

# Algebraic Graph Theory Algorithms For Modern Computer Architectures

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Algorithms play a big role in AGT. Examples of tasks that are solved algorithmically include the following:

- Isomorphism tests of coherent configurations;
- Finding the full automorphism group of a coherent configuration;
- Stabilization procedures such as Weisfeiler-Lehman, which finds the smallest coherent configuration containing a given set of relations;
- Enumeration of mergings of coherent configurations.

Whereas computers used to be modeled using a single central processing unit having access to a uniform random access memory this description is no longer accurate.

Modern computers, even in the consumer PC range, provide parallelism on several levels. Wide registers can accommodate lots of data and the possibility of SIMD processing (Single instruction, multiple data). Processors contain several more or less independent processing units or cores. Moreover, single cores can execute interleave several threads of instruction, leading to apparent parallel execution (SMT, simultaneous multithreading). Computers appear which contain several processors, and finally, large numbers of computers are interconnected in networks.

Since each of the processors has its own memory the assumption of uniform memory the assumption of uniform memory access is not fulfilled either. But even on a single processor we deal with the fact that faster memory is more expensive than slower memory. This leads to a hierarchical organization of memory, with six or more levels of slower and more abundant memory, ranging from hundreds of bytes of registers over caches and DRAM to terabytes on a hard drive.

Since memory access dominates arithmetical computations in many problems in AGT, the layout of data in memory is crucial for high performant algorithms.

We look at implementations of two algorithms:

The algorithm for finding coherent configurations by Weisfeiler-Lehman [3] was originally stated in terms of matrices:

1. Replace entries of the given matrix with non-commuting indeterminates

2. Compute the square of the matrix
3. Repeat until the number of distinct entries is stable.

Two implementations have been described [1], with different practical and theoretical complexity properties. We give a new implementation which is practically faster on many examples and has moderate space requirements.

S-rings are particular instances of association schemes. They are invariant under a regular permutation group. S-rings over a group  $H$  are thus mergings of the centralizer ring of a regular action of  $H$ . These correspond to certain partitions of  $H$ .

Ziv-Av has enumerated all S-rings over groups of order up to 63 [4]. The elementary abelian group of order 64 was previously dealt with by the author.

The enumeration of mergings proceeds in two stages:

1. Enumeration of "good" subsets of  $H$ .
2. Constructing adequate partitions from those sets.

For most groups the first stage is the hardest part. Here we need to consider all subsets of  $H$ , so the search space has the shape of a hypercube. We consider the following optimizations:

- Using the automorphism group to reduce the search tree.
- Coarse parallelism, processing different parts of the tree simultaneously.
- Using the self-similarity of the search space. By reordering the search we can make use of SIMD instructions.

## References

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