Some new computer-aided models for the exceptional Zara graph on 126 vertices

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The exceptional Zara graph Z has the following properties:

- it is regular connected undirected graph on 126 vertices of valency 45;
- it contains a maximal clique C of size 6;
- each vertex x outside of clique C is adjacent to the same number $e_c = 2$ of neighbours in C.

The number e_c is called the *nexus* of C. (In fact all maximal cliques of Z have the same size 6 and nexus equal to 2.)

It was proved in [1] that these properties define unique, up to isomorphism, graph, namely the strongly regular graph Z with the parameters (126,45,80,12,18). The automorphism group $G = Aut(Z)$ is a rank 3 group of order 13063680.

Our interest to the graph Z stems from the investigation of so-called total graph coherent configurations. Namely, it was proved in [5] that an SRG Γ_1 with the same parameters and order of group appears as a suitable merging of the total graph coherent configuration, defined by the triangular graph $T(7)$. On this way we get a new model Γ_1 of Z, which is invariant with respect to S_7, having orbits of length 21 and 105 on vertices and 6 orbits on edges. This was established via the use of computer package COCO [2]. It is known that the graph Z has exactly 567 maximal cliques. For the created model Γ_1 these cliques split into three orbits of lengths 105, 210, 252; the members of each orbit have a nice combinatorial interpretation in terms of considered action of S_7.

The group $G = Aut(Z)$ contains as a subgroup of index 4 simple group $PSU(4,9)$ aka $U_4(3)$. This simple group is isomorphic to $P\Omega^-(6,3)$. Exactly this latter group was investigated by W.L. Edge in [3], where its primitive actions of degree 126 and 567 were clearly explained in classical terms of finite geometries.

In our attempts to create a reasonably clear model of Z, starting from a relatively small subgroup of G, acting transitively on the point set of Z, our attention was attracted to two conjugacy classes of subgroups of $U_4(3)$, both isomorphic to $PSU(3,3)$ of order 6048. In fact, for each of these two classes an overgroup $H = P\Gamma U(3,3)$ of order 12096 is also a subgroup of G.

First, the group H was regarded as the group $Aut(H(3))$ of the automorphisms of the classical hermitian unital with 28 points and 63 blocks of size 4. This unital $H(3)$ has exactly one orbit of spreads of length 63. Some properties of these spreads, that is partitions of the vertex set into 7 blocks, were carefully investigated. Using GAP [4], two conjugacy classes of subgroups L_1 and L_2 of H of order 96, both having orbits of length 4 and 24 on the points of $H(3)$, were detected and interpreted in ad hoc combinatorial terms. Transitive actions of H of degree 126 on cosets of L_1 and L_2 have rank 6 and 8 respectively. In each of the appearing association schemes there exists a rank 3 merging, with basic graphs Γ_2 and Γ_3, both isomorphic to Z.

Finally, a suitable amalgam of groups S_7 and H in G is investigated. It allows to outline a computer free proof of the fact that all the three graphs Γ_i, $i = 1,2,3$, are isomorphic to Z.

References

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