graphs of Lubotzky-Phillips-Sarnak (LPS) [3]: first construct a free group on some generators of the integral octonions (say over $\mathbb{Z}$) of given norm p, yielding an infinite regular tree, and by reducing modulo another prime q, to obtain finite quotients of the infinite regular tree.

- For each odd prime p, there is a distinguished family $\mathscr{P}(p) \subset \mathbb{O}(\mathbb{Z})$ of p^3+1 integral octonions of norm p whose Cayley graph is an infinite regular tree.
- for each prime $q > p$, let $\mathscr{S}_{p,q}$ be the canonical image of $\mathscr{P}(p)$ in $\mathbb{O}^\star(\mathbb{F}_q)/\mathscr{Z}$: this yields a Cayley graph

  $$\mathscr{X}_{p,q} = \mathscr{C}ay(\mathscr{S}_{p,q}, \mathbb{O}^\star(\mathbb{F}_q)/\mathscr{Z}), \tag{1}$$

  of degree p^3+1, connected bipartite if $\left(\frac{p}{q}\right) = -1$ and non-bipartite with two connected components of same order otherwise. The order is $|\mathbb{O}^\star(\mathbb{F}_q)/\mathscr{Z}| = q^7 - q^3$.

Despite these analogies with the LPS Ramanujan graphs, studying the properties of the graphs $\mathscr{X}_{p,q}$ is much more difficult than for Cayley graphs on $PSL_2(\mathbb{F}_p)$. We conjecture that:

1. the graphs $\mathscr{X}_{p,q}$ are not vertex-transitive
2. the bipartite graphs $\mathscr{X}_{p,q}$ are semi-symmetric (edge-transitive, non vertex transitive).

However describing even a single non-trivial automorphism is not easy (note that the automorphism obtained by the multiplication by a group element in Cayley graphs on groups does not exist in Cayley graphs on loops). The sole construction of non-vertex transitive Cayley graphs on Moufang loops is in [2] where the authors used the notion of regular maps, thereby constraining to degree 3 regular graphs.

The investigation of the properties of the graphs $\mathscr{X}_{p,q}$ has motivated an implementation [1] in MAGMA; And in order to check the implementation, of the LPS Ramanujan graphs as well for which theoretical results are known and thus can be verified. Due the rapidly increasing order/size of these graphs when q or p grows, it becomes quickly impossible to build or even store the whole adjacency table of the graphs. We could compute however the second largest eigenvalue using the power method, and the girth (they are not Ramanujan graphs, neither have they large girth as the LPS Ramanujan graphs). However, these computations support the conjecture above (the girth is not uniform as in vertex-transitive graphs).

References

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[1] See the web-page `http://xdahan.sakura.ne.jp/Package/graph.html` for some brief instructions and to download the source code